

GEOMETRY ASSOCIATED WITH SEMISIMPLE FLAT HOMOGENEOUS SPACES⁽¹⁾

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Abstract. Our object is Cartan connections with semisimple flat homogeneous spaces as standard spaces. We study these from the viewpoint of G -structures of second order. This allows us especially to treat classical projective and conformal connections in the unifying manner. We also consider its application to the problem of certain geometric transformations.

Introduction. Let M be a manifold of dimension n . We denote by $\mathcal{F}^1(M)$ the frame bundle of M , and by $A(\mathcal{F}^1(M))$ the set of all linear connections without torsion on $\mathcal{F}^1(M)$. The projective geometry of linear connections is the study of invariants of projectively equivalent classes in $A(\mathcal{F}^1(M))$. It is the classical theory by E. Cartan that for each projectively equivalent class there corresponds uniquely a projective Cartan connection [4].

Examining the algebraic structure of projective spaces, N. Tanaka introduced the notion of a l -system l and the homogeneous space $M_l = L(l)/L_0(l)$ [29]. Generalizing the notion of projective equivalence, he gave the notion of l -equivalence in the set $A(P)$ of all $G_0(l)$ -connections without torsion on a given $G_0(l)$ structure P over M , where $G_0(l)$ is the linear isotropy group of $M = L(l)/L_0(l)$. Then he showed that for each l -equivalence class there corresponds uniquely a Cartan connection of type M_l [29].

On the other hand S. Kobayashi and T. Nagano found that the projective equivalence is also defined by using the bundle $\mathcal{F}^2(M)$ of 2nd order frames of M [13]. And they reconstructed the above theory of E. Cartan [13].

Our first purpose in this paper is to rebuild the theory of N. Tanaka from the viewpoint of 2nd order structures, thus generalizing the results in [13]. The Spencer cohomology group of a graded Lie algebra plays an important role in the constructions.

Next, we deal with the geometric tensors associated with our spaces. As one of the examples, a generalization of Weyl's projective tensors is given. It is proved that its vanishing is equivalent to the integrability of certain geometric structures.

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Our geometry depends heavily upon the theory of semisimple graded Lie algebras. §§1 through 7 are devoted to a study of graded Lie algebras and their Spencer cohomology groups in detail for our construction and theorem.

As a final purpose, we give some applications to the problem of geometric transformations. We generalize the well-known relationship between the group of isometries and that of projective transformations of a riemannian manifold with a parallel Ricci-tensor (cf. [20] and [21]).

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Notations. Throughout the paper the following standard conventions have been adopted. \mathbf{Z} denotes the ring of integers. \mathbf{R} (resp. \mathbf{C}) denotes the field of real (resp. complex) numbers and \mathbf{F} denotes \mathbf{R} or \mathbf{C} . For a finite dimensional vector space V over \mathbf{F} , we denote by V^* the dual space of V . Also we denote with pointed brackets $(x \cdot f)$ the value of the canonical bilinear mapping $b: V \times V^* \rightarrow \mathbf{F}$ taken at $\alpha \in V$ and $f \in V^*$. We denote by $\Lambda^p V$ (resp. ΛV) the p th exterior product of V (resp. the sum $\sum \Lambda^p V$). For a real vector space V , $V^{\mathbf{C}}$ is the complexification of V , and for a complex vector space V , $V_{\mathbf{R}}$ is the restriction of the field to \mathbf{R} .

For a point x of a manifold M , $T(M)_x$ is the tangent space of M at x . For a differentiable mapping f , f_* and f^* are the differential and the codifferential of f respectively. For a Lie group G , its Lie algebra is written by the corresponding German letter \mathfrak{g} . The adjoint representation of G (resp. \mathfrak{g}) into \mathfrak{g} is denoted by Ad (resp. ad). For a vector space V over \mathbf{F} , $GL(V)$ denotes the general linear group, and $\mathfrak{gl}(V)$ denotes its Lie algebra. For a G -principal bundle $P \rightarrow M$, R_a denotes the right action of an element a of G on P . For an element A of \mathfrak{g} , A^* denotes the fundamental vector field on P corresponding to A .

1. Outline of the results. We consider the l -system due to N. Tanaka from a different aspect, that is, from the viewpoint of graded Lie algebras (cf. [26]).

(a) A homogeneous space L/L_0 is called *flat of order r* if the Lie algebra \mathfrak{l} of L has a graded Lie algebra structure $\mathfrak{l} = \sum_{p \in \mathbf{Z}} \mathfrak{g}_p$ ($\mathfrak{g}_p = 0$ if and only if $p \leq -2$ or $p \geq r$) such that $\mathfrak{l}_0 = \sum_{p \geq 0} \mathfrak{g}_p$ is the Lie subalgebra corresponding to L_0 . L/L_0 is called *semisimple flat* if L is furthermore semisimple. For a semisimple flat homogeneous space L/L_0 , we shall prove:

(a-1) the order is two, i.e., $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$;

(a-2) L_0 is a semidirect product $L_0 = G'_0 \cdot G_1$ of G'_0 and a connected Lie subgroup G_1 such that the Lie algebra of G'_0 and G_1 are \mathfrak{g}_0 and \mathfrak{g}_1 respectively;

(a-3) G_1 is identical to the kernel of the linear isotropy representation, i.e., G'_0 is isomorphic to the linear isotropy group G_0 . By this we identify G'_0 and G_0 .

Homogeneous spaces M_l defined by N. Tanaka are examples of semisimple flat homogeneous spaces. For those M_l , the above results were proved in [29].

(b) \mathfrak{g}_{-1} is an abelian Lie subalgebra of \mathfrak{l} . We consider the Lie algebra cohomology

groups $H(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l}) = \sum_{q \geq 0} H^q(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})$ of \mathfrak{g}_{-1} with respect to its adjoint representation in \mathfrak{l} . The graded Lie algebra structure $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ induces another decomposition

$$H^q(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l}) = \sum_{p \geq 0} H^{p,q}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l}) \quad \text{for } q \geq 0.$$

The cohomology groups $H(\mathfrak{g}_{-1}^{\mathcal{C}}, \text{ad}^{\mathcal{C}}, \mathfrak{l}^{\mathcal{C}}) = \sum H^q(\mathfrak{g}_{-1}^{\mathcal{C}}, \text{ad}^{\mathcal{C}}, \mathfrak{l}^{\mathcal{C}})$ has been studied by B. Kostant [16] and Y. Matsushima-S. Murakami [17], using a certain *harmonic theory* with respect to the canonically defined inner-product. Using their results, we shall prove:

- (b-1) $H^{2,1}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})$ vanishes for almost all L/L_0 ;
- (b-2) $H^{2,2}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})$ vanishes for almost all L/L_0 ; and we shall give
- (b-3) the necessary and sufficient condition for $H^{1,1}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})$ to vanish.

The equivalent statements of (b-1) and (b-3) are shown also in [29].

(c) The results of (a) permit us to consider the principal bundle $L \rightarrow L/L_0$ as a subbundle of the bundle of frames of 2nd order contact of L/L_0 . For a manifold M , we denote by $\mathcal{F}^r(M)$ the bundle of frames of r th order contact [12]. Its structure group is denoted by $G^r(n)$ if $\dim M = n$. $\mathcal{F}^1(M)$ is usually called the *frame bundle* of M and $G^1(n)$ is isomorphic to $GL(n, \mathbf{R})$. $GL(n, \mathbf{R})$ can be considered as a subgroup of $G^r(n)$. We shall prove:

- (c-1) $L_0 = G_0 \cdot G_1$ is a subgroup of $G^2(n)$;
- (c-2) $L_0 \cap GL(n, \mathbf{R}) = G_0$;
- (c-3) $L \rightarrow L/L_0$ is a L_0 -subbundle of $\mathcal{F}^2(L/L_0) \rightarrow L/L_0$.

(d) Let P be a G_0 -structure on M i.e., a G_0 -subbundle of $\mathcal{F}^1(M)$. From a theorem of S. Kobayashi [12], each G_0 -connection on P without torsion gives a G_0 -subbundle of $\mathcal{F}^2(M) \rightarrow M$. Since $G^2(n) \supset L_0 \supset G_0$, we have a L_0 -subbundle of $\mathcal{F}^2(M) \rightarrow M$ extending the structure group. Thus each G_0 -connection Γ on P without torsion (if it exists) naturally gives a L_0 -subbundle $Q(\Gamma)$ of $\mathcal{F}^2(M) \rightarrow M$. The above consideration gives the *L_0 -equivalence relation* in the set $A(P)$ of all G_0 -connections on P without torsions as follows: two elements Γ and Γ' of $A(P)$ are equivalent if and only if $Q(\Gamma) = Q(\Gamma')$. It is easy to see that the above L_0 -equivalence is equivalent to the l -equivalence due to N. Tanaka [29]. Then we shall prove:

(d-1) A fixed equivalence class in $A(P)$ gives rise to a one-to-one correspondence between the equivalence class of $A(P)$ and $H^{1,1}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})$.

(e) For each L_0 -equivalence class α in $A(P)$, we denote by $Q(\alpha)$ the L_0 -subbundle of $\mathcal{F}^2(M) \rightarrow M$ defined by any element of α . $Q(\alpha)$ is called an *L_0 -structure of 2nd order* associated with α . From a theorem of C. Ehresmann we easily see that there exists at least one Cartan connection on $Q(\alpha)$ of type L/L_0 , i.e., an infinitesimal connection (satisfying one additional condition) on the L -principal bundle associated with the L_0 -principal bundle $Q(\alpha)$. We can single out the most natural one among others by the following results:

- (e-1) For each Cartan connection ω on $Q(\alpha)$ its curvature defines a cocycle

$c(\omega)_x$ in $H^{1,2}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})$ ($x \in Q(\alpha)$) and for each point $x \in Q(\alpha)$ its cohomology class is independent of the choice of a particular ω ;

(e-2) If $H^{2,1}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})=0$, there exists one and only one Cartan connection ω such that $c(\omega)_x$ is harmonic for each point $x \in Q(\alpha)$ (see (b)). We call the Cartan connection in (e-2) the *normal Cartan connection* of an L_0 -equivalence class α . The above construction of normal Cartan connections seems to give a better interpretation of N. Tanaka's normal Cartan connection of type M_1 .

(f) Using the normal Cartan connection, we shall define the Weyl tensor $W(\alpha)$ of an L_0 -equivalence class α . $W(\alpha)$ is a tensor of type (1.3) on M . We shall prove:

(f-1) $Q(\alpha) \rightarrow M$ is locally equivalent to $L \rightarrow L/L_0$ if and only if $W(\alpha)$ vanishes, provided $H^{2,2}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})=0$.

If $H^{1,1}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})=0$, then by (d-1) we have the well-defined tensor $W(P)$ for each G_0 -structure provided $A(P)$ is not empty. We shall prove:

(f-2) If $H^{1,1}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})=0$, G_0 -structure P is locally flat if and only if $W(P)=0$.

(g) Let Γ be an element of an L_0 -equivalence class α . We denote by $\text{Aut}(P, \alpha)$ (resp. $\text{Aut}(P, \Gamma)$) the group of all G_0 -automorphisms preserving α (resp. Γ). It is plausible to conjecture: if Γ is Ricci-parallel and $\text{Aut}(P, \alpha) \neq \text{Aut}(P, \Gamma)$, M is diffeomorphic to L/L_0 and $\text{Aut}(P, \alpha)=L$ (up to covering). We shall prove the conjecture under some restrictive additional conditions:

(g-1) If $\text{Aut}(P, \alpha)$ is transitive and irreducible, the above conjecture is true.

(h) Let us point out some connections with classical results. Examples of semi-simple flat homogeneous spaces are given by:

- (1) a projective space and the group of its projective transformations, and
- (2) a Moebius space and the group of its conformal transformations.

When L/L_0 is a projective space, G_0 is $GL(n, \mathbf{R})$ and L_0 -equivalence is the projective equivalence. For a projective equivalence class α , our Weyl tensor $W(\alpha)$ coincides with the classical Weyl's projective tensor [30]. When L/L_0 is a Moebius space, G_0 is $\text{CO}(n)$. In this case we have $H^{1,1}(\mathfrak{g}_{-1}, \text{ad}, \mathfrak{l})=0$. Then the Weyl tensor $W(P)$ for any $\text{CO}(n)$ -structure coincides with the classical Weyl's conformal tensor [30]. It is easily seen that in both cases our conjecture is true from [20] and [21].

2. Graded Lie algebras and Spencer cohomologies. A finite dimensional Lie algebra \mathfrak{g} over F ($H=C$ or \mathbf{R}), is called a *graded Lie algebra* over F if \mathfrak{g} is a direct sum of subspaces \mathfrak{g}_p ($p \in \mathbf{Z}$) such that

$$(2-1). \quad \mathfrak{g}_p = 0 \text{ for } p \leq -2,$$

$$(2-2). \quad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \text{ for all } p \text{ and } q,$$

$$(2-3). \quad \text{for an } x \in \mathfrak{g}_p \text{ (} p \geq 0, x \neq 0 \text{), } [\mathfrak{g}_{-1}, x] \neq 0.$$

Two graded Lie algebras (over F) $\mathfrak{g} = \sum \mathfrak{g}_p$ and $\mathfrak{g}' = \sum \mathfrak{g}'_p$ are called *isomorphic* if there exists a Lie algebra isomorphism $\phi: \mathfrak{g} \rightarrow \mathfrak{g}'$ such that $\phi(\mathfrak{g}_p) = \mathfrak{g}'_p$ for all p . By a *graded Lie subalgebra* $\mathfrak{h} = \sum \mathfrak{h}_p$ of the graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_p$, we mean a subalgebra \mathfrak{h} of \mathfrak{g} such that $\mathfrak{h}_p \subset \mathfrak{g}_p$. If $\mathfrak{g}_p \neq 0$, then from (2-3) we have $\mathfrak{g}_q \neq 0$ for $-1 \leq q \leq p$. Thus there exists uniquely an integer r such that $\mathfrak{g}_p \neq 0$ for $-1 \leq p \leq r-1$

and $\mathfrak{g}_p = 0$ for $p \geq r$. The integer r is called the *order* of the graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_p$. For two graded Lie algebras $\mathfrak{g} = \sum \mathfrak{g}_p$ and $\mathfrak{g}' = \sum \mathfrak{g}'_p$ we define their direct sum $\tilde{\mathfrak{g}} = \sum \tilde{\mathfrak{g}}_p$ in the following natural manner; $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$ (direct sum) and $\tilde{\mathfrak{g}}_p = \mathfrak{g}_p \oplus \mathfrak{g}'_p$ (direct sum). For a graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_p$ over \mathbf{R} (resp. \mathbf{C}) we can naturally define its complexification $\mathfrak{g}^{\mathbf{C}}$ (resp. the restriction of the field $\mathfrak{g}_{\mathbf{R}}$).

LEMMA 2-1. *For a graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_p$, we have*

- (2-4). \mathfrak{g}_{-1} is an abelian Lie subalgebra of \mathfrak{g} ,
- (2-5). \mathfrak{g}_0 is a Lie subalgebra of \mathfrak{g} ,
- (2-6). $\sum_{p \geq 0} \mathfrak{g}_p$ is a Lie subalgebra of \mathfrak{g} ,
- (2-7). $\sum_{p \geq k} \mathfrak{g}_p$ is an ideal of $\sum_{p \geq 0} \mathfrak{g}_p$,
- (2-8). the adjoint representation $\text{ad}_{\mathfrak{g}} \mathfrak{g}_0$ of \mathfrak{g}_0 into \mathfrak{g} leaves each \mathfrak{g}_p invariant.

Proof. These are trivial from the definition.

By (2-8) we have a representation $\text{ad}_{\mathfrak{g}} \mathfrak{g}_0|_{\mathfrak{g}_p}$ of \mathfrak{g}_0 into \mathfrak{g}_p . The representation $\text{ad}_{\mathfrak{g}} \mathfrak{g}_0|_{\mathfrak{g}_{-1}}$ is especially called the *linear isotropy representation* of $\mathfrak{g} = \sum \mathfrak{g}_p$. By (2-3) this representation is faithful. The graded Lie algebra is called *irreducible* if its linear isotropy representation is irreducible.

For a graded Lie algebra (over F) $\mathfrak{g} = \sum \mathfrak{g}_p$, the Lie algebra cohomology $H(\mathfrak{g})^{(2)} = H(\mathfrak{g}_{-1}, \text{ad}_{\mathfrak{g}}|_{\mathfrak{g}_{-1}}, \mathfrak{g})$ of the abelian Lie algebra \mathfrak{g}_{-1} with respect to its adjoint representation into \mathfrak{g} is called the *Spencer cohomology* of $\mathfrak{g} = \sum \mathfrak{g}_p$. With the graded structure in mind, let us recall the definition. Let $C^{p,q}(\mathfrak{g})$ be the vector space of all \mathfrak{g}_{p-1} -valued q -forms on \mathfrak{g}_{-1} (i.e., $C^{p,q}(\mathfrak{g}) = \mathfrak{g}_{p-1} \otimes \Lambda^q(\mathfrak{g}_{-1})^*$) and put $C(\mathfrak{g}) = \sum C^{p,q}(\mathfrak{g})$. Now we define coboundary operator $\partial: C(\mathfrak{g}) \rightarrow C(\mathfrak{g})$ to be $\partial C^{p,q}(\mathfrak{g}) \subset C^{p-1,q+1}(\mathfrak{g})$ and

(2-9). $(\partial c)(x_1, \dots, x_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} [x_i, c(x_1, \dots, \hat{x}_i, \dots, x_{q+1})]$ for all $c \in C^{p,q}(\mathfrak{g})$ and $x_1, \dots, x_{q+1} \in \mathfrak{g}_{-1}$. Then we have $\partial^2 = 0$. Thus we have the cohomology $H(\mathfrak{g})$ by

(2-10). $H(\mathfrak{g}) = \partial^{-1}(0)/\partial(\mathfrak{g})$.

