A GALOIS THEORY FOR
INSEPARABLE FIELD EXTENSIONS(1)

BY

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Abstract. A Galois theory is obtained for fields $k$ of characteristic $p \neq 0$ in which the Galois subfields $h$ are those for which $k/h$ is normal, modular, and for some nonnegative integer $r$, $k(k^p^{r+1})/h$ is separable. The related automorphism groups $G$ are subgroups of the group $A$ of automorphisms $a$ on $k[X]=k[X]/X^{p^{r+1}}k[X]$, $X$ an indeterminate, such that $a(X)=X$. A subgroup $G$ of $A$ is Galois if and only if $G$ is a semidirect product of subgroups $G_k$ and $G_0$, where $G_k$ is a Galois group of automorphisms on $k$ (classical separable theory) and $G_0$ is a Galois group of rank $p^r$ higher derivations on $k$ (Jacobson-Davis purely inseparable theory). Implications of certain invariance conditions on a Galois subgroup of a Galois group are also investigated.

I. Introduction. Let $k$ be a field having characteristic $p \neq 0$. The Jacobson Galois correspondence between finite-dimensional restricted $p$ Lie algebras of derivations on $k$ and their fields of constants [6] has been extended recently by R. L. Davis [1], [2] to groups of higher derivations of finite rank and their fields of constants.

In this paper we exhibit an automorphism group invariant field correspondence which incorporates both the Krull infinite Galois theory [7, p. 147] and the purely inseparable theory referred to above. The invariant subfields $h$ are those for which $k/h$ is algebraic, normal, modular (as defined by Sweedler [8, Definition, p. 404]) and the purely inseparable part has finite exponent (Theorem 3.1). The associated automorphism groups are subgroups of the automorphism group of the local ring $k[X]$ described below. They can also be described as groups of rank $p^r$ higher derivations as described in §II with the modification that $d^{(0)}$ is an automorphism on $k$ rather than restricting $d^{(0)}$ to be the identity map. The higher derivation approach has the advantage of making unnecessary the introduction of the local ring $k[X]$. This seems to be outweighed by the advantage of familiarity of automorphisms and related constructions.

Let $H < G$ be Galois groups with invariant fields $k^H \supseteq k^G$. §IV is concerned with those conditions on $H$ relative to $G$ which are equivalent to $k^G$ being a Galois

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subfield of \( k^H \). A partial result is obtained (Theorem 4.1). A complete solution awaits an analysis of the above question in the purely inseparable Jacobson-Davis theory.

II. Definitions and preliminary results. Let \( \overline{X} \) denote the coset of the indeterminate \( X \) in the quotient ring \( k[\overline{X}] = k[X]/X^{p^r+1}k[X] \), \( r \geq 0 \). We will use the following notation.

\( A \): Group of automorphisms \( \alpha \) on \( k[\overline{X}] \) such that \( \alpha(\overline{X}) = \overline{X} \).

For \( G \) a subgroup of \( A \),

\[
G_0 = \{ \alpha \in G \mid \alpha(k) \subseteq k \},
\]

\[
G_0^0 = \{ \alpha \in G \mid \alpha(a) - a \in \overline{X}k[\overline{X}] \text{ for } a \in k \},
\]

\[
k^G = \{ a \in k \mid \alpha(a) = a \text{ for } \alpha \in G \},
\]

and, for \( h \) a subfield of \( k \),

\[
G^h = \{ \alpha \in G \mid \alpha(a) = a \text{ for } a \in h \}.
\]

For \( f(\overline{X}) \in k[\overline{X}] \) let \( \zeta(f(\overline{X})) = f(0) \). Then, for \( \alpha \in A \), \( \alpha^* (= \zeta|_k) \) is an automorphism on \( k \). For \( \beta \) an automorphism on \( k \), \( \beta^* \) will denote its unique extension to \( A \). The map \( \alpha \rightarrow \alpha^* \) \((= \alpha^*)\) is a homomorphism of \( A \) onto \( A_k \). With a subgroup \( G \) of \( A \) we associate the groups \( G^c = \{ \alpha^c \mid \alpha \in G \} \) and \( G^c_0 = \{ \alpha^c \mid \alpha \in G \} \).

A rank \( p^r \) higher derivation on \( k \) is a sequence \( d = \{ d^{(i)} | 0 \leq i \leq p^r \} \) of additive maps of \( k \) into \( k \) such that \( d^{(i)}(ab) = \sum (d^{(i)}(a)d^{(j)}(b) | i+j=n) \) and \( d^{(0)} \) is the identity map. The set \( \mathcal{H} \) of all rank \( p^r \) higher derivations on \( k \) is a group with respect to the composition "\( \circ \)" where \( d \circ e = f \) and \( f^{(0)} = \sum \{ d^{(0)}e^{(i)} | i+j=t \} \) for \( t=0, \ldots, p^r \). [5, Theorem 1, p. 33].

(2.1) **Proposition.** The map \( \Delta : \mathcal{H} \rightarrow A_0 \) given by \( \Delta(d)|_k = \sum \{ \overline{X}^{i}d^{(i)} | i=0, \ldots, p^r \} \) and \( \Delta(d)(\overline{X}) = \overline{X} \) is an isomorphism of \( \mathcal{H} \) with \( A_0 \).

**Proof.** For \( \alpha \in A_0 \) and \( a \in k \), \( \alpha(a) = \sum a_i \overline{X}^i \) with \( a_0 = a \). For \( i=0, \ldots, p^r \) let \( d^{(i)}(a) = a_i \). Then \( d = \{ d^{(i)} \} \) is in \( \mathcal{H} \) and \( \Delta(d) = \alpha \). This and the fact that \( \sum \overline{X}^id^{(i)} \) is an isomorphism for \( d \) in \( \mathcal{H} \) were essentially observed by Hasse and Schmidt [3]. Also,

\[
\Delta(d)\Delta(e)|_k = \sum \{ \overline{X}^{i}d^{(i)}(\overline{X}^{i}e^{(i)}) | 0 \leq i, j \leq p^r \}
\]

\[
= \sum \{ \overline{X}^{(i)}(d \circ e)^{(i)} | i = 0, \ldots, p^r \} = \Delta(d \circ e)|_k.
\]

For \( \mathcal{H} \) a subgroup of \( \mathcal{H} \) let \( k^\mathcal{H} = \{ a \in k \mid d^{(i)}(a) = 0, i > 0, d \in \mathcal{H} \} \); \( k^\mathcal{H} \) is called the field of constants of \( \mathcal{H} \). For \( h \) a subfield of \( k \) let

\[
\mathcal{H}^h = \{ d \in \mathcal{H} \mid d^{(i)}(a) = 0, i > 0, a \in h \}.
\]

The following is an immediate consequence of the definitions involved.

(2.2) **Proposition.** For \( \mathcal{H} \) a subgroup of \( \mathcal{H} \), \( k^\mathcal{H} = k^\mathcal{H}^h \), and, for \( h \) a subfield of \( k \), \( \Delta(\mathcal{H}^h) = A_0^h \).
A familiar property of higher derivations states that if $d \in \mathbb{H}$ then $d^{(i)}