Putting

(2-11). $H^{p,q}(\mathfrak{g}) = \partial^{-1}(0) \cap C^{p,q}(\mathfrak{g})/\partial(C^{p+1,q-1}(\mathfrak{g}))$, $H(\mathfrak{g})$ is naturally isomorphic to $\sum H^{p,q}(\mathfrak{g})$.

LEMMA 2-2. *We have*

- (2-12). $H^{0,0}(\mathfrak{g}) \cong \mathfrak{g}_{-1}$,
- (2-13). $H^{p,0}(\mathfrak{g}) = 0$ if $p \neq 0$,
- (2-14). $H^{0,1}(\mathfrak{g}) \cong \mathfrak{gl}(\mathfrak{g}_{-1})/\mathfrak{g}_0$.

Proof. These are easy to see. (2-13) comes from (2-3).

The following are standard facts on Lie algebra cohomologies [25].

(2) We should write $H(\sum \mathfrak{g}_p)$. But we take this notation if there is no danger of confusion.

LEMMA 2-3. Let $\mathfrak{g} = \sum \mathfrak{g}_p$ be the direct sum of two graded Lie algebras $\mathfrak{g} = \sum \mathfrak{g}_p$ and $\mathfrak{g}' = \sum \mathfrak{g}'_p$. Then we have

$$H^{p,q}(\mathfrak{g}) = \left(\sum_{s+t=q} H^{p,s}(\mathfrak{g}) \otimes \Lambda^t(\mathfrak{g}'_{-1})^* \right) \oplus \left(\sum_{s+t=q} \Lambda^s(\mathfrak{g}_{-1})^* \otimes H^{p,t}(\mathfrak{g}') \right).$$

LEMMA 2-4. Let \mathfrak{g}^C be the complexification of a real graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_p$. Then we have $H^{p,q}(\mathfrak{g}^C) \cong H^{p,q}(\mathfrak{g}) \otimes \mathbb{C}$ for all p, q .

The Lie subalgebra \mathfrak{g}_0 has a natural representation ρ on each $C^{p,q}(\mathfrak{g})$. Then it is easy to see

LEMMA 2-5. We have $\partial(\rho(a)c) = \rho(a)(\partial c)$ for all $c \in C(\mathfrak{g})$ and $a \in \mathfrak{g}_0$.

By the above lemma, there is a natural representation ρ of \mathfrak{g}_0 into

$$H(\mathfrak{g}) = \sum H^{p,q}(\mathfrak{g}).$$

3. Semisimple graded Lie algebras. A graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_p$ over F is called *semisimple* (resp. *simple*) if \mathfrak{g} is semisimple (resp. simple) over F . Unless otherwise stated, we assume, in this section, all graded Lie algebras considered are semisimple (or simple) over F .

PROPOSITION 3-1. The order of $\mathfrak{g} = \sum \mathfrak{g}_p$ is 2.

Proof. Take $z \in \mathfrak{g}_2$. Then $\text{ad } x \text{ ad } z: \mathfrak{g} \rightarrow \mathfrak{g}$ is nilpotent for all $x \in \mathfrak{g}$. Denoting by B the Killing form of \mathfrak{g} , we have $B(x, \mathfrak{g}) = 0$. Since B is nondegenerate, we have $x = 0$.

LEMMA 3-1. The linear endomorphism α of $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ defined by

$$\alpha(x+y+z) = -x+y-z \quad \text{for } x \in \mathfrak{g}_{-1}, y \in \mathfrak{g}_0, z \in \mathfrak{g}_1,$$

is an involutive automorphism of \mathfrak{g} .

Proof. The proof is straightforward.

LEMMA 3-2. Denote by B the Killing form of \mathfrak{g} . Then

(3-1). $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ and \mathfrak{g}_0 are perpendicular to each other with respect to B , so that both $B|_{\mathfrak{g}_{-1} \oplus \mathfrak{g}_1}$ and $B|_{\mathfrak{g}_0}$ are nondegenerate;

(3-2). $B|_{\mathfrak{g}_{-1}} = 0$ and $B|_{\mathfrak{g}_1} = 0$;

(3-3). \mathfrak{g}_{-1} is the dual vector space of \mathfrak{g}_1 under the pairing $(x, z) \rightarrow B(x, z)$ ($x \in \mathfrak{g}_{-1}, z \in \mathfrak{g}_1$).

Proof. (i) Let α be the involutive automorphism of \mathfrak{g} defined in Lemma (3-1). Since B is invariant by α , we have

$$B(x+z, y) = B(\alpha(x+z), \alpha(y)) = B(-x-z, y) \quad \text{for } x \in \mathfrak{g}_{-1}, y \in \mathfrak{g}_0 \text{ and } z \in \mathfrak{g}_1.$$

Hence $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ and \mathfrak{g}_0 are perpendicular with respect to B . Thus, since B is nondegenerate, so are its restrictions to $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ and \mathfrak{g}_0 .

(ii) Let $x, x' \in \mathfrak{g}_{-1}$. Then $\text{ad } x \text{ ad } x': \mathfrak{g} \rightarrow \mathfrak{g}$ is nilpotent. Hence $B(x, x')=0$ proving $B|\mathfrak{g}_{-1}=0$. Similarly $B|\mathfrak{g}_1=0$.

(iii) Let $x \in \mathfrak{g}_{-1}$, and assume $B(x, \mathfrak{g}_1)=0$. By (3-1) and (3-2), this implies $B(x, \mathfrak{g})=0$. Hence $x=0$. Similarly, if $z \in \mathfrak{g}_1$, and $B(\mathfrak{g}_{-1}, z)=0$, then $z=0$.

LEMMA (3-3). *The representation $\text{ad}_{\mathfrak{g}} \mathfrak{g}_0|\mathfrak{g}_{-1}$, and $\text{ad}_{\mathfrak{g}} \mathfrak{g}_0|\mathfrak{g}_1$ of \mathfrak{g}_0 are dual to each other with respect to the Killing form B .*

Proof. This follows from the invariance of B under $\text{ad } y$ ($y \in \mathfrak{g}_0$), i.e.

$$B(\text{ad } (y)x, z) = -B(x, \text{ad } (y)z) \quad \text{for } x \in \mathfrak{g}_{-1}, z \in \mathfrak{g}_1.$$

LEMMA (3-4). *There exists uniquely an element e in \mathfrak{g}_0 such that*

$$\begin{aligned} [e, x] &= -x && \text{for } x \in \mathfrak{g}_{-1}, \\ [e, y] &= 0 && \text{for } y \in \mathfrak{g}_0, \\ [e, z] &= z && \text{for } z \in \mathfrak{g}_1. \end{aligned}$$

Proof [14]. Let E be the linear endomorphism defined by $E(x)=-x$ for $x \in \mathfrak{g}_{-1}$, $E(y)=0$ for $y \in \mathfrak{g}_0$ and $E(z)=z$ for $z \in \mathfrak{g}_1$. Then it is easy to see that E is a derivation of \mathfrak{g} . Since every derivation of a semisimple Lie algebra is inner, there is a unique element e in \mathfrak{g} such that $E=\text{ad } e$ (the uniqueness follows from the fact that \mathfrak{g} has no center). Let α be the involutive automorphism defined in Lemma (3-1). We can easily see that $\text{ad } (\alpha(e))$ coincides with E . Hence $\alpha(e)=e$ proving $e \in \mathfrak{g}_0$.

COROLLARY (3-1). *An abelian Lie subalgebra \mathfrak{a} of \mathfrak{g} containing e (of Lemma (3-4)) is contained in \mathfrak{g}_0 . In particular there exists a Cartan subalgebra \mathfrak{h} of \mathfrak{g} contained in \mathfrak{g}_0 .*

Proof. Take an element $a=x+y+z$ ($a \in \mathfrak{a}$, $x \in \mathfrak{g}_{-1}$, $y \in \mathfrak{g}_0$, $z \in \mathfrak{g}_1$). Then $0=[e, a]=[e, x+y+z]=-x+z$. Hence $x=0, z=0$, i.e., $\mathfrak{a} \subset \mathfrak{g}_0$. Since $\text{ad } e$ is F -diagonalizable, there exists a Cartan subalgebra \mathfrak{h} containing e . \mathfrak{h} being abelian, we get $\mathfrak{h} \subset \mathfrak{g}_0$.

PROPOSITION (3-2) [29]. (i) *Let $\mathfrak{g}=\mathfrak{F}_1 \oplus \dots \oplus \mathfrak{F}_r$ be the decomposition of \mathfrak{g} into its simple factors. Then*

$$\mathfrak{F}_j = (\mathfrak{g}_{-1} \cap \mathfrak{F}_j) \oplus (\mathfrak{g}_0 \cap \mathfrak{F}_j) \oplus (\mathfrak{g}_1 \cap \mathfrak{F}_j)$$

and $\mathfrak{F}_j = \sum (\mathfrak{g}_p \cap \mathfrak{F}_j)$ is a simple graded Lie algebra.

(ii) $\mathfrak{g} = \sum \mathfrak{g}_p$ is isomorphic to the direct sum of $\mathfrak{F}_j = \sum (\mathfrak{g}_p \cap \mathfrak{F}_j)$ ($j=1, \dots, r$).

Proof. Take an element $a=x+y+z$ ($a \in \mathfrak{F}_j$, $x \in \mathfrak{g}_{-1}$, $y \in \mathfrak{g}_0$, $z \in \mathfrak{g}_1$). Then $\mathfrak{F}_j \ni [e, a]=-x+z$ and $\mathfrak{F}_j \ni [e, [e, a]]=x+z$. Hence $\mathfrak{F}_j = (\mathfrak{g}_{-1} \cap \mathfrak{F}_j) \oplus (\mathfrak{g}_0 \cap \mathfrak{F}_j) \oplus (\mathfrak{g}_1 \cap \mathfrak{F}_j)$. The rest of our assertions are straightforward.

PROPOSITION (3-3). *A graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_p$ is simple if and only if it is irreducible and the order is > 1 .*

Proof. See [14], [23] or [29].

COROLLARY (3-2). *We have $B(x, z) = 2(\text{the trace of ad } [x, z]|_{\mathfrak{g}_1})$ for all $x \in \mathfrak{g}_{-1}$ and $z \in \mathfrak{g}_1$.*

Proof [29]. $2(\text{the trace of ad } [x, z]|_{\mathfrak{g}_1}) = B([x, z], e) = B(x, [z, e]) = B(x, z)$.

COROLLARY (3-3). *We have $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$ and \mathfrak{g}_0 is a reductive Lie algebra.*

Proof. From Proposition (3-2), clearly we can assume that \mathfrak{g} is simple. Then, by Proposition 3-3, \mathfrak{g}_0 has a faithful irreducible representation. Therefore \mathfrak{g}_0 must be reductive. It is easy to see that $\mathfrak{g}_{-1} \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus \mathfrak{g}_1$ is an ideal of \mathfrak{g} . Therefore we have $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$.

PROPOSITION (3-4). *$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}' = \mathfrak{g}'_{-1} \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ are isomorphic in the sense in §2 if and only if $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ are isomorphic as abstract Lie algebras.*

Proof. Let $\phi: \mathfrak{g}_0 \oplus \mathfrak{g}_1 \rightarrow \mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ be an isomorphism. We denote by \mathfrak{c} and \mathfrak{c}' the center of \mathfrak{g}_0 and \mathfrak{g}'_0 respectively. Then clearly $[\mathfrak{g}_0, \mathfrak{g}_0] \oplus (\mathfrak{c} \oplus \mathfrak{g}_1)$ and $[\mathfrak{g}'_0, \mathfrak{g}'_0] \oplus (\mathfrak{c}' \oplus \mathfrak{g}'_1)$ are the Levi decomposition of $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ respectively. Since the Levi decomposition is unique up to conjugation, we can assume $\phi([\mathfrak{g}_0, \mathfrak{g}_0]) = [\mathfrak{g}'_0, \mathfrak{g}'_0]$ and $\phi(\mathfrak{c} \oplus \mathfrak{g}_1) = \mathfrak{c}' \oplus \mathfrak{g}'_1$, changing ϕ if necessary. Then we have

$$\phi(\mathfrak{g}_1) = \phi([\mathfrak{c} \oplus \mathfrak{g}_1, \mathfrak{c} \oplus \mathfrak{g}_1]) = [\mathfrak{c}' \oplus \mathfrak{g}'_1, \mathfrak{c}' \oplus \mathfrak{g}'_1] = \mathfrak{g}'_1.$$

We shall show that for a nonzero element y of \mathfrak{c} the \mathfrak{c}' -component of $\phi(y)$ is not zero. In fact, otherwise, for any $z \in \mathfrak{g}_1$, we have $\phi([y, z]) = [\phi(y), \phi(z)] = 0$. Hence $[y, z] = 0$ since $\phi(\mathfrak{g}_1) = \mathfrak{g}'_1$. From this we get $y = 0$. We define an isomorphism $\phi': \mathfrak{c} \rightarrow \mathfrak{c}'$ by $\phi(y) = \phi'(y) \bmod \mathfrak{g}'_1$. Then we define a new mapping $\phi'': [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{c} \oplus \mathfrak{g}_1 \rightarrow [\mathfrak{g}'_0, \mathfrak{g}'_0] \oplus \mathfrak{c}' \oplus \mathfrak{g}'_1$ by $\phi''(x + y + z) = \phi(x) + \phi'(y) + \phi(z)$ ($x \in [\mathfrak{g}_0, \mathfrak{g}_0]$, $y \in \mathfrak{c}$ and $z \in \mathfrak{g}_1$). It is easy to see that ϕ'' is an isomorphism from $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ onto $\mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ such that $\phi''(\mathfrak{g}_0) = \mathfrak{g}'_0$ and $\phi''(\mathfrak{g}_1) = \mathfrak{g}'_1$. Therefore we can assume that our isomorphism ϕ satisfies $\phi(\mathfrak{g}_0) = \mathfrak{g}'_0$ and $\phi(\mathfrak{g}_1) = \mathfrak{g}'_1$. Using the duality of \mathfrak{g}_{-1} and \mathfrak{g}_1 (Lemma (3-2) and Lemma (3-3)), we can extend ϕ to a linear endomorphism from \mathfrak{g}_{-1} onto \mathfrak{g}'_{-1} . In fact, it suffices to show $\phi([x, z]) = [\phi(x), \phi(z)]$ for $x \in \mathfrak{g}_{-1}$, $z \in \mathfrak{g}_1$. We denote by B' the Killing form of \mathfrak{g}' . For $y \in \mathfrak{g}_0$, we have

$$\begin{aligned} B'(\phi([x, z]), \phi(y)) &= B([x, z], y) && \text{(easy to see)} \\ &= B(x, [z, y]) = B'(\phi(x), \phi([z, y])) && \text{(from the definition)} \\ &= B'(\phi(x), [\phi(z), \phi(y)]) = B'([\phi(x), \phi(z)], \phi(y)) && \text{(Lemma 3-2).} \end{aligned}$$

COROLLARY (3-4). *$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}' = \mathfrak{g}'_{-1} \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ are isomorphic if and only if \mathfrak{g}_0 and \mathfrak{g}'_0 are isomorphic and their linear isotropy representations are equivalent.*

Proof. This is a direct consequence of Proposition (3-4).

LEMMA (3-5). *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be real simple, If $\mathfrak{g}^{\mathbb{C}}$ is not simple, there exists a complex simple graded Lie algebra $\mathfrak{h} = \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1$ such that $\mathfrak{g} = \mathfrak{h}_R$. Further $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{h} \oplus \mathfrak{h}$.*

Proof. Since $\mathfrak{g}^{\mathbb{C}}$ is not simple, it is known that \mathfrak{g} is a complex simple Lie algebra. We shall show \mathfrak{g}_0 and \mathfrak{g}_1 are complex subalgebras. Let $[\mathfrak{g}_0 \oplus \mathfrak{g}_1]$ be the minimal complex subalgebra of \mathfrak{g} containing $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\{\mathfrak{g}_0 \oplus \mathfrak{g}_1\}$ be the maximal complex subalgebra contained in $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. Since $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is irreducible, $[\mathfrak{g}_0 \oplus \mathfrak{g}_1]$ is either $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ or \mathfrak{g} . In the first case $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a complex subalgebra of \mathfrak{g} . Now suppose $[\mathfrak{g}_0 \oplus \mathfrak{g}_1]$ is \mathfrak{g} . Since $\{\mathfrak{g}_0 \oplus \mathfrak{g}_1\}$ is an ideal of $[\mathfrak{g}_0 \oplus \mathfrak{g}_1]$ (see the next lemma), $[\mathfrak{g}_0 \oplus \mathfrak{g}_1] = 0$, i.e., $\mathfrak{g} = (\mathfrak{g}_0 \oplus \mathfrak{g}_1) \oplus \sqrt{-1}(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$. This is absurd from the consideration of dimension (cf. (3-3)). Thus we have shown that $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a complex subalgebra of \mathfrak{g} . Since $\mathfrak{g}_1 = \{y \in \mathfrak{g}_0 \mid [\mathfrak{g}, y] \subset \mathfrak{g}_0\}$, \mathfrak{g}_1 is also a complex space. The rest of the proof is straightforward.

LEMMA (3-6). *$\{\mathfrak{g}_0 \oplus \mathfrak{g}_1\}$ is an ideal of $[\mathfrak{g}_0 \oplus \mathfrak{g}_1]$.*

Proof. Let $x \in \{\mathfrak{g}_0 \oplus \mathfrak{g}_1\}$ and $y \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then for an arbitrary complex number λ , $\lambda[x, y] = [\lambda x, y] \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Therefore $[x, y] \in \{\mathfrak{g}_0 \oplus \mathfrak{g}_1\}$. Now let $z \in [\mathfrak{g}_0 \oplus \mathfrak{g}_1]$. Then $z = z_1 + \sqrt{-1} z_2$, where $z_1, z_2 \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and we have

$$[x, z] = [x, z_1 + \sqrt{-1} z_2] = [x, z_1] + \sqrt{-1} [x, z_2] \in \{\mathfrak{g}_0 \oplus \mathfrak{g}_1\}.$$

PROPOSITION (3-5). *There exists an involutive automorphism σ of \mathfrak{g} such that*

$$\sigma(\mathfrak{g}_{-1}) = \mathfrak{g}_1, \quad \sigma(\mathfrak{g}_0) = \mathfrak{g}_0, \quad \sigma(\mathfrak{g}_1) = \mathfrak{g}_{-1},$$

and the bilinear form B^ on \mathfrak{g} defined by $B^*(x, y) = -B(x, \sigma(y))$ ($x, y \in \mathfrak{g}$ and B is the Killing form of \mathfrak{g}) is positive definite or positive definite hermitian symmetric, according as $F = \mathbb{R}$ or $F = \mathbb{C}$.*

Proof. See [14] or [29].

LEMMA (3-7). *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a simple graded Lie algebra over F . Let \mathfrak{h} be an irreducible subalgebra of \mathfrak{g}_0 ($\subset \mathfrak{g}(\mathfrak{g}_{-1})$). If there exists a nonzero element z in \mathfrak{g}_1 such that $[\mathfrak{g}_{-1}, z] \subset \mathfrak{h}$, then \mathfrak{h} is necessary identical to \mathfrak{g}_0 .*

Proof. Let \mathfrak{h}_1 be $\{z' \in \mathfrak{g}_1 \mid [\mathfrak{g}_{-1}, z'] \subset \mathfrak{h}\}$. Then clearly $\mathfrak{g}_{-1} \oplus \mathfrak{h} \oplus \mathfrak{h}_1$ is a graded Lie algebra. Since the linear isotropy representation is irreducible, $\mathfrak{g}_{-1} \oplus \mathfrak{h} \oplus \mathfrak{h}_1$ is simple by Proposition (3-3). Then by Lemma (3-2), the dimension of \mathfrak{h}_1 must be equal to that of \mathfrak{g}_{-1} . In particular we have $\mathfrak{h}_1 = \mathfrak{g}_1$. Then by Corollary (3-3), we see $\mathfrak{h}_0 = \mathfrak{g}_0$.

PROPOSITION (3-6). *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a simple graded Lie algebra over F . And let \mathfrak{z} be a subalgebra of the Lie algebra \mathfrak{g} such that*

$$(1) \dim \mathfrak{z}/(\mathfrak{g}_0 \oplus \mathfrak{g}_1) = \dim \mathfrak{g}_{-1},$$

- (2) the natural representation of $\mathfrak{s} \cap (\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ into $\mathfrak{s}/(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ is irreducible,
- (3) $\mathfrak{s} \cap \mathfrak{g}_1 \neq \{0\}$.

Then \mathfrak{s} is necessary identical to \mathfrak{g} .

Proof. We put $\mathfrak{h} = \mathfrak{s} \cap (\mathfrak{g}_0 + \mathfrak{g}_1)/\mathfrak{s} \cap \mathfrak{g}_1$. Then \mathfrak{h} can be considered as a subalgebra of \mathfrak{g}_0 satisfying the condition of Lemma (3-7). Therefore

$$\mathfrak{s} \cap (\mathfrak{g}_0 \oplus \mathfrak{g}_1)/\mathfrak{s} \cap \mathfrak{g}_1 = \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{s} \cap \mathfrak{g}_1 = \mathfrak{g}_1.$$

Thus we have $\mathfrak{s} \supset \mathfrak{g}_0$. Hence $\mathfrak{s} = \mathfrak{g}$ by the condition (1).

4. Harmonic theory in the Spencer cohomologies (cf. [16]). Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a semisimple graded Lie algebra over F . We define a formal adjoint operator $\partial^* : C(\mathfrak{g}) \rightarrow C(\mathfrak{g})$ of ∂ to be $\partial^* C^{p,q}(\mathfrak{g}) \rightarrow C^{p+1,q-1}(\mathfrak{g})$, and

(4-1). $(\partial^*c)(x_1, \dots, x_{q-1}) = \sum_i [Z^i, c(X_i, x_1, \dots, x_{q-1})]$ for all $c \in C^{p,q}(\mathfrak{g})$ and $x_1, \dots, x_{q-1} \in \mathfrak{g}_{-1}$, where $\{X_i\}$ is a basis of \mathfrak{g}_{-1} and $\{Z^i\}$ is the dual basis of $\{X_i\}$ in \mathfrak{g}_1 (cf. Lemma (3-2) and Lemma (3-3)). It is evident that ∂^* is well defined (i.e., independent of the choice of the basis $\{X_i\}$).

LEMMA (4-1). We have $\partial^*(\rho(a)c) = \rho(a) \partial^*c$ for $c \in C(\mathfrak{g})$ and $a \in \mathfrak{g}_0$, here ρ is the natural representation of \mathfrak{g}_0 into $C(\mathfrak{g})$.

Proof. This is easy to see (cf. Lemma (3-3)).

We define the *Laplacian* \square by $\square = \partial^* \partial + \partial \partial^*$. Then \square is a linear endomorphism of $C^{p,q}(\mathfrak{g})$ for each p, q . Then the kernel of $\square : C^{p,q}(\mathfrak{g}) \rightarrow C^{p,q}(\mathfrak{g})$ is denoted by $\mathcal{H}^{p,q}(\mathfrak{g})$ ⁽³⁾ and an element of $\mathcal{H}^{p,q}(\mathfrak{g})$ is called a *harmonic form*.

LEMMA (4-2). We have $\rho(a) \square c = \square(\rho(a)c)$ for $c \in C(\mathfrak{g})$.

Proof. This follows from Lemma (2-5) and Lemma (4-1).

PROPOSITION (4-1). We have

(4-2). $\partial C^{p,q}(\mathfrak{g}) \cap \mathcal{H}^{p-1,q+1}(\mathfrak{g}) = \{0\}$,

(4-3). $H^{p,q}(\mathfrak{g}) \cong \mathcal{H}^{p,q}(\mathfrak{g})$.

Proof. To prove this it suffices to show the existence of an inner product (resp. hermitian inner product) $(\ , \)$ on $C(\mathfrak{g})$ (according as $F = \mathbf{R}$ or $F = \mathbf{C}$) such that ∂^* is the adjoint operator with respect to the inner product $(\ , \)$ i.e., $(\partial c, c') = (c, \partial^*c')$ for $c, c' \in C(\mathfrak{g})$ [16]. For that purpose we make use of the bilinear forms B^* in Proposition (3-5). We define an inner product (resp. hermitian inner product) in $C(\mathfrak{g})$ by

(4-4). $(c, c') = (1/q!) \sum B^*(c(X_{i_1}, \dots, X_{i_q}), c'(X_{i_1}, \dots, X_{i_q}))$ for all $c, c' \in C^{p,q}(\mathfrak{g})$, where X_i is an orthonormal basis of \mathfrak{g}_{-1} with respect to B^* . We put $(c, c') = 0$ for $c \in C^{p,q}(\mathfrak{g}), c' \in C^{p',q'}(\mathfrak{g})$ ($p \neq p'$ or $q \neq q'$). Now we prove

⁽³⁾ We should write $\mathcal{H}^{p,q}(\sum \mathfrak{g}_p)$, but we adopt this notation if there is no danger of confusion.

(4-5). $(\partial c, c') = (c, \partial^* c')$ for $c, c' \in C(\mathfrak{g})$. In fact, for $c \in C^{p,q}(\mathfrak{g})$ and $c' \in C^{p-1,q+1}(\mathfrak{g})$, we get

$$\begin{aligned} (\partial c, c') &= -\frac{1}{(q+1)!} \sum B((\partial c)(X_{i_1}, \dots, X_{i_{q+1}}), \sigma \cdot c'(X_{i_1}, \dots, X_{i_{q+1}})) \\ &= -\frac{1}{(q+1)!} \sum \sum (-1)^j B([c(X_{i_1}, \dots, \hat{X}_{i_j}, \dots, X_{i_{q+1}}), X_{i_j}], \\ &\hspace{20em} \sigma \cdot c'(X_{i_1}, \dots, X_{i_{q+1}})) \\ &= -\frac{1}{(q+1)!} \sum \sum (-1)^j B(c(X_{i_1}, \dots, \hat{X}_{i_j}, \dots, X_{i_{q+1}}), \\ &\hspace{20em} \sigma[\sigma X_{i_j}, c'(X_{i_1}, \dots, X_{i_{q+1}})]) \\ &= \frac{1}{(q+1)!} \sum \sum B(c(X_{i_1}, \dots, \hat{X}_{i_j}, \dots, X_{i_{q+1}}), \\ &\hspace{20em} \sigma[-\sigma X_{i_j}, c'(X_{i_j}, X_{i_1}, \dots, \hat{X}_{i_j}, \dots, X_{i_{q+1}})]) \\ &= (c, \partial^* c'). \end{aligned}$$

Here we remark that $\{-\sigma X_i\}$ is the dual basis of $\{X_i\}$.

5. **The Spencer cohomologies of complex simple graded Lie algebras** (cf. [16], [17], [18], [19]). To obtain the Spencer cohomologies of semisimple graded Lie algebras over F , we have only to obtain that of complex one (see Lemma (2-4)). Then by Lemma (2-3) and Proposition (3-2), it is sufficient to consider complex simple one.

In this section $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ denotes a complex simple graded Lie algebra. And e is the elements of Lemma (3-4).

LEMMA (5-1). *We have*

(5-1). $\mathfrak{g}_0 = Ce \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$,

(5-2). $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a semisimple Lie algebra.

Proof. This follows from the fact that \mathfrak{g}_0 has an irreducible faithful representation (cf. Proposition (3-3)).

From now on we fix once for all a Cartan subalgebra \mathfrak{h} of \mathfrak{g} such that $\mathfrak{g}_0 \supset \mathfrak{h} \ni e$ (cf. Corollary (3-1)). We denote by Δ the set of nonzero roots of \mathfrak{g} with respect to \mathfrak{h} . For $\alpha \in \Delta$, \mathfrak{g}_α denotes the eigenspace of α . The restriction of the Killing form B of \mathfrak{g} onto \mathfrak{h} is still nondegenerate. Therefore we define H_α for each $\alpha \in \Delta$ by $B(H_\alpha, H) = \alpha(H)$, $H \in \mathfrak{h}$. Let \mathfrak{h}_0 be the real subspace of \mathfrak{h} spanned by $\{H_\alpha \mid \alpha \in \Delta\}$. It is known that \mathfrak{h}_0 is characterized as the maximal real subspace of \mathfrak{h} on which all $\alpha \in \Delta$ has real value. Thus we have $e \in \mathfrak{h}_0$. The restriction of B onto \mathfrak{h}_0 is positive definite. Thus we introduce an inner product $(,)$ in \mathfrak{h}_0^* . Then $(\alpha, \beta) = B(H_\alpha, H_\beta)$ ($\alpha, \beta \in \Delta$).

Since \mathfrak{h} is contained in \mathfrak{g}_0 , Δ is a disjoint union of three subsets Δ_{-1} , Δ_0 and Δ_1 such that

$$\mathfrak{g}_{-1} = \sum_{\alpha \in \Delta_{-1}} \mathfrak{g}_\alpha, \quad \mathfrak{g}_0 = \mathfrak{h} + \sum_{\beta \in \Delta_0} \mathfrak{g}_\beta, \quad \mathfrak{g}_1 = \sum_{\gamma \in \Delta_1} \mathfrak{g}_\gamma.$$

LEMMA (5-2). *We have $\Delta_{-1} = -\Delta_1$.*

Proof. This follows from Lemma (3-3).

Let \mathfrak{g}_u be a real compact form of \mathfrak{g} such that $\mathfrak{g}_u \cap \mathfrak{h} = \sqrt{-1} \mathfrak{h}_0$. Let σ be the conjugation of \mathfrak{g} with respect to \mathfrak{g}_u . Then $\sigma(H_\alpha) = -H_\alpha$ for all $\alpha \in \Delta$. Thus we have $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. Therefore we have

$$\sigma(\mathfrak{g}_{-1}) = \mathfrak{g}_1, \quad \sigma(\mathfrak{g}_0) = \mathfrak{g}_0, \quad \sigma(\mathfrak{g}_1) = \mathfrak{g}_{-1}.$$

REMARK (5-1). Since the bilinear form B^* defined by $B^*(x, y) = -B(x, \sigma(y))$ is hermitian symmetric, our σ satisfies the conditions in Proposition (3-5).

LEMMA (5-3). For $\alpha \in \Delta_{-1} \cup \Delta_1$, we can choose a root vector $X_\alpha \in \mathfrak{g}_\alpha$ such that

- (i) $\{X_\alpha\}_{\alpha \in \Delta_{-1}}$ is an orthonormal basis of \mathfrak{g}_{-1} with respect to B^* .
- (ii) $\{X_{-\alpha}\}_{\alpha \in \Delta_{-1}}$ is the dual basis of $\{X_\alpha\}_{\alpha \in \Delta_{-1}}$ in \mathfrak{g}_1 with respect to B .
- (iii) $\sigma(X_\alpha) = -X_{-\alpha}$.
- (iv) $[X_\alpha, X_{-\alpha}] = H_\alpha$.

Proof. This follows from the fact that for $\alpha, \beta \in \Delta_{-1} \cup \Delta_1$, $B(X_\alpha, X_\beta) = 0$ unless $\alpha + \beta = 0$, $[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha})H_\alpha$, and $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.

For the matter of convenience, we denote by $\{\alpha_1, \dots, \alpha_N\}$ be the element in Δ_1 . We write X_i and X_i^- instead of X_{α_i} and $X_{-\alpha_i}$ respectively. Since $\sigma(\mathfrak{g}_0) = \mathfrak{g}_0$, we can choose a basis $\{Y_a\}_{a=1, \dots, r}$ of \mathfrak{g}_0 such that $B(Y_a, Y_b) = -\delta_{ab}$. We fix once for all those X_i, Y_a . Then it is easy to see that Laplacian $\square: C^{p,q}(\mathfrak{g}) \rightarrow C^{p,q}(\mathfrak{g})$ is expressed by

$$(5-3) \quad \begin{aligned} (\square c)(X_{i_1}, \dots, X_{i_q}) &= \sum_{k=1}^N [X_k [X_k c(X_{i_1}, \dots, X_{i_q})]] \\ &+ \sum_{k=1}^N \sum_{u=1}^q (-1)^{u+1} [[X_{i_u}, X_k] c(X_k, X_{i_1}, \dots, \hat{X}_{i_u}, \dots, X_{i_q})]. \end{aligned}$$

For an $x \in \mathfrak{g}_1$, $\varepsilon(x)$ is the operator of exterior multiplication by x on $\Lambda \mathfrak{g}_1$, and for X_k^0 , $i(X_k)$ is the operator on $\Lambda \mathfrak{g}_1$ defined as usual by the formula

$$i(X_k)(z_1 \cdots z_p) = \sum_{u=1}^p (-1)^u \langle X_k, z_u \rangle z_1 \wedge \cdots \wedge \hat{z}_u \wedge \cdots \wedge z_p.$$

We denote by ad_+ (resp. ad_+^q) the representation of \mathfrak{g}_0 into $\Lambda \mathfrak{g}_1$ (resp. $\Lambda^q \mathfrak{g}_1$).

LEMMA (5-4). We have

$$(5-4) \quad \text{ad}_+(Y_a) = \sum_{k=1}^N \varepsilon([Y_a, X_k]) i(X_k).$$

Proof. It suffices to show this on $\Lambda^1 \mathfrak{g}_1 = \mathfrak{g}_1$.

Now for $X_j \in \mathfrak{g}_1$,

$$\left(\sum_{k=1}^N \varepsilon([Y_a, X_k]) i(X_k) \right) (X_j) = \sum_{k=1}^N B(X_k, X_j) [Y_a, X_k] = [Y_a, X_j] = (\text{ad}_+ Y_a)(X_j).$$

We put $[X_i, X_j] = \sum C_{aj} Y_a$ ($i, j = 1, \dots, N$). From the relation

$$B([X_i, Y_a], X_j) + B(Y_a, [X_i, X_j]) = 0,$$

we get $[Y_a, X_i] = -\sum_{j=1}^N C_{aj} X_j$.

We use the following two nontrivial lemmas without proof (see [17]).

LEMMA (5-5). *We have*

(5-5). $e = 2 \sum_{\alpha \in \Delta_1} H_\alpha,$

(5-6). $\sum_{a=1}^r (\text{ad}_+^q(Y_a))^2 = -\sum_{k=1}^N \text{ad}_+^q([X_k, X_k]) = -(q/2)\mathbf{1}$ on $\Lambda^q \mathfrak{g}_1,$

here $\mathbf{1}$ denotes the identity map.

Since \mathfrak{g}_{-1} and \mathfrak{g}_1 are dual with each other, we can identify $C^{p,q}(\mathfrak{g}) = \mathfrak{g}_{p-1} \otimes \Lambda^q(\mathfrak{g}_{-1})^*$ with $\mathfrak{g}_{p-1} \otimes \Lambda^q \mathfrak{g}_1$ by

$$c \in C^{p,q}(\mathfrak{g}) \mapsto \frac{1}{q!} \sum c(X_{i_1}, \dots, X_{i_q}) X_{i_1} \wedge \dots \wedge X_{i_q}.$$

Then the Laplacian \square can be expressed on $\mathfrak{g} \otimes \Lambda \mathfrak{g}_1$ by

(5-7) $\square = \sum_{k=1}^N \text{ad } X_k \cdot \text{ad } X_k \otimes \mathbf{1} + \sum_{k,u=1}^N \text{ad } [X_u, X_k] \otimes \varepsilon(X_u)i(X_k).$

LEMMA (5-6). *The Laplacian \square has the following expression on $\mathfrak{g} \otimes \Lambda \mathfrak{g}_1$:*

(5-8) $\square = \sum_{k=1}^N \text{ad } (X_k) \text{ad } (X_k) \otimes \mathbf{1} + \sum_{a=1}^r \text{ad } (Y_a) \otimes \text{ad}_+ (Y_a).$

Proof.

$$\begin{aligned} & \sum_{k,u=1}^N \text{ad } [X_u, X_k] \otimes \varepsilon(X_u)i(X_k) \\ &= - \sum_{a,k,u} C_{ak\bar{u}} \text{ad } (Y_a) \otimes \varepsilon(X_u)i(X_k) \quad (\text{see the remark after Lemma (5-4)}) \\ &= - \sum_{a,k,u} \text{ad } (Y_a) \otimes \varepsilon(C_{ak\bar{u}} X_u)i(X_k) \\ &= \sum_{a,k} \text{ad } (Y_a) \otimes \varepsilon[Y_a, Y_k]i(X_k) = \sum_a \text{ad } Y_a \otimes \text{ad}_+ (Y_a) \quad (\text{by Lemma (5-4)}). \end{aligned}$$

Let now c be the element of the universal enveloping algebra \mathcal{E} of \mathfrak{g} defined by

$$c = \sum_{k=1}^N (X_k X_k + X_k X_k) - \sum_{a=1}^r Y_a^2.$$

The element c is called the *Casimir element* of \mathcal{E} and is in the center of \mathcal{E} . We put $c' = -\sum_a Y_a^2$. Then c' is an element in the center of the enveloping algebra \mathcal{E}' of \mathfrak{g}_0 . Extending the representation of \mathfrak{g} (resp. \mathfrak{g}_0) to those of \mathcal{E} (resp. \mathcal{E}'), we get the following formula.

LEMMA (5-7). *The Laplacian \square has the following expression on \mathfrak{g} .*

(5-8) $2\square = \text{ad } (c) \otimes \mathbf{1} - (\text{ad} \otimes \text{ad}_+)(c') + \frac{1}{2}(\text{ad} \otimes \text{ad}_+)(e).$

Here $\text{ad} \otimes \text{ad}_+$ denotes the product of two representations ad and ad_+ .

Proof. $2 \sum \text{ad}(X_k) \text{ad}(X_k) = \text{ad}(c) - \text{ad}(c') + \sum \text{ad}([X_k, X_k])$. On the other hand $(\text{ad} \otimes \text{ad}_+)(Y_a) = \text{ad}(Y_a) \otimes \mathbf{1} + \mathbf{1} \otimes \text{ad}_+(Y_a)$. Thus

$$((\text{ad} \otimes \text{ad}_+)(Y_a))^2 = (\text{ad}(Y_a))^2 \otimes \mathbf{1} + \mathbf{1} \otimes (\text{ad}_+(Y_a))^2 + 2 \text{ad}(Y_a) \otimes \text{ad}_+(Y_a).$$

Hence

$$\begin{aligned} 2 \sum \text{ad}(Y_a) \otimes \text{ad}_+(Y_a) &= \sum ((\text{ad} \otimes \text{ad}_+)(Y_a))^2 - \sum (\text{ad}(Y_a))^2 \otimes \mathbf{1} - \sum \mathbf{1} \otimes (\text{ad}_+(Y_a))^2 \\ &= -(\text{ad} \otimes \text{ad}_+)(c') + \text{ad}(c') \otimes \mathbf{1} + \sum \mathbf{1} \otimes \text{ad}_+[X_k, X_k] \end{aligned}$$

(see Lemma (5-5)). From this we have a desired expression.

Now we choose, once and for all, an ordering of the roots such that the roots belonging to Δ_1 are all positive. The existence of such an order is well known. Let Θ be the set of all positive roots α of \mathfrak{g} such that $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$. Then $\Delta^+ = \Delta_1 \cup \Theta$ is the set of all positive roots. We denote by $\{\gamma_1, \dots, \gamma_0\}$ the set of all simple roots. Then we know that we may assume $\gamma_1 \in \Delta_1$ and $\gamma_2, \dots, \gamma_l \in \Theta$ [9]. Then $\gamma_2, \dots, \gamma_l$ are also simple as the roots of the complex reductive Lie algebra \mathfrak{g}_0 . Under these notations, if ρ (resp. τ) is an irreducible representation of \mathfrak{g} (resp. \mathfrak{g}_0) with the lowest weight Λ' (resp. μ'), then $\rho(c)$ and $\tau(c')$ are scalar operators which is known to be given by

$$(5-9). \quad \rho(c) = \{(\Lambda', \Lambda') - \sum_{\alpha \in \Delta^+} (\Lambda', \alpha)\} \cdot \mathbf{1},$$

$$(5-10). \quad \tau(c') = \{(\mu', \mu') - \sum_{\beta \in \Theta} (\mu', \beta)\} \cdot \mathbf{1}.$$

Let

$$(5-11) \quad \mathfrak{g} \otimes \Lambda \mathfrak{g}_1 = \sum n_{\xi'} U_{\xi'}$$

be a decomposition of the \mathfrak{g}_0 -module $\mathfrak{g} \otimes \Lambda \mathfrak{g}_1$ into direct sum of irreducible \mathfrak{g}_0 -modules $U_{\xi'}$ where ξ' denotes the lowest weight of the irreducible representation $\tau_{\xi'}$ of \mathfrak{g}_0 into $U_{\xi'}$. Since e is in the center \mathfrak{g}_0 , the last term in the expression (5-8) is a scalar operator on each $U_{\xi'}$. We denote by Λ' the lowest root of \mathfrak{g} and put $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

PROPOSITION (5-1) (B. KOSTANT [16]). *We have*

$$(5-12) \quad \square |U_{\xi'} = \{(\Lambda' - \delta, \Lambda' - \delta) - (\xi' - \delta, \xi' - \delta)\} \cdot \mathbf{1}.$$

Proof. This follows from (5-8), (5-9), (5-10) and the above remark.

COROLLARY (5-1). *Let $\mathfrak{g} \otimes \Lambda \mathfrak{g}_1 = \sum n_{\xi'} U_{\xi'}$ be as above. Then we have*

$$\mathcal{H}(\mathfrak{g}) \cong \sum_{|\Lambda' - \delta| = |\xi' - \delta|} n_{\xi'} U_{\xi'}.$$

Here $|\Lambda' - \delta|^2 = (\Lambda' - \delta, \Lambda' - \delta)$.

Proof. This follows from proposal (5-1) and Lemma (4-2).

For a subset Φ of Δ , we denote by $\langle \Phi \rangle$ the sum of the roots belonging to Φ .

Let W be the Weyl group of \mathfrak{g} . And for an element T of W , we put $\Phi_T = T(-\Delta^+) \cap \Delta^+$. Put $W^1 = \{T \in W \mid \Phi_T \subset \Delta_1\}$. Let ζ' be the lowest weight of an irreducible representation $\rho_{\zeta'}$ of \mathfrak{g} with the representation space V . And let

$$(5-13) \quad V \otimes \Lambda_{\mathfrak{g}_1} = \sum l_{\zeta'} U_{\zeta'}$$

be a decomposition of the \mathfrak{g}_0 -module $V \otimes \Lambda_{\mathfrak{g}_1}$ into direct sum of irreducible \mathfrak{g}_0 -module $U_{\zeta'}$. For $T \in W^1$, we put $\zeta'_T = T\zeta' + \langle \Phi_T \rangle$. Clearly ζ'_T is a weight of the \mathfrak{g}_0 -module $V \otimes \Lambda_{\mathfrak{g}_1}$.

LEMMA (5-8). *We have*

(1) *Let η be a weight of the \mathfrak{g}_0 -module $V \otimes \Lambda_{\mathfrak{g}_1}$, then $|\Lambda' - \delta| \geq |\eta - \delta|$ and the equality holds if and only if there exists $T \in W^1$ such that $\eta = \zeta'_T$.*

(2) *The map $T \mapsto \zeta'_T$ is a bijection of W^1 into the set of the lowest weight ξ' appearing in the decomposition (5-13) and satisfying the equality $|\Lambda' - \delta| = |\xi' - \delta|$.*

Proof. See [19].

PROPOSITION (5-2) (B. KOSTANT [16]). *We have $\mathcal{H}(\mathfrak{g}) = \sum_{T \in W^1} U_{\Lambda'_T}$. Here Λ' is the lowest root of \mathfrak{g} and $\Lambda'_T = T\Lambda' + \langle \Phi_T \rangle$.*

Proof. This follows from Corollary (5-1) and Lemma (5-8).

Let $n(T)$ denote the number of the root belonging to Φ_T .

COROLLARY (5-2). *We have $\sum_p \mathcal{H}^{p,q}(\mathfrak{g}) = \sum_{T \in W^1, n(T)=q} U_{\Lambda'_T}$.*

PROPOSITION (5-3) (B. KOSTANT [16]). *The \mathfrak{g}_0 -module $\Lambda_{\mathfrak{g}_1}$ and $\Lambda^q \mathfrak{g}_1$ decompose into a direct sum of \mathfrak{g}_0 -irreducible modules*

$$\Lambda_{\mathfrak{g}_1} = \sum_{T \in W^1} U_{\langle \Phi_T \rangle}, \quad \Lambda^q \mathfrak{g}_1 = \sum_{T \in W^1, n(T)=q} U_{\langle \Phi_T \rangle}.$$

Proof. This follows from Lemma (5-5) and Lemma (5-8).

Let $\mathfrak{g} = \sum m_{\zeta'} \mathfrak{g}_{\zeta'}$ be a decomposition of the \mathfrak{g}_0 -module \mathfrak{g} into direct sum of irreducible \mathfrak{g}_0 -module $\mathfrak{g}_{\zeta'}$ with the lowest weight ξ' . Combining this with the decomposition $\Lambda_{\mathfrak{g}_1} = \sum_{T \in W^1} U_{\langle \Phi_T \rangle}$, we have

$$\mathfrak{g} \otimes \Lambda_{\mathfrak{g}_1} = \sum_{\zeta'} \sum_{T \in W^1} m_{\zeta'} \mathfrak{g}_{\zeta'} \otimes U_{\langle \Phi_T \rangle}.$$

LEMMA (5-9). *Suppose that there exists an element $c \in \mathfrak{g}_{\zeta'} \otimes U_{\langle \Phi_T \rangle}$ such that $c \neq 0$ and $\square c = 0$. Then $\xi'_T = T\Lambda'$.*

Proof [19]. By the assumption, the intersection of $\mathfrak{g}_{\zeta'} \otimes U_{\langle \Phi_T \rangle}$ with $\sum_{T \in W^1} U_{\Lambda'_T}$ is not empty. Therefore there exists $S \in W^1$ such that $U_{\Lambda'_S}$ is contained in $\mathfrak{g}_{\zeta'} \otimes U_{\langle \Phi_T \rangle}$. Then Λ'_S is a weight of the \mathfrak{g}_0 -module $\mathfrak{g}_{\zeta'} \otimes U_{\langle \Phi_T \rangle}$ and hence there exists a weight ν in $\mathfrak{g}_{\zeta'}$ and a subset Φ of Δ_1 such that $\Lambda'_S = \nu + \langle \Phi \rangle$. Then by Lemma (5-8) we get $\nu = S\Lambda'$ and $\Phi = \Phi_S$. We remark now that $R^{-1}(\Theta) \subset \Theta$ for an $R \in W^1$. Therefore $(S\Lambda', \beta) = (\Lambda', S^{-1}\beta) \leq 0$ for all $\beta \in \Theta$ because Λ' is the lowest root of

\mathfrak{g}_0 . It follows from this that $\nu = S\Lambda'$ is the lowest weight in $\mathfrak{g}_{\xi'}$ and hence $\xi' = S\Lambda'$. On the other hand, since $\langle \Phi_S \rangle$ is the lowest weight in $U_{\langle \Phi_S \rangle}$, we have $\langle \langle \Phi_S \rangle, \beta \rangle \leq 0$ for $\beta \in \Theta$. Therefore, as $\langle \Phi_S \rangle$ is a weight in $U_{\langle \Phi_T \rangle}$, we get $\langle \Phi_T \rangle = \langle \Phi_S \rangle$. Then by the lemma, we get $T = S$.

COROLLARY (5-3) (MATSUSHIMA-MURAKAMI [18]). *Let q_{ad} be the numbers of the roots in $\{\alpha \in \Delta_1 \mid \langle \alpha, \mu' \rangle > 0\}$, here μ' is the lowest root in Δ_1 . Then we have*

$$H^{2,q}(\mathfrak{g}) = 0 \quad \text{if } q < q_{ad}.$$

Proof. By Proposition (4-1) and Corollary (5-2), we have

$$H^{2,q}(\mathfrak{g}) = \mathcal{H}^{2,q}(\mathfrak{g}) = \sum_p \mathcal{H}^{p,q}(\mathfrak{g}) \cap \mathfrak{g}_1 \otimes \Lambda \mathfrak{g}_1.$$

By Lemma (5-9), $\sum_p \mathcal{H}^{p,q}(\mathfrak{g}) \cap \mathfrak{g}_1 \otimes U_{\langle \Phi_T \rangle} \neq 0$ only if $\mu' = T\Lambda'$ and $n(T) = q$. If $\langle \alpha, \mu' \rangle > 0$ ($\alpha \in \Delta_1$), then $0 < \langle \alpha, \mu' \rangle = \langle \alpha, T\Lambda' \rangle = \langle T^{-1}\alpha, \Lambda' \rangle$. Since Λ' is the lowest weight, $T^{-1}\alpha < 0$, i.e., $\alpha \in \Phi_T$. Thus $q_{ad} \leq n(T)$.

LEMMA (5-10). *If $H^{1,1}(\mathfrak{g}) \neq 0$, then \mathfrak{g}_0 has an element of rank 1, \mathfrak{g}_0 being considered as the subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$.*

Proof. Suppose $H^{1,1}(\mathfrak{g}) \neq 0$. Then $\mathfrak{g}_0 \otimes \mathfrak{g}_1$ contains $U_{\Lambda'_T}$ for some $T \in W^1$. Taking an eigenvector of the lowest weight $\Lambda'_T = T\Lambda' + \langle \Phi_T \rangle$, we can assume $\mathcal{H}^{1,1}(\mathfrak{g})$ contains an element of the form $y \otimes z$ ($y \in \mathfrak{g}_0, z \in \mathfrak{g}_1$). Since $\partial(y \otimes z) = 0$, for $x, x' \in \mathfrak{g}_{-1}$, we have

$$B(x, z)[y, x'] = B(x', z)[y, x].$$

Taking x' as $B(x, z) = 1$, we have $[y, x] = B(x, z)[y, x']$, i.e., y is an element of rank 1.

REMARK. Simpler proof may be obtained using Guillemin, Quillen, Sternberg, *Classification of the irreducible algebras of infinity type*, J. Analyse Math. **23** (1967).

6. Semisimple homogeneous spaces. Let L/L_0 be a connected homogeneous space on which a semisimple Lie group L acts effectively and transitively. Here L is not necessary connected. L/L_0 is called a semisimple *flat homogeneous space* if the Lie algebra \mathfrak{l} of L has a graded Lie algebra structure $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the Lie algebra of L_0 . We call the graded Lie algebra $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the *associated graded Lie algebra* of a semisimple homogeneous space L/L_0 (cf. Proposition (3-4)). Let G_1 (resp. G_{-1}) be the connected Lie subgroup of L corresponding to \mathfrak{g}_1 (resp. \mathfrak{g}_{-1}). We define G_0 as the normalizer of \mathfrak{g}_0 in L_0 , i.e., $G_0 = \{a \in L_0 \mid \text{Ad}(a)(\mathfrak{g}_0) = \mathfrak{g}_0\}$.

LEMMA (6-1). *The exponential mapping $\exp: \mathfrak{g}_1 \rightarrow G_1$ and $\exp: \mathfrak{g}_{-1} \rightarrow G_{-1}$ are bijective.*

Proof. Since \mathfrak{g}_1 is abelian, \exp is surjective. Let \mathfrak{n} be the kernel of $\exp: \mathfrak{g}_1 \rightarrow G_1$. Since \mathfrak{n} is discrete we have $[\mathfrak{g}_0, \mathfrak{n}] = 0$. This is contradiction (see Lemma (3-4)).

LEMMA (6-2). G_0 is identical to the normalizer of \mathfrak{g}_{-1} in L_0 , i.e.,

$$G_0 = \{a \in L_0 \mid \text{Ad}(a)(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}\}.$$

Proof. Let G'_0 be $\{a \in L_0 \mid \text{Ad}(a)(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}\}$. Take an element a in G_0 . Since $\text{Ad}(a)(\mathfrak{g}_0) = \mathfrak{g}_0$ and $\text{Ad}(a)(\mathfrak{g}_1) = \mathfrak{g}_1$, ($\mathfrak{g}_1 = \{x \in \mathfrak{g}_0 \oplus \mathfrak{g}_1 \mid [x, x] \in \mathfrak{g}_0\}$) by the argument in Proposition (3-4), there exists an isomorphism ϕ of \mathfrak{g} such that $\phi(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}$ and $\phi|_{\mathfrak{g}_0 + \mathfrak{g}_1} = \text{Ad}(a)|_{\mathfrak{g}_0 + \mathfrak{g}_1}$. Then $\phi \cdot \text{Ad}(a)^{-1}$ is, in particular, the identity on \mathfrak{g}_0 . Since \mathfrak{g}_0 contains a Cartan subalgebra \mathfrak{h} of \mathfrak{g} , there exists an element H of \mathfrak{h} , such that $\phi \cdot \text{Ad}(a)^{-1} = \exp \text{ad } H$. From this we have $\text{Ad}(a)(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}$. Take $a \in G'_0$, then

$$\text{Ad}(a)(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1.$$

Since \mathfrak{g}_0 is the orthogonal complement of $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ with respect to the Killing form B , thus $\text{Ad}(a)(\mathfrak{g}_0) = \mathfrak{g}_0$ since $\text{Ad}(a)$ leaves B invariant.

LEMMA (6-3). We have $G_0 \cap G_1 = \{\text{id}\}$.

Proof. Let $Z \in \mathfrak{g}_1$ be such that $\exp Z \in G_0 \cap G_1$. For $X \in \mathfrak{g}_{-1}$,

$$\text{Ad}(\exp Z)(X) = X + [Z, X] + \frac{1}{2}[Z[Z, X]].$$

Since $\text{Ad}(\exp Z)(\mathfrak{g}_{-1}) \subset \mathfrak{g}_{-1}$, we have $[Z, \mathfrak{g}_{-1}] = 0$. This is a contradiction (see (2-3)).

LEMMA (6-4). We have $L_0 = G_0 \cdot G_1$.

Proof. Let c be the center of \mathfrak{g}_0 . From Lemma (5-1) we can easily see that $\text{ad } c$ ($c \in c$) is semisimple. From the argument in Proposition (3-4) for $a \in L_0$ there exists an element Z in \mathfrak{g}_1 such that $\text{Ad}(\exp Z) \text{Ad}(a)$ preserves $\mathfrak{g}_1, \mathfrak{g}_1 + c$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$. From the above remark we see $\text{Ad}(\exp Z) \text{Ad}(a)(c) = c$. Thus we have $\text{Ad}(\exp Z) \text{Ad}(a)(\mathfrak{g}_0) = \mathfrak{g}_0$. Then $\exp Z \cdot a \in G_0$ i.e., $L_0 = G_1 \cdot G_0 = G_0 \cdot G_1$.

COROLLARY (6-1). L_0 is the semidirect product of G_0 and G_1 .

We have a natural representation l of L_0 into \mathfrak{g}_{-1} (so-called the linear isotropy representation of L_0). It is defined by

$$l(a)x = \text{Ad}(a)x \text{ mod } (\mathfrak{g}_0 \oplus \mathfrak{g}_1) \quad \text{for } a \in L_0.$$

LEMMA (6-5). G_1 is identical to the kernel of l , or equivalently l is faithful on G_0 .

Proof. It is clear that G_1 is contained in the kernel of l . Take $a \in G_0$ such that $\text{Ad}(a)|_{\mathfrak{g}_1}$ is the identity. Then by Lemma (3-3), $\text{Ad}(a)|_{\mathfrak{g}_{-1}}$ is also the identity. Since $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$, $\text{Ad}(a)$ is the identity on l , i.e., a commutes with the identity connected component of L . This is a contradiction because the operation of L is effective.

The group G_0 is called the *linear isotropy subgroup* of the semisimple flat homogeneous space L/L_0 .

7. **Classification of simple graded Lie algebras.** We can classify all semisimple graded Lie algebras over F ($F = \mathbf{R}$ or \mathbf{C}). By Proposition (3-2), we have only to classify simple graded Lie algebras. This has been done in [14]. And the value q_{ad} (cf., Corollary (5-3)) for complex simple Lie algebras has been computed by A. Borel [1]. We here only write the table. For the details, see [14].

Let K be the field \mathbf{R} of real numbers, the field \mathbf{C} of complex numbers or the field \mathbf{Q} of quaternions. In a natural way, $\mathbf{R} \subset \mathbf{C} \subset \mathbf{Q}$. For each element x of K , we define the elements \bar{x} and \tilde{x} as follows.

If $x = x_0 + x_1i + x_2j + x_3k \in \mathbf{Q}$ with $x_0, x_1, x_2, x_3 \in \mathbf{R}$. Then

$$\bar{x} = x_0 - x_1i - x_2j - x_3k, \quad \tilde{x} = x_0 + x_1i - x_2j - x_3k.$$

We use the following notations.

- (1) $\mathfrak{gl}(n; K) = \{\text{all } n \times n \text{ matrices over the field } K\}$.
- (2) $\mathfrak{sl}(n; K) = \{\text{the semisimple part of } \mathfrak{gl}(n; K)\}$.
- (3) $\mathfrak{so}(p, q; K) = \{A \in \mathfrak{gl}(p+q; K) \mid {}^t \bar{A} I_{p,q} + I_{p,q} A = 0\}$, where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

$$\mathfrak{so}(n; K) = \mathfrak{so}(n, 0; K).$$

- (4) $\mathfrak{u}(p, q; K) = \{A \in \mathfrak{gl}(p+q; K) \mid {}^t \bar{A} I_{p,q} + I_{p,q} A = 0\}$.

$$\mathfrak{u}(n; K) = \mathfrak{u}(n, 0; K).$$

- (5) $\mathfrak{su}(p, q; K) = \mathfrak{u}(p, q; K) \cap \mathfrak{sl}(p+q; K)$.

- (6) $\mathfrak{sp}(n; K) = \{A \in \mathfrak{gl}(2n; K) \mid {}^t \bar{A} J + J A = 0\}$.

I. *Complex simple graded Lie algebras* $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$.

type	$I_{p,q}(p \geq q \geq 1)$	$II_p(p \geq 2)$	$III_p(p \geq 2)$	IV_p	V	VI
\mathfrak{g}	$\mathfrak{sl}(p+q; \mathbf{C})$	$\mathfrak{so}(2p; \mathbf{C})$	$\mathfrak{sp}(n; \mathbf{C})$	$\mathfrak{so}(p+q+2; \mathbf{C})$	$E_6^{\mathfrak{C}}$	$E_7^{\mathfrak{C}}$
\mathfrak{g}_0	$\mathfrak{sl}(p; \mathbf{C}) + \mathfrak{sl}(q; \mathbf{C}) + \mathbf{C}$	$\mathfrak{gl}(p; \mathbf{C})$	$\mathfrak{gl}(n; \mathbf{C})$	$\mathfrak{so}(p+q; \mathbf{C}) + \mathbf{C}$	$\mathfrak{so}(10; \mathbf{C}) + \mathbf{C}$	$E_8^{\mathfrak{C}} + \mathbf{C}$
q_{ad}	$p+q-1$	$2p-3$	p	$p+q-1$	11	17

II. *Real simple graded Lie algebras* $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$.

	$K = \mathbf{R}$	$K = \mathbf{C}$	$K = \mathbf{Q}$
\mathfrak{g}	$\mathfrak{sl}(p+q; \mathbf{R})$	$\mathfrak{sl}(p+q; \mathbf{C})$	$\mathfrak{su}^*(2p+2q)$
\mathfrak{g}_0	$\mathfrak{sl}(p; \mathbf{R}) + \mathfrak{sl}(q; \mathbf{R}) + \mathbf{R}$	$\mathfrak{sl}(p; \mathbf{C}) + \mathfrak{sl}(q; \mathbf{C}) + \mathbf{C}$	$\mathfrak{su}^*(2p) + \mathfrak{su}^*(2q) + \mathbf{R}$
\mathfrak{g}		$\mathfrak{su}(p, p)$	
\mathfrak{g}_0		$\mathfrak{sl}(p; \mathbf{C}) + \mathbf{R}$	
\mathfrak{g}	$\mathfrak{so}(n, n)$	$\mathfrak{so}(2n; \mathbf{C})$	$\mathfrak{so}^*(4n)$
\mathfrak{g}_0	$\mathfrak{gl}(n; \mathbf{R})$	$\mathfrak{gl}(n; \mathbf{C})$	$\mathfrak{u}^*(2n)$
\mathfrak{g}	$\mathfrak{so}(p+1, q+1)$	$\mathfrak{so}(p+q+2; \mathbf{C})$	
\mathfrak{g}_0	$\mathfrak{so}(p+q) + \mathbf{R}$	$\mathfrak{so}(p+q; \mathbf{C}) + \mathbf{C}$	
\mathfrak{g}	$\mathfrak{sq}(n; \mathbf{R})$	$\mathfrak{sq}(n; \mathbf{C})$	$\mathfrak{sq}(n, n)$
\mathfrak{g}_0	$\mathfrak{gl}(n; \mathbf{R})$	$\mathfrak{gl}(n; \mathbf{C})$	$\mathfrak{su}(2n) + \mathbf{R}$

The above gives the table for classical Lie algebras. The table for exceptional Lie algebras is given as following. For the detailed definition, see also [14].

\mathfrak{g}	E_8^1	E_6^C	E_6^4
\mathfrak{g}_0	$\mathfrak{so}(5.5) + \mathbf{R}$	$\mathfrak{so}(10; \mathbf{C}) + \mathbf{C}$	$\mathfrak{so}(1.9) + \mathbf{R}$
\mathfrak{g}	E_7^1	E_7^C	E_7^3
\mathfrak{g}_0	$E_6^1 + \mathbf{R}$	$E_6^C + \mathbf{C}$	$E_6^4 + \mathbf{R}$

PROPOSITION (7-1). *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a real simple graded Lie algebra such that \mathfrak{g} is different from $\mathfrak{sl}(2, \mathbf{R})$ and $\mathfrak{sl}(2, \mathbf{C})$, then $H^{2,1}(\mathfrak{g}) = 0$.*

Proof. This follows from Lemma (2-3), Lemma (3-6) and Corollary (5-3).

PROPOSITION (7-2). *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a real simple graded Lie algebra such that \mathfrak{g} and \mathfrak{g}_0 are different from*

\mathfrak{g}	$(\mathfrak{sl}(p+q; \mathbf{R}))$	$\mathfrak{sl}(p+q; \mathbf{C})$	$\mathfrak{su}(1, 1)$
\mathfrak{g}_0	$\mathfrak{sl}(p; \mathbf{R}) + \mathfrak{sl}(q; \mathbf{R}) + \mathbf{R}$ $p+q \leq 3$	$\mathfrak{sl}(p; \mathbf{C}) + \mathfrak{sl}(q; \mathbf{C}) + \mathbf{C}$ $p+q \leq 3$	$\mathfrak{sl}(1, \mathbf{C}) + \mathbf{R}$
\mathfrak{g}	$\mathfrak{so}(p+1, q+1)$	$\mathfrak{so}(p+q+2, \mathbf{C})$	
\mathfrak{g}_0	$\mathfrak{so}(p, q) + \mathbf{R}$ $p+q \leq 3$	$\mathfrak{so}(p+q, \mathbf{C})$ $p+q \leq 3$	

Then $H^{2,2}(\mathfrak{g}) = 0$.

Proof. This follows from Lemma (2-3), Lemma (3-6) and Corollary (5-3).

PROPOSITION (7-3). *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a real simple graded Lie algebra. Then $H^{1,1}(\mathfrak{g}) = 0$ except the cases $\mathfrak{g} = \mathfrak{sl}(p+1, \mathbf{R})$, $\mathfrak{g}_0 = \mathfrak{sl}(p, \mathbf{R}) + \mathbf{R}$ and $\mathfrak{g} = \mathfrak{sl}(p+1, \mathbf{C})$, $\mathfrak{g}_0 = \mathfrak{sl}(p, \mathbf{C}) + \mathbf{C}$.*

Proof. This follows from Lemma (5-11) and the table above.

8. **Frames of higher order contact.** Let M be a manifold of dimension n . Let U and V be neighborhoods of the origin 0 of \mathbf{R}^n . Two local diffeomorphisms $f: U \rightarrow M$ and $g: V \rightarrow M$ such that $f(0) = g(0)$ give rise to the same *frame of r th order contact* (or *r -frame*) at $x = f(0) = g(0)$ if they have the same partial derivatives up to order r at 0 . Here $1 \leq r \leq \infty$. The r -frame given by f is denoted by $j^r(f)$. We denote by $\mathcal{F}^r(M)$ the set of all r -frames of M . It is a principal fibre bundle over M with the natural projection $\pi; \pi(j^r(f)) = f(0)$. We shall describe its structure group and the right action of the structure group on $\mathcal{F}^r(M)$.

Let $G^r(n)$ be the set of r -frames of \mathbf{R}^n at 0 , i.e., $G^r(n) = \{j^r(g) \in \mathcal{F}^r(\mathbf{R}^n) \mid g(0) = 0\}$. Then $G^r(n)$ is a group with the natural multiplication defined by $j^r(g) \cdot j^r(g') = j^r(g \circ g')$. The group $G^r(n)$ acts on $\mathcal{F}^r(M)$ from the right by $j^r(f) \cdot j^r(g) = j^r(f \circ g)$ for $j^r(f) \in \mathcal{F}^r(M)$ and $j^r(g) \in G^r(n)$. It is easy to verify that $\mathcal{F}^r(M)$ is a principal fibre bundle over M with group $G^r(n)$. $\mathcal{F}^r(M)$ is called the *bundle of r -frame* of M .

Let $h: M \rightarrow N$ be a diffeomorphism from M onto a manifold N . Then h induces a bundle isomorphism $h^{(r)}: \mathcal{F}^r(M) \rightarrow \mathcal{F}^r(N)$ defined by

$$h^{(r)}(j^r(f)) = j^r(h \circ f) \in \mathcal{F}^r(N) \quad \text{for } j^r(f) \in \mathcal{F}^r(M).$$

We call $h^{(r)}$ the *r*th prolongation of h .

From now on we shall be mainly interested in $\mathcal{F}^2(M)$ and $\mathcal{F}^1(M)$. We have a natural projection $\alpha: G^2(n) \rightarrow G^1(N)$ and $\nu: \mathcal{F}^2(M) \rightarrow \mathcal{F}^1(M)$ defined by $\alpha(j^2(g)) = j^1(g)$ ($j^2(g) \in G^2(n)$) and $\nu(j^2(f)) = j^1(f)$ ($j^2(f) \in \mathcal{F}^2(M)$).

LEMMA (8-1). $\alpha: G^2(n) \rightarrow G^1(n)$ is a group homomorphism and $\nu: \mathcal{F}^2(M) \rightarrow \mathcal{F}^1(M)$ is a bundle homomorphism associated with $\alpha: G^2(n) \rightarrow G^1(n)$;

$$\begin{array}{ccc} \mathcal{F}^2(M) & \xrightarrow{\nu} & \mathcal{F}^1(M) \\ \downarrow \pi & & \downarrow \pi \\ M & \xlongequal{\quad} & M \end{array}$$

Proof. It is easy to see.

REMARK (8-1). We have a natural inclusion of $GL(n; \mathbf{R})$ into $G^1(n)$, defined by $g \rightarrow j^1(g)$ for a $g \in GL(n; \mathbf{R})$. In particular $GL(n; \mathbf{R})$ and $G^1(n)$ are isomorphic by this inclusion. From now on we identify $GL(n; \mathbf{R})$ with $G^1(n)$ and consider $GL(n; \mathbf{R})$ as a subgroup of $G^r(n)$ by this inclusion. Then the unit element e of $GL(n; \mathbf{R})$ corresponds to $j^r(\text{id}) \in G^r(n)\mathcal{F}^r(\mathbf{R}^n)$, here id means the identity transformation.

For a subgroup G of $GL(n; \mathbf{R})$ a G -subbundle P of $\mathcal{F}^1(M)$ is called a G -structure on M . Let P and P' be G -structures on M and a manifold M' respectively. A diffeomorphism h of M onto M' is called an *isomorphism* of P and P' if the first prolongation $h^{(1)}$ satisfies $h^{(1)}(P) = P'$. Especially an isomorphism of P itself is called a G -automorphism of P .

We shall now define an \mathbf{R}^n -values 1-form θ on $\mathcal{F}^1(M)$. Let X be a vector tangent to $\mathcal{F}^1(M)$ at a point $j^1(f) \in \mathcal{F}^1(M)$. f induces an isomorphism f_*^{-1} of the tangent space $T(M)_{\pi(f)}$ onto $T(\mathbf{R}^n)_0$, and $\pi_*(X_p)$ is a vector in $T(M)_{\pi(f)}$. We define θ by

$$\langle X_p, \theta \rangle = f_*^{-1}(\pi_*(X_p)).$$

The restriction of θ onto a G -structure P is also denoted by θ and called the *canonical form* of P .

LEMMA (8-2). Let P and P' be G -structures on M and M' respectively, and θ and θ' be their canonical forms respectively. If $h: M \rightarrow M'$ is an isomorphism of P and P' , $(h^{(1)})^*\theta' = \theta$. Conversely if a bundle isomorphism $H: P \rightarrow P'$ satisfies $H^*\theta' = \theta$, then H is the first prolongation $h^{(1)}$ of the diffeomorphism h of the base manifolds M and M' induced by H .

Proof. It is easy to see (cf. [8]).

LEMMA (8-3). Let P be a G -structure on M and θ be its canonical form. Then we have

(8-1). $\langle A^*, \theta \rangle = 0$ for $A \in \mathfrak{g}$,

(8-2). $R_a^* \theta = a^{-1}$ for $a \in G$.

Here \mathfrak{g} denotes the Lie algebra of G .

Proof. It is easy to verify (cf. [8]).

Let $\mathcal{T}_s^r(\mathbf{R}^n)$ be the vector space of all tensors of type (r, s) on \mathbf{R}^n , i.e., $\mathcal{T}_s^r(\mathbf{R}^n) = (\otimes^r \mathbf{R}^n) \otimes (\otimes^s (\mathbf{R}^n)^*)$. The group $GL(n; \mathbf{R})$ naturally operates on $\mathcal{T}_s^r(\mathbf{R}^n)$. For a G -structure P on M , a *tensor field of type (r, s)* on M is a mapping \mathfrak{t} of P into $\mathcal{T}_s^r(\mathbf{R}^n)$ satisfying $\mathfrak{t}_{pa} = a^{-1} \cdot (\mathfrak{t}_p)$ for all $a \in G$ and $p \in P$. This definition of a tensor field on M is equivalent to the usual one.

Let H be a subgroup of $G^2(n)$. An H -subbundle Q of $\mathcal{F}^2(M)$ is called an H -structure of 2nd order on M . Then $\nu(Q) \subset \mathcal{F}^1(M)$ is an $\alpha(H)$ -structure on M , which is called the *associated $\alpha(H)$ -structure* of Q . Let Q and Q' be H -structure of 2nd order on M and M' respectively. A diffeomorphism $h: M \rightarrow M'$ is called an isomorphism of Q and Q' if $h^{(2)}(Q) = Q'$. Especially an isomorphism of Q itself is called an H -automorphism of Q .

We shall now define a 1-form Θ on $\mathcal{F}^2(M)$ with values in $T(\mathcal{F}^1(\mathbf{R}^n))_e$. Let X be a vector tangent to $\mathcal{F}^2(M)$ at $j^2(f)$. Since f is a diffeomorphism of a neighborhood of the origin $0 \in \mathbf{R}^n$ onto a neighborhood of $f(0) \in M$, it induces a diffeomorphism $f^{(1)}$ of a neighborhood of $e \in \mathcal{F}^1(\mathbf{R}^n)$ onto a neighborhood of $j^1(f) \in \mathcal{F}^1(M)$. $f^{(1)}$ then induces an isomorphism $f_*^{(1)}$ of the tangent space $T(\mathcal{F}^1(\mathbf{R}^n))_e$ at e onto the tangent space of $\mathcal{F}^1(M)$ at $\nu(j^2(f)) = j^1(f)$. Then form Θ is defined by

$$\langle X, \Theta \rangle = (f_*^{(1)})^{-1}(\nu_*(X)),$$

and is called the canonical form of $\mathcal{F}^2(M)$. For an H -structure of 2nd order Q , the restriction of Θ onto Q is also denoted by Θ and called the *canonical form* of Q .

LEMMA (8-4). *Let Θ and Θ' be the canonical forms of $\mathcal{F}^2(M)$ and $\mathcal{F}^2(M')$ respectively. Let $h: M \rightarrow M'$ be a diffeomorphism. Then the second prolongation $h^{(2)}$ of h satisfies*

$$(h^{(2)})^* \Theta' = \Theta.$$

Conversely if $H: \mathcal{F}^2(M) \rightarrow \mathcal{F}^2(M')$ is a bundle isomorphism such that $H^ \Theta' = \Theta$, then H is the second prolongation $h^{(2)}$ of the diffeomorphism H of the base manifolds M and M' induced by H .*

Proof. See [8].

We define an action of $G^2(n)$ on $T(\mathcal{F}^1(\mathbf{R}^n))_e$. Let $j^2(g) \in G^2(n)$ and $j^1(f) \in \mathcal{F}^1(\mathbf{R}^n)$. The mapping of a neighborhood of e onto a neighborhood of e defined by

$$j^1(f) \mapsto j^1(g \circ f \circ g^{-1})$$

induces a linear isomorphism of the tangent space $T(\mathcal{F}^1(\mathbf{R}^n))_e$ onto itself. This linear isomorphism depends only on $j^2(g)$ and is denoted by $\text{Ad}(j^2(g))$.

LEMMA (8-5). Let Θ be the canonical form on $\mathcal{F}^2(M)$. Then

(8-3). $\langle A^*, \Theta \rangle = \alpha_*(A)$ for $A \in \mathfrak{g}^2(n)$,

(8-4). $(R_a^*)\Theta = \text{Ad}(a^{-1}) \cdot \Theta$ for $a \in G^2(n)$.

Here $\mathfrak{g}^2(n)$ denotes the Lie algebra of $G^2(n)$ and α_* is the linear mapping: $\mathfrak{g}^2(n) \rightarrow \mathfrak{gl}(n, \mathbf{R})$ induced by $\alpha: G^2(n) \rightarrow GL(n; \mathbf{R})$.

Proof. See [8].

LEMMA (8-6). For an $A \in \mathfrak{g}^2(n)$ we have $\nu_*(A^*) = (\alpha_*A)^*$.

Proof. The proof is straightforward.

LEMMA (8-7). Let Θ and θ be the canonical forms of $\mathcal{F}^2(M)$ and $\mathcal{F}^1(M)$ respectively. Then we have

$$\nu'_*(\Theta) = \nu^*\theta.$$

Here $\nu: \mathcal{F}^2(M) \rightarrow \mathcal{F}^1(M)$ and $\nu': \mathcal{F}^2(\mathbf{R}^n) \rightarrow \mathcal{F}^1(\mathbf{R}^n)$.

Proof. Let X be a vector tangent to $\mathcal{F}^2(M)$ at $j^2(f)$. Then

$$\langle X, \nu'_*(\Theta) \rangle = \nu'_*\langle X, \Theta \rangle = \nu'_*((f_*^{(1)})^{-1}(\nu_*X)) = f_*^{-1}(\pi_*(\nu_*X)) = \langle X, \nu^*\theta \rangle.$$

Let $A(n; \mathbf{R})$ be the affine group acting on \mathbf{R}^n . Considering $A(n; \mathbf{R})$ as a principal bundle over $\mathbf{R}^n = A(n; \mathbf{R})/GL(n; \mathbf{R})$ with structure group $GL(n; \mathbf{R})$, we have a natural bundle isomorphism between $A(n; \mathbf{R})$ and $\mathcal{F}^1(\mathbf{R}^n)$:

$$\begin{array}{ccc} A(n; \mathbf{R}) & \longleftrightarrow & \mathcal{F}^1(\mathbf{R}^n) \\ \downarrow & & \downarrow \\ \mathbf{R}^n & \xlongequal{\text{id}} & \mathbf{R}^n \end{array}$$

Under this isomorphism the identity e of $A(n; \mathbf{R})$ (or of $GL(n; \mathbf{R})$) corresponds to $j_0^1(\text{id})$. The tangent space of $\mathcal{F}^1(\mathbf{R}^n)$ at e , corresponds to the Lie algebra

$$\mathfrak{a}(n; \mathbf{R}) = \mathbf{R}^n + \mathfrak{gl}(n; \mathbf{R}) \text{ of } A(n; \mathbf{R}).$$

Let Θ be the canonical form of $\mathcal{F}^2(M)$. We denote by Θ_{-1} and Θ_0 the \mathbf{R}^n - and $\mathfrak{gl}(n; \mathbf{R})$ -component of Θ .

LEMMA (8-8). We have $d\Theta_{-1} + [\Theta_{-1}, \Theta_0] = 0$.

Proof. See [12].

9. **Flat L_0 -structures of 2nd order.** For a linear subgroup G of $GL(n, \mathbf{R})$, the subgroup $\mathbf{R}^n \cdot G \subset A(n, \mathbf{R})$ can be considered a G -structure on \mathbf{R}^n . This structure is called a *flat G -structure*. A G -structure P over M is called flat if it is locally equivalent to the flat G -structure.

Let L/L_0 be a semisimple homogeneous space, and $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be its associated graded Lie algebra. We denote by \bar{o} the point of $\{L_0\}$. Then the mapping Exp from \mathfrak{g}_{-1} into L/L_0 defined by $\text{Exp}(x) = \exp x \cdot \bar{o}$ is a diffeomorphism into.

We have a natural mapping ι from L into $\mathcal{F}^2(L/L_0)$ and a group homomorphism $\iota': L/L_0 \rightarrow G^2(n)$ by

$$\begin{aligned} l \in L &\mapsto \iota(l) = j^2(l \cdot \text{Exp}), \\ a \in L_0 &\mapsto \iota'(a) = j^2(\text{Exp}^{-1} \cdot a \cdot \text{Exp}). \end{aligned}$$

LEMMA (9-1). ι' is injective. In particular ι is injective.

Proof. Let N be the kernel of ι' , and \mathfrak{n} be the Lie algebra of N . Since N is contained in the kernel of the linear isotropy representation, we get $N \subset G_1$, $\mathfrak{n} \subset \mathfrak{g}_1$ (Lemma (6-5)). And it is clear $\mathfrak{n} = \{x \in \mathfrak{g}_1 \mid [x, \mathfrak{g}_1] \subset \mathfrak{g}_1\}$. Then by the same argument in Proposition (3-1), we have $\mathfrak{n} = 0$. Thus N is a discrete group. In view of Lemma (6-1), we put $N = \exp \mathfrak{m}$. It is clear that G_0 normalizes N . Therefore, N being discrete, we have $[\mathfrak{g}_0, \mathfrak{m}] = 0$. Hence $\mathfrak{m} = 0$ by Lemma (3-4).

Thus $L \rightarrow L/L_0$ can be naturally considered as an L_0 -structure of 2nd order (if we fix \mathfrak{g}_{-1}). We call this structure the flat L_0 -structure of 2nd order. An L_0 -structure of 2nd order is called flat if it is locally equivalent of the flat L_0 -structure.

We recall that $G^2(n)$ operates on $\mathfrak{g}_{-1} \oplus \mathfrak{gl}(\mathfrak{g}_{-1})$, identifying \mathfrak{g}_{-1} with \mathbb{R}^n ($n = \dim \mathfrak{g}_{-1}$). With this in mind, we have the following.

LEMMA (9-2). The group L_0 ($\subset G^2(n)$) leaves $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ ($\subset \mathfrak{g}_{-1} \oplus \mathfrak{gl}(\mathfrak{g}_{-1})$) invariant. And its operation onto $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is identical to the operation of $L_0: \mathfrak{l}/\mathfrak{g}_{-1} \rightarrow \mathfrak{l}/\mathfrak{g}_{-1}$ induced by $\text{Ad}(a): \mathfrak{l} \rightarrow \mathfrak{l}$.

Proof. This is trivial.

Let Ξ be the Mauer-Cartan form of the group L . We write Ξ as $\Xi = \Xi_{-1} \oplus \Xi_0 \oplus \Xi_1$ according to the decomposition $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

LEMMA (9-3). Let $\Theta = \Theta_{-1} \oplus \Theta_0$ be the canonical form of the flat L_0 -structure of 2nd order $L \rightarrow L/L_0$. Then we have $\Theta_{-1} = \Xi_{-1}$ and $\Theta_0 = \Xi_0$.

Proof. This follows from the definition of the Mauer-Cartan form.

Let $L \rightarrow L/L_0$ be the flat L_0 -structure of 2nd order. Then $\nu(L) \subset \mathcal{F}^1(L/L_0)$ is a G_0 -structure, here G_0 is the linear isotropy subgroup and $\nu: \mathcal{F}^2(L/L_0) \rightarrow \mathcal{F}^1(L/L_0)$.

LEMMA (9-4). The G_0 -structure $\nu(L) \rightarrow L/L_0$ is flat.

Proof. This follows from Lemma (6-5).

10. Linear connections and 2nd order structures. An affine connection Γ on a manifold M of dimension n is a $\mathfrak{gl}(n; \mathbb{R})$ -values 1-form on $\mathcal{F}^1(M)$ satisfying the conditions

(10-1). $\langle A^*, \Gamma \rangle = A$ for $A \in \mathfrak{gl}(n; \mathbb{R})$,

(10-2). $R_c^* \Gamma = \text{Ad}(a^{-1}) \Gamma$ for $a \in GL(n; \mathbb{R})$.

For an element $u \in \mathbb{R}^n$, there exists uniquely a vector field V_u on $\mathcal{F}^1(N)$ such that $\langle V_u, \Gamma \rangle = 0$ and $\langle V_u, \theta \rangle = u$, where θ is the canonical form of $\mathcal{F}^1(M)$. We call this

vector field V_u the horizontal vector field of $u \in \mathbb{R}^n$. The torsion form $T(\Gamma)$ and the curvature form $R(\Gamma)$ are defined by

$$(10-3). \quad d\theta = [\theta, \Gamma] + T(\Gamma),$$

$$(10-4). \quad d\Gamma = \frac{1}{2}[\Gamma, \Gamma] + R(\Gamma).$$

If $T(\Gamma) = 0$, Γ is called an affine connection with torsion. We define the curvature tensor field τ of type (1.3) by

$$\tau_p(u, v) = \langle V_u \wedge V_v, R(\Gamma) \rangle \quad \text{for } u, v \in \mathbb{R}^n.$$

We define the covariant derivative of a tensor field ξ of type (r, s) on M . At each point $p \in \mathcal{F}^1(M)$, let ∇ be the linear mapping of \mathbb{R}^n into $\mathcal{F}_s^r(\mathbb{R}^n)$ defined by

$$\nabla_u \xi = (V_u)_p \cdot \xi.$$

Then $\nabla \xi$ is a tensor field of type $(r, s+1)$.

A cross-section $s: \mathcal{F}^1(M) \rightarrow \mathcal{F}^2(M)$ is called admissible if we have

$$(10-5). \quad s(p \cdot a) = s(p) \cdot a \quad \text{for } p \in \mathcal{F}^1(M) \text{ and } a \in GL(n; \mathbb{R}).$$

THEOREM (10-1) (S. KOBAYASHI [12]). *For an admissible cross-section $s: \mathcal{F}^1(M) \rightarrow \mathcal{F}^2(M)$, $s^* \Theta_0$ is an affine connection on M without torsion. And this gives us one-to-one correspondence between affine connections on M without torsion and admissible cross-sections.*

Let P be a G -structure on M . An affine connection Γ on M is called a G -connection if the restriction of Γ onto P has values in \mathfrak{g} (the Lie algebra of G).

Let L/L_0 be a semisimple flat homogeneous space. We fix once and for all its associated graded Lie algebra $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then L_0 is a subgroup of $G^2(n)$. We denote by G_0 the linear isotropy subgroup of L/L_0 ; $G_0 \subset L_0 \subset G^2(n)$.

Let P be a G_0 -structure on M and Γ be a G_0 -connection on P without torsion. By Theorem (10-1), P is a G_0 -subbundle of $\mathcal{F}^2(M)$. By extending the structure group G_0 into L_0 , we have a L_0 -structure of 2nd order, which is called the L_0 -structure of 2nd order associated with a G_0 -connection Γ , and denoted by $Q(\{\Gamma\})$.

Two G_0 -connections Γ and Γ' on P without torsion are called L_0 -equivalent if $Q(\Gamma) = Q(\Gamma')$. This is clearly equivalence relations. Denote by $A(P)$ the set of all G_0 -connection on P without torsion, each equivalence class α in $A(P)$ define uniquely a L_0 -structure of 2nd order, which is denoted by $Q(\alpha)$.

LEMMA (10-1). *Let L/L_0 be a semisimple flat homogeneous space with the associated graded Lie algebra $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and G_0 be its linear isotropy group. Let $P \rightarrow M$ be a G_0 -structure and α an L_0 -equivalence class in $A(P)$. If we denote by $\Theta = \Theta_{-1} \oplus \Theta_0$ the canonical form of $Q(\pi)$, the associated L_0 -structure of π , then we have*

- (1) Θ_{-1} is \mathfrak{g}_{-1} -valued and Θ_0 is \mathfrak{g}_0 -valued,
- (2) $\langle A^*, \Theta_0 \rangle = A'$ (the \mathfrak{g} -component of A), for $A \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$,
- (3) $R_a^*(\Theta_{-1} \oplus \Theta_0) \equiv \text{Ad}(a^{-1})(\Theta_{-1} + \Theta_0) \pmod{\mathfrak{g}_1}$.

Proof. (1) Let $s: P \rightarrow Q(\alpha)$ be the imbedding which gives one of G_0 -connection in α . Then the restriction of Θ_{-1} and Θ_0 have their values in \mathfrak{g}_{-1} and \mathfrak{g}_0 respectively. Our assertion follows from Lemma (9-2). The rest of our assertion follows also from Lemma (9-2) and the definitions.

Let $f: M \rightarrow M$ be a G_0 -automorphism of a G_0 -structure P . Then for each element Γ in $A(P)$, $f_*^{(1)}\Gamma$ is also a G_0 -connection on P . For each equivalence class α of $A(P)$, f is called α -automorphism if $f_*^{(1)}\alpha = \alpha$.

PROPOSITION (10-1). *If $H^{1,1}(\mathfrak{l})$ is zero, then there is only one equivalence class in $A(P)$ if $A(P) \neq \emptyset$.*

Proof. Let Γ and Γ' be two connections in $A(P)$. Since $\Gamma - \Gamma'$ is zero on the fibres of $P \rightarrow M$, at each point p of P , there exists a function F_p from \mathbb{R}^n into \mathfrak{g}_0 such that $\langle X, \Gamma - \Gamma' \rangle = F_p(\langle X, \theta \rangle)$ for each vector. We consider F_p as an element of $C^{1,1}(\mathfrak{l})$. Since Γ and Γ' are torsion free, we have $\partial F_p = 0$ for $p \in P$. Since $H^{1,1}(\mathfrak{l}) = 0$ there exists $F'_p \in C^{2,0}(\mathfrak{l})$ such that $\partial F'_p = F_p$. If s and \bar{s} are admissible cross-sections which give Γ and Γ' respectively, then it is easy to see that $s'(p) = \bar{s}(p) \exp F'_p$.

REMARK. For the geometric meaning of his proposition, see 7.

11. Normal Cartan connections. Let L/L_0 be a connected homogeneous space on which a Lie group L operates effectively and transitively. We denote by \mathfrak{l} and \mathfrak{l}_0 the Lie algebra of L and L_0 . A Cartan connection of type L/L_0 on an L_0 -principal bundle $Q \rightarrow M$ is, by definition, an \mathfrak{l} -valued 1-form ω on Q satisfying the conditions;

(11-0). $\dim M = \dim L/L_0$,

(11-1). $\langle A^*, \omega \rangle = A$ for $A \in \mathfrak{l}_0$,

(11-2). $R_a^* \omega = \text{Ad}(a^{-1})\omega$ for $a \in L_0$,

(11-3). $\langle X, \omega \rangle \neq 0$ for every nonzero vector of Q .

Let $H: Q \rightarrow Q$ be a bundle isomorphism. We call H an isomorphism of the Cartan connection ω if we have $H^*\omega = \omega$. We define the *curvature form* $\Omega = \Omega(\omega)$ of ω by $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$.

LEMMA (11-1). *We have*

(1) $R_a^* \Omega = \text{Ad}(a^{-1})\Omega$,

(2) $i(A^*)\Omega = 0$ for $A \in \mathfrak{l}_0$, here $i(A^*)\Omega$ means the inner product of Ω and A^* .

Proof. This is well known [15].

EXAMPLE (11-1). Let ω_0 be the Maurer-Cartan form of L . Then ω_0 is a Cartan connection on the L_0 -principal bundle $L \rightarrow L/L_0$.

PROPOSITION (11-1). *Let ω be a Cartan connection on an L_0 -principal bundle $Q \rightarrow M$. If the curvature form Ω of ω is zero, then at each point $m \in M$, there exists a local bundle isomorphism H between Q and L , defined on some neighborhood of m such that $H^*\omega_0 = \omega$.*

Proof. Since Ω is zero, the 1-form ω satisfies $d\omega + \frac{1}{2}[\omega, \omega] = 0$. It follows that, given a point u of Q , there exists a local diffeomorphism H of a neighborhood N'

of the identity of L onto a neighborhood N of u which sends ω into the Maurer-Cartan form ω_0 . Then it is easy to see that in an obvious manner, we extend H to a diffeomorphism $H: N' \cdot L_0 \rightarrow H \cdot L_0$. Let $U' = \pi'(N')$ and $U = \pi(N)$, where $\pi': L \rightarrow L/L_0$ and $\pi: Q \rightarrow M$. Then $\pi'^{-1}(U') = N' \cdot L_0$, and $\pi^{-1}(U) = N \cdot L_0$. By construction $H: \pi'^{-1}(U') \rightarrow \pi^{-1}(U)$ is a bundle isomorphism.

From now on we assume furthermore that L/L_0 is a semisimple flat homogeneous space. We denote by $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ its associated graded Lie algebras and by G_0 its linear isotropy subgroup of L_0 . Then a Cartan connection ω and its curvature form Ω can be written respectively as

$$\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1 \quad \text{and} \quad \Omega = \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$$

according to the decomposition; $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then clearly we have the following relations.

$$(11-4). \quad \Omega_{-1} = d\omega_{-1} + [\omega_0, \omega_{-1}].$$

$$(11-5). \quad \Omega_0 = d\omega_0 + \frac{1}{2}[\omega_{-1}, \omega_1].$$

$$(11-6). \quad \Omega_1 = d\omega_1 + [\omega_1, \omega_0].$$

LEMMA (11-2). *If $\Omega_{-1} = 0$, for an element $a \cdot \exp z \in L_0$ ($a \in G_0, z \in \mathfrak{g}_{-1}$), we have*

$$(11-7). \quad (R_a \cdot \exp z)^* \Omega_0 = \text{Ad}(a^{-1})\Omega_0,$$

$$(11-8). \quad (R_a \cdot \exp z)^* \Omega_1 = -[z, \Omega_0] + \text{Ad}(a^{-1})\Omega_1.$$

Proof. By Lemma (11-1), we have

$$\begin{aligned} (R_a \exp z)^*(\Omega_0 \oplus \Omega_1) &= \text{Ad}(\exp(-z) \cdot a^{-1})(\Omega_0 \oplus \Omega_1) \\ &= \text{Ad}(\exp(-z)) \cdot \text{Ad}(a^{-1})(\Omega_0 \oplus \Omega_1) \\ &= \exp \text{ad}(-z)(\text{Ad}(a^{-1})\Omega_0 + \text{Ad}(a^{-1})\Omega_1) \\ &= \text{Ad}(a^{-1})\Omega_0 + \text{Ad}(a^{-1})\Omega_1 - [z, \Omega_0]. \end{aligned}$$

LEMMA (11-3). *Let $P \rightarrow M$ be an G_0 -structure on M , and α be a L_0 -equivalence class in $A(P)$. Then there exists at least one Cartan connection $\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1$, of type L/L_0 on the associated L_0 -structure $Q(\alpha)$ of 2nd order such that $\omega_{-1} = \Theta_{-1}$ and $\omega_0 = \Theta_0$, here $\Theta = \Theta_{-1} \oplus \Theta_0$ is the canonical forms of $Q(\alpha)$.*

Proof. This is a local problem. In fact let $\{U_\lambda\}$ be a locally finite open covering of M with a partition of unity $\{f_\lambda\}$. If ω_λ is a Cartan connection on $Q(\alpha)U_\lambda$ with the given Θ_{-1} and Θ_0 , then $\sum_\lambda (f_\lambda \circ \pi)\omega_\lambda$ is a Cartan connection on $Q(\alpha)$ satisfying our condition, where $\pi: Q(\alpha) \rightarrow M$ is a projection. Hence our problem is reduced to the case where $Q(\alpha)$ is a trivial bundle. Fix a cross-section $s: P \rightarrow Q$ such that $s(pa) = s(p)a$ for $p \in P$ and $a \in G_0$, and set $\langle X, \omega_1 \rangle = 0$ for every vector tangent to $s(P)$. If Y is an arbitrary vector tangent to $Q(\alpha)$, then we can write uniquely

$$(11-9). \quad U = R_a(X) + V$$

where X is a vector tangent to $s(P)$ and $a \in \exp \mathfrak{g}_1$. And finally V is a vector tangent to a fibre of $Q \rightarrow P$ so that V can be extended to a unique fundamental vector field A^* on Q with $A \in \mathfrak{g}_1$. Then we put

$$(11-10) \quad \langle Y, \omega_1 \rangle = \mathfrak{g}_1\text{-component of } \text{Ad}(a^{-1})\langle X, \Theta_{-1} \oplus \Theta_0 \rangle + A.$$

Then $\omega = \Theta_{-1} \oplus \Theta_0 \oplus \omega_1$ is a Cartan connection. In fact for $A \in \mathfrak{g}_0$ and $B \in \mathfrak{g}_1$

$$\langle A^* + B^*, \omega \rangle = A + \langle A^* + B^*, \omega_1 \rangle = A + B \quad (\text{Lemma (8-6)}).$$

For $\exp z$ ($z \in \mathfrak{g}_1$) and Y in (11-9),

$$\begin{aligned} \langle Y, (R \exp z)^* \omega \rangle &= \langle (R \exp z)_* Y, \omega \rangle = \langle (R_a \exp z)_* X + A^*, \omega \rangle \\ &= \langle (R_1 \exp z)_* X + A, \Theta_{-1} \oplus \Theta_0 \rangle \\ &\quad + \text{the } \mathfrak{g}_1\text{-component of } \text{Ad}(\exp(-z)a^{-1})\langle X, \Theta_{-1} \oplus \Theta_0 \rangle + A \\ &= \text{Ad}(\exp(-z)a^{-1})\langle X, \Theta_{-1} \oplus \Theta_0 \rangle + A \quad (\text{from Lemma (9-2)}) \\ &= \text{Ad}(\exp(-z))\langle Y, \omega \rangle. \end{aligned}$$

Hence $(R_{\exp z})^* \omega = \text{Ad}(\exp(-z))\omega$. For a $g \in G_0$,

$$\begin{aligned} \langle Y, R_g^* \omega \rangle &= \langle R_g Y, \omega \rangle = \langle R_g R_a X + R_g A, \omega \rangle \\ &= \langle R_g R_a R_{g^{-1}} R_g X + (\text{Ad}(g^{-1})A)^*, \omega \rangle \\ &= \text{Ad}(g^{-1}ag)^{-1} \langle R_g X, \omega \rangle + \text{Ad}(g^{-1})A \quad (g^{-1}ag \in \exp \mathfrak{g}_1) \\ &= \text{Ad}(g^{-1}ag)^{-1} \langle R_g X, \Theta_{-1} \oplus \Theta_0 \rangle + \text{Ad}(g^{-1})A \\ &\hspace{15em} (\text{since } R_g X \text{ is tangent to } S(P)) \\ &= \text{Ad}(g^{-1}ag)^{-1} \text{Ad}(g^{-1})\langle X, \Theta_{-1} \oplus \Theta_0 \rangle + \text{Ad}(g^{-1})A \\ &= \text{Ad}(g^{-1}a^{-1})\langle X, \Theta_{-1} \oplus \Theta_0 \rangle + \text{Ad}(g^{-1})A \\ &= \text{Ad}(g^{-1})\{\text{Ad}(a^{-1})\langle X, \Theta_{-1} \oplus \Theta_0 \rangle + A\} \\ &= \text{Ad}(g^{-1})\langle Y, \omega \rangle. \end{aligned}$$

Thus we have shown that the conditions (11-1) and (11-2) are satisfied. Now since $\langle Y, \omega \rangle = \text{Ad}(a^{-1})\langle X, \Theta_{-1} \oplus \Theta_0 \rangle + A$, $\langle Y, \omega \rangle = 0$ implies $Y = 0$, proving the condition (11-3).

A Cartan connection which satisfies the assumption of Lemma (11-3) is called an *admissible Cartan connection* on $Q(a)$.

Let now $\omega = \Theta_{-1} \oplus \Theta_0 \oplus \omega_1$ be an admissible Cartan connection on $Q(a)$, and $\Omega = \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$ be its curvature form.

LEMMA (11-4). *We have $\Omega_{-1} = 0$. And there exists a form $\tilde{\Omega}$ on P such that $\nu^* \tilde{\Omega} = \Omega_0$.*

Proof. By (11-4), $\Omega_{-1} = d\Theta_{-1} + [\Theta_0, \Theta_{-1}]$. Then by Lemma (8-8) we have $\Omega_{-1} = 0$. The rest of our assertion follows from (11-1).

At each point $q \in Q$, we define a subspace H_q of $T(Q)_q$ by

$$H_q = \{X \in T(Q)_q \mid \langle X, \Theta_0 \rangle = 0, \langle X, \omega_1 \rangle = 0\}.$$

LEMMA (11-5). *We have $T(Q)_q = H_q \oplus (\mathfrak{g}_0 \oplus \mathfrak{g}_1)^*$ (direct sum), and Θ_{-1} is a linear isomorphism from H_q onto \mathfrak{g}_{-1} .*

Proof. From the condition (11-3), $H_q \cap (\mathfrak{g}_0 \oplus \mathfrak{g}_1)^* = \{0\}$. Take any element $X \in T(Q)_q$. Then $X - \langle X, \Theta_0 + \omega_1 \rangle^*$ belongs to H_q . The surjectivity of Θ_{-1} follows from Lemma (8-7), and the injectivity follows from the condition (11-3).

Now we define mappings $c = c(\omega)$ and $v = b(\omega)$ from Q into $C^{1,2}(I)$ and $C^{2,2}(I)$ respectively by

$$c_q(u, v) = \langle X \wedge Y, \Omega_0 \rangle \quad \text{and} \quad b_q(u, v) = \langle X \wedge Y, \Omega_1 \rangle,$$

where $X, Y \in H_q$ are defined by $\langle X, \Theta_{-1} \rangle = u$ and $\langle Y, \Theta_{-1} \rangle = v$.

LEMMA (11-6). *We have*

(1) $c_{q \cdot a \exp z} = \rho(a)^{-1} c_q$ for $a \in G_0, z \in \mathfrak{g}_1$,

(2) $\partial c_q = 0$ for $q \in Q(a)$,

(3) if $\Omega_0 = 0$, then $\partial b_q = 0$ for $q \in Q(a)$.

Proof. First we remark that $c_q(u, v) = \langle X' \wedge Y', \Omega_0 \rangle$ if $\langle X', \Theta_{-1} \rangle = u$ and $\langle Y', \Theta_{-1} \rangle = v$. In fact let $X \in H_q$ and $Y \in H_q$ be such that $\langle X, \Theta_{-1} \rangle = u$ and $\langle Y, \Theta_{-1} \rangle = v$. Then $X - X' = A^*$ and $Y - Y' = B^*$ for $A, B \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then our remark follows from Lemma (11-1). Combining this with the fact

$$\langle (R_{a \exp z})_* X, \Theta_{-1} \rangle = \text{Ad}(a^{-1}) \langle X, \Theta_{-1} \rangle$$

our first assertion can be proved. From Lemma (11-4), differentiating (11-4), we have $0 = [d\Theta_0, \Theta_{-1}] - [\Theta_0, d\Theta_{-1}]$. From (11-5), we have

$$\begin{aligned} [\Theta_{-1}, \Omega_0] &= [\Theta_{-1}, d\Theta_0] + \frac{1}{2}[\Theta_{-1}, [\Theta_0, \Theta_0]] + [\Theta_{-1}[\Theta_{-1}, \Theta_1]] \\ &= [\Theta_0, d\Theta_{-1}] + \frac{1}{2}[\Theta_{-1}[\Theta_0, \Theta_0]] + [\Theta_{-1}[\Theta_{-1}, \Theta_1]]. \end{aligned}$$

Therefore the restriction of $[\Theta_{-1}, \Omega_0]$ onto H_q is zero, this is nothing but $\partial c_q = 0$. Analogously we can prove (3).

Let $\omega' = \Theta_{-1} \oplus \Theta_0 + \omega'_1$ be another admissible Cartan connection on $Q(a)$. As before we define H'_q by

$$H'_q = \{X \in T(Q)_q \mid \langle X, \Theta_0 \rangle = 0, \langle X, \omega'_1 \rangle = 0\}.$$

Take $u \in \mathfrak{g}_{-1}$. If X_u (resp. X'_u) is the element in H_q (resp. H'_q) such that $\langle X_u, \Theta_{-1} \rangle = u$ (resp. $\langle X'_u, \Theta_{-1} \rangle = u$), we have $\langle X_u - X'_u, \Theta_{-1} \rangle = 0$ and $\langle X_u - X'_u, \Theta_0 \rangle = 0$. Therefore there exists an element $f_q(u)$ in \mathfrak{g}_1 such that $X_u - X'_u = (f_q(u))^*$. Clearly the mapping $f_q: \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1$ can be considered as an element of $C^{2,1}(I)$. Thus with two admissible Cartan connections ω and ω' , we associate a function $f(\omega, \omega')$ from Q into $C^{2,1}(I)$.

LEMMA (11-7). *Let $\omega = \Theta_{-1} \oplus \Theta_0 \oplus \omega_1$ and $\omega' = \Theta_{-1} \oplus \Theta_0 \oplus \omega'_1$ be two admissible Cartan connections on $Q(a)$. Then we have $c(\omega)_q - (c\omega')_q = \partial f(\omega, \omega')_q$ for each $q \in Q(a)$.*

Proof. Let Ω_0 and Ω'_0 be the \mathfrak{g}_0 -component of the curvature forms of ω and ω' respectively. From (11-5) we have $\Omega_0 - \Omega'_0 = [\Theta_{-1}, \omega_1 - \omega'_1]$. Take $u, v \in \mathfrak{g}_{-1}$. And

let $X_u, Y_v \in H_q$ and $X'_u, Y'_v \in H'_q$ be such that $\langle X_u, \Theta_{-1} \rangle = \langle X'_u, \Theta_{-1} \rangle = u$ and $\langle Y_v, \Theta_{-1} \rangle = \langle Y'_v, \Theta_{-1} \rangle = v$. Then

$$\begin{aligned} c(\omega)_q(u, v) - c(\omega')_q &= \langle X_u \wedge Y_v, \Omega_0 \rangle - \langle X'_u \wedge Y'_v, \Omega'_0 \rangle \\ &= \langle X'_u + (f(\omega, \omega')_q(u))^* \wedge Y'_v + (f(\omega, \omega')_q(v))^*, \Omega_0 \rangle \\ &\quad - \langle X'_u \wedge Y'_v, \Omega'_0 \rangle \\ &= \langle X'_u \wedge Y'_v, \Omega_0 \rangle - \langle X'_u \wedge Y'_v, \Omega'_0 \rangle \quad (\text{from Lemma (11-1)}) \\ &= \langle X'_u \wedge Y'_v, \Omega_0 - \Omega'_0 \rangle = \langle X'_u \wedge Y'_v, [\Theta_{-1}, \omega_1 - \omega'_1] \rangle. \end{aligned}$$

From this our assertion follows if we remark that

$$\langle X'_u, \omega_1 - \omega'_1 \rangle = f(\omega, \omega')_q(u) \quad \text{and} \quad \langle Y'_v, \omega_1 - \omega'_1 \rangle = f(\omega, \omega')_q(v).$$

The function $c(\omega)$ from $Q(a)$ into $C^{1,2}(I)$ is called *Weyl cocycle* for ω .

THEOREM (11-1) (N. TANAKA). *Let L/L_0 be a semisimple flat homogeneous space such that $H^{2,1}(I) = 0$. Given a G_0 -structure P on a manifold M , here G_0 denotes the linear isotropy group of L/L_0 . Then for each L_0 -equivalence class α in the set of all G_0 -connections on P , there exists uniquely an admissible Cartan connection on the associated L_0 -structure $Q(\alpha)$ of α such that its Weyl cocycle $c(\omega)_q$ is harmonic at each point q of $Q(\alpha)$ (cf., the definition after Lemma (4-1)).*

Proof. (Uniqueness). Let $\omega = \Theta_{-1} \oplus \Theta_0 \oplus \omega_1$ and $\omega' = \Theta_{-1} \oplus \Theta_0 \oplus \omega'_1$ be two admissible Cartan connections such that their Weyl cocycle $c(\omega)$ and $c(\omega')$ are harmonic, or equivalently $\partial c(\omega) = \partial^* c(\omega) = 0$. From Lemma (11-7), we have $c(\omega) - c(\omega') = \partial f(\omega, \omega')$. Thus we have $\partial^* \partial f(\omega, \omega') = \square f(\omega, \omega') = 0$. Since $H^{2,1}(I) = 0$, we have $f(\omega, \omega') = 0$. This is nothing but $\omega_1 = \omega'_1$.

(Existence). Let $\bar{\omega} = \Theta_{-1} \oplus \Theta_0 \oplus \bar{\omega}_1$ be an arbitrary admissible Cartan connection on $Q(\alpha)$ (cf. Lemma (11-3)). Let J be a $C^{2,1}(I)$ -valued function on $Q(\alpha)$ determined by $\square J = \partial^* \partial J = \partial^* c(\bar{\omega})$. J is well determined since $H^{2,1}(I) = 0$ implies that \square is an isomorphism of $C^{2,1}(I)$. For $a \cdot \exp z \in L_0$ ($a \in G_0, z \in \mathfrak{g}_1$), we have $J_{q \cdot a \cdot \exp z} = \rho(a)^{-1} J_q$ since $\rho(a) \square = \square \rho(a)$ and $c(\omega)_{q \cdot a \cdot \exp z} = \rho(a) c(\bar{\omega})_q$ (cf. Lemma (11-6)). We define a I -valued 1-form ω on $Q(\alpha)$ by

$$\langle X, \omega \rangle = \langle X, \bar{\omega} \rangle + J(\langle X, \Theta_{-1} \rangle),$$

for a vector tangent to Q . Then ω is the required admissible Cartan connection. In fact

- (i) $\langle A^*, \omega \rangle = \langle A^*, \bar{\omega} \rangle + J(\langle A^*, \Theta_{-1} \rangle) = \langle A^*, \bar{\omega} \rangle = A$.
- (ii) If $\langle X, \omega \rangle = 0$, then

$$0 = \langle X, \omega \rangle + J(\langle X, \Theta_{-1} \rangle) = \langle X, \Theta_{-1} \rangle + \langle X, \Theta_0 \rangle + \{ \langle X, \bar{\omega}_1 \rangle + J(\langle X, \Theta_{-1} \rangle) \}.$$

Thus $\langle X, \Theta_{-1} \rangle = \langle X, \Theta_0 \rangle = 0$. Therefore X is of the form A^* for some $A \in \mathfrak{g}_1$. Since $\langle A^*, \omega \rangle = A$, A must be 0.

(iii) For $a \cdot \exp z \in L_0$ ($a \in G_0, z \in \mathfrak{g}_1$) and a vector X_q in $T(Q)_q$, we have

$$\begin{aligned} \langle X_q, (R_{a \cdot \exp z})^* \omega \rangle &= \langle (R_{a \cdot \exp z})_* X_q, \omega \rangle \\ &= \langle (R_{a \cdot \exp z})_* X_q, \omega \rangle + J_{q \cdot a \cdot \exp z}(\langle (R_{a \cdot \exp z})_* \cdot, \Theta_{-1} \rangle) \\ &= \text{Ad}(a \exp z)^{-1} \langle X_q, \omega \rangle + J_{q \cdot a \cdot \exp z}(\text{Ad}(a^{-1} \langle X, \Theta_{-1} \rangle)) \\ &= \text{Ad}(a \exp z)^{-1} \langle X_q, \omega \rangle + \text{Ad}(a^{-1})(\rho(a) J_{q, a \exp z})(\langle X, \Theta_{-1} \rangle) \\ &= \text{Ad}(a \exp z)^{-1} \langle X_q, \omega \rangle + \text{Ad}(a^{-1}) \cdot J_q(\langle X, \Theta_{-1} \rangle) \\ &= \text{Ad}(a \exp z)^{-1} \langle X_q, \omega \rangle + \text{Ad}(a \exp z)^{-1} \cdot J_q(\langle X, \Theta_{-1} \rangle) \\ &= \text{Ad}(a \exp z)^{-1} \langle X_q, \omega \rangle. \end{aligned}$$

Thus we have shown that ω is a Cartan connection. Clearly this is admissible. Now the Weyl cocycle $c(\omega)$ of ω is given by $c(\omega) = c(\omega) - \partial J$. Therefore $\partial^* c(\omega) = \partial^* c(\omega) - \partial^* \partial J$. Since $\partial^* \partial J = \partial^* c(\omega)$, we have $\partial^* c(\omega) = 0$, i.e. $c(\omega) = 0$.

The admissible Cartan connection, which is assured to exist in Theorem (11-1) is called the *normal Cartan connection* of the L_0 -equivalence class.

Now we preserve our notations in Theorem (11-1). Let $H: M \rightarrow M$ be a G_0 -automorphism of P , preserving α . Then 2nd prolongation $h^{(2)}$ of h is an isomorphism of $Q(\alpha)$. Then by the uniqueness, $h^{(2)}$ is an isomorphism of the normal Cartan connection. Thus we have a natural mapping from the group of G_0 -automorphism preserving α into the group of isomorphisms of the normal Cartan connections.

LEMMA (11-8). *The mapping defined above is bijective.*

Proof. The injectivity is trivial. The surjectivity follows from Lemma (8-4).

COROLLARY (11-1). *The group of all G_0 -automorphisms of P preserving α is a finite dimensional Lie group.*

Proof. This follows from Lemma (11-8) and the fact that the automorphism group of a Cartan connection is a finite dimensional Lie group [11].

Now let $L \rightarrow L/L_0$ be the flat L_0 -structure. And as in §9, $\Xi = \Xi_{-1} \oplus \Xi_0 \oplus \Xi_1$ be the Mauer-Cartan form of L . Then Ξ is clearly the normal Cartan connection of the natural equivalence class on the flat G_0 -structure.

LEMMA (11-9). *The automorphism group of the normal Cartan connection of $L \rightarrow L/L_0$ is L .*

Proof. Clearly L is contained in the automorphism group since L preserves the Mauer-Cartan form. Conversely let h be a bundle isomorphism preserving the normal Cartan connection. Then $h^{(2)}$ preserves the Mauer-Cartan connection. Then $L_{h^{(2)}(e)}^{-1} \cdot h^{(2)}$ also preserves the Mauer-Cartan form and the identity of L . Therefore $L_{h^{(2)}(e)}^{-1} \cdot h$ is the identity on the connected component of the identity. Then $h^{(2)}$ is $L_{h^{(2)}(e)}$ from Lemma (8-4).

LEMMA (11-10). *A local automorphism of the normal Cartan connection of $L \rightarrow L/L_0$ can be uniquely extended to a global automorphism of $L \rightarrow L/L_0$.*

Proof. This can be proved as Lemma (11-9).

12. The Weyl tensors. Let L/L_0 be a semisimple flat homogeneous space such that $H^{2,1}(\mathfrak{l})=0$. Given a G_0 -structure P on a manifold M , here G_0 is the linear isotropy subgroup of L/L_0 . Let α be a L_0 -equivalence class in $A(P)$, the set of all G_0 -connections on P without torsion. Let ω be the normal Cartan connection on $Q(\alpha)$; the associated L_0 -structure of α . Then by Lemma (11-1) and Lemma (11-6), the Weyl cocycle $c(\omega)$ can be considered as a tensor field of type (1.3). We call the tensor field the *Weyl tensor field* of the family α , and denoted by $W(\alpha)$. If L/L_0 satisfies furthermore $H^{1,1}(\mathfrak{l})=0$, we can uniquely define an L_0 -structure of 2nd order since there is only one equivalence class in $A(P)$ (see Proposition (10-1)). In this case we call $W(\alpha)$ the Weyl tensor of a G_0 -structure P .

THEOREM (12-1). *Let L/L_0 be a semisimple flat homogeneous space such that $H^{2,1}(\mathfrak{l})=0$ and $H^{2,2}(\mathfrak{l})=0$. Given an L_0 -equivalence class α on a G_0 -structure P , G_0 being the linear isotropy subgroup of L/L_0 . Then the following are equivalent to each other:*

- (1) $W(\alpha)=0$,
- (2) $Q(\alpha)$ is flat.

If L/L_0 satisfies $H^{1,1}(\mathfrak{l})=0$, then the following are equivalent to each other:

- (1') The Weyl tensor of P vanishes.
- (2') P is flat.

Proof. Since the Weyl tensor of the flat L_0 -structure $L \rightarrow L/L_0$ vanishes, (2) clearly implies (1). By Lemma (11-6), $W(\alpha)=0$ implies the curvature form of the normal Cartan connection vanishes since $H^{2,2}(\mathfrak{l})=0$. Then by Proposition (9-1), (1) implies (2). The second assertion follows from Lemma (7-4).

PROPOSITION (12-1). *Let L/L_0 be a semisimple flat homogeneous space such that $H^{2,1}(\mathfrak{l})=0$. Let Γ be a G_0 -connection in a L_0 -equivalence class α in $A(P)$. We denote by s the admissible cross-section corresponding to Γ . Let $\omega = \Theta_{-1} \oplus \Theta_0 \oplus \omega_1$ be the normal Cartan connection of α .*

- (1) *There exists a tensor field $S: P \rightarrow C^{2,1}(\mathfrak{l})$ such that $\langle X, s^*\omega_1 \rangle = S(\langle X, \theta \rangle)$ for a vector tangent to P , here θ is the canonical form of P .*
- (2) $W(\alpha) = \mathfrak{r}(\Gamma) \oplus \partial S$, $\mathfrak{r}(\Gamma)$ is the curvature tensor field of Γ .
- (3) $S = -\square^{-1}(\partial^*(\Gamma))$.

Proof. Since $s^*\omega$, is zero on the fibre of $P \rightarrow M$, $s^*\omega$ can be expressed by θ . (2) From (11-5), we have

$$\begin{aligned} s^*\Omega_0 &= s^*(d\Theta_0 + \frac{1}{2}[\Theta_0, \Theta_0] + [\Theta_{-1}, \omega_1]) \\ &= ds^*\Theta_0 + \frac{1}{2}[s^*\Theta_0, s^*\Theta_0] + [s^*\Theta_{-1}, s^*\omega_1]. \end{aligned}$$

Since $s^*\omega_0 = \Gamma$, $s^*\Theta_{-1} = \theta$, we get $s^*\Omega_0 = \mathfrak{r}(\Gamma) + [\theta, s^*\omega_1]$. From this (2) is proved. (3) Applying ∂^* to (2), we have $\partial^*W(\alpha) = \partial^*\mathfrak{r}(\Gamma) + \partial^*\partial S$. Since $\partial^*W(\alpha) = 0$, we have the desired formula.

13. **Applications to geometric transformations.** Let L/L_0 be a semisimple flat homogeneous space such that $H^{2,1}(L)=0$. Given a G_0 -structure P on a manifold M , G_0 being the linear isotropy subgroup of L/L_0 . Let α be an L_0 -equivalence class in $A(P)$. Let $\Omega = \Omega_0 + \Omega_1$ be the curvature form of the normal Cartan connection of α . Let $s: P \rightarrow Q(\alpha)$ be an admissible cross-section, and Γ be the G_0 -connection corresponding to s . Then $\partial^*(\mathfrak{r}(\Gamma))$ is called the *Ricci tensor field* of Γ , here $\mathfrak{r}(\Gamma)$ is the curvature tensor field of Γ . Here we remark that \mathfrak{g}_1 and \mathfrak{g}_{-1} are naturally dual from Lemma (3-3), therefore we can consider $\partial^*\mathfrak{r}(\Gamma)$ as a tensor field.

THEOREM (13-1). *Let $s: P \rightarrow Q(\alpha)$ be a G_0 -connection in α . If the Ricci tensor field is parallel (i.e. $\nabla(\partial^*\mathfrak{r}(\Gamma))=0$), then $s^*\Omega_1=0$.*

Proof. From (11-6), we have

$$s^*\Omega_1 = s^*(d\omega_1 + [\omega_1, \Theta_0]) = ds^*\omega_1 + [s^*\omega_1, \Gamma] \quad (\Gamma = s^*\Theta_0).$$

Since $s^*\Omega_1$ is zero on the fibres of $P \rightarrow M$, for tangent vectors X and Y , we have

$$\begin{aligned} \langle X \wedge Y, s^*\Omega_1 \rangle &= \langle V_{\theta(X)} \wedge V_{\theta(Y)}, s^*\Omega_1 \rangle = \langle V_{\theta(X)} \wedge V_{\theta(Y)}, ds^*\omega_1 \rangle \\ &= V_{\theta(X)}\langle V_{\theta(Y)}, s^*\omega_1 \rangle - V_{\theta(Y)}\langle V_{\theta(X)}, s^*\omega_1 \rangle - \langle [V_{\theta(X)}, V_{\theta(Y)}], s^*\omega_1 \rangle. \end{aligned}$$

Here $V_{\theta(X)}$ and $V_{\theta(Y)}$ denote the horizontal vector fields corresponding $\theta(X)$ and $\theta(Y)$ respectively. Since Γ is torsion free, $[V_{\theta(X)}, V_{\theta(Y)}]$ is tangent to the fibres. Thus we get

$$\langle V_u \wedge V_v, s^*\Omega_1 \rangle = (\nabla_u S)(v) - (\nabla_v S)(u)$$

for $u, v \in \mathfrak{g}_{-1}$, here S is defined in Proposition (12-1). By Proposition (12-1), we know $S = -\square^{-1}(\partial^*\mathfrak{r}(\Gamma))$. In particular $\nabla\partial^*\mathfrak{r}(\Gamma)=0$ implies $\nabla S=0$. Thus $s^*\Omega_1=0$.

THEOREM (13-2). *Let $s: P \rightarrow Q(\alpha)$ and $t: P \rightarrow Q(\alpha)$ be two G_0 -connections in α . Then there exists a tensor field $A: P \rightarrow \mathfrak{g}_1$ such that $A(p)=s(p) \cdot \exp A_p$. If the Ricci tensor fields of s and t are parallel, then $[A_p, W(\alpha)_p]=0$ for $p \in P$.*

Proof. The first assertion is easy to see. For tangent vectors X_p and Y_p at p , we have

$$\begin{aligned} \langle X_p \wedge Y_p, t^*\Omega_1 \rangle &= \langle X_p \wedge Y_p, (R_{\exp A \cdot s})^*\Omega_1 \rangle \\ &= \langle (R_{\exp A \cdot s})_*(X_p \wedge Y_p), \Omega_1 \rangle. \end{aligned}$$

Since $(R_{\exp A \cdot s})_*X_p = (R_{\exp A})_*(s_*X_p) + \{\text{a vertical vector}\}$, we have

$$\langle X_p \wedge Y_p, t^*\Omega_1 \rangle = \langle (R_{\exp A})_*s_*(X_p \wedge Y_p), \Omega_1 \rangle \quad \text{from Lemma (11-1).}$$

Then by Lemma (11-2), we have

$$\begin{aligned} \langle X_p \wedge Y_p, t^*\Omega_1 \rangle &= \langle s_*(X_p \wedge Y_p), -[A_p, \Omega_0] + \Omega_1 \rangle \\ &= \langle X_p \wedge Y_p, -[A_p, s^*\Omega_0] + s^*\Omega_1 \rangle. \end{aligned}$$

Since $s^*\Omega_1$ and $t^*\Omega_1=0$, we have $[A_p, s^*\Omega_0]=0$. Since $s^*\Omega_0$ defines the Weyl tensor $W(\alpha)$, we have the desired result.

For a G_0 -connection Γ in α , we denote by $\text{Aut}(P, \Gamma)$ the subgroup of the automorphism group $\text{Aut}(\alpha)$ of the L_0 -equivalence class α preserving Γ . We suspect:

CONJECTURE. We assume L/L_0 is simple. If Γ is complete, Ricci-parallel and $\text{Aut}(P, \Gamma)$ is strictly different from $\text{Aut}(\alpha)$, then (up to covering) M is diffeomorphic to L/L_0 and L_0 -structure $Q(\alpha) \rightarrow M$ coincides with the flat L_0 -structure $L \rightarrow L/L_0$. This conjecture is true when L/L_0 is a projective space (resp. a Moebius space) and L is the group of projective transformations (resp. conformal transformations), provided Γ is the Lemi-Civita connection of some riemannian metric [20], [21].

THEOREM (13-3). *Let L/L_0 be a simple flat homogeneous space such that $H^{2,1}(\Gamma) = 0$. Given a G_0 -structure P on M , G_0 being the linear isotropy group of L/L_0 , let Γ be a complete G_0 -connection on P whose Ricci-tensor is parallel. If we have*

- (1) $\text{Aut}(\{\Gamma\})$ is transitive on M ,
- (2) the connected component of the linear isotropy group is irreducible,
- (3) $\text{Aut}(P, \{\Gamma\}) = \text{Aut}(\{\Gamma\})$,

then M is diffeomorphic to L/L_0 (up to covering) and $\text{Aut}(P, \{\Gamma\})$ is identical to L .

Proof. We may assume M is simply connected. Let f be an element in $\text{Aut}(\{\Gamma\})$ such that $f^*\Gamma \neq \Gamma$. Let $p \in P$ be a point such that $f^*\Gamma_p \neq \Gamma_p$. Taking as s and t in Theorem (13-2) the cross-sections corresponding to $f^*\Gamma$ and Γ respectively, we get the tensor field A such that $[A_p, W(\{\Gamma\})_p] = 0$. We denote by $\text{Ad}(g)$ the linear isotropy representation of the isotropy subgroup $\text{Aut}(\{\Gamma\})_p$ of $\text{Aut}(\{\Gamma\})$ at p . For each g in $\text{Aut}(\{\Gamma\})_p$, replacing f by g we also have

$$[{}^t\text{Ad}(g)^{-1}A_p, W(\{\Gamma\})_p] = 0.$$

Since the linear isotropy representation is irreducible, we have $W(\{\Gamma\})_p = 0$. Then the homogeneity implies $W(\{\Gamma\}) = 0$. By Proposition (12-1), we have

$$\tau(\Gamma) = \partial(\square^{-1}(\partial^*\tau(\Gamma))).$$

Since the Ricci-tensor is parallel, so is the curvature tensor $\tau(\Gamma)$. Thus (M, Γ) is a global affine symmetric space, and so is $(M, f^*\Gamma)$. Let x be a point in M and τ_x be the symmetry of (M, Γ) at x . If G_0 contains the element $-I$, then τ_x preserves P since Γ is a G_0 -connection on P and τ_x preserves the fibre of P at x . When G_0 does not contain $-I$, we consider the bundle extension to $P \cup P \cdot (-I)$ (disjoint). Then $\tau_x(P) = P \cdot (-I)$ and $\tau_x(P \cdot (-I)) = P$. Now there exists a point z in M such that $\tau_z \cdot \tau_z \neq \text{id}$ here τ'_z is the symmetry of $(M, f^*\Gamma)$ at z . In fact, otherwise, the symmetries of $(M, f^*\Gamma)$ coincide at every point M . From this we see that $\Gamma = f^*\Gamma$. Then from the above remark, we see $\tau_z \tau'_z$ preserves P . Then it is easy to see that $\tau_z \tau'_z$ is an element of $\text{Aut}(\{\Gamma\})$. Let \mathfrak{h} be the Lie algebra of $\text{Aut}(\{\Gamma\})$. Then \mathfrak{h} can be considered as a germ of vector field at z . Since $Q(\{\Gamma\})$ is locally equivalent to L/L_0 , we can consider \mathfrak{h} is a germ of vector field at the origin of L/L_0 . Then using Lemma

(11-10), we can consider \mathfrak{h} is a subalgebra of \mathfrak{l} . Also using (11-10), $\tau_z \cdot \tau'_z$ can be considered as an element of L . Furthermore $\tau_z \cdot \tau'_1$ belongs to $\exp \mathfrak{g}_1$ since its linear isotropy representation is the identity. We write $\tau_z \tau'_z$ as $\tau_z \cdot \tau'_z = \exp Z$ ($Z \neq 0$). Then clearly $\mathfrak{h} \oplus \{Z\}$ forms a subalgebra of \mathfrak{l} such that its linear isotropy representation is irreducible. Then using Proposition (3-6), we conclude, $\mathfrak{h} = \mathfrak{l}$. Therefore $M = L/L_0$.

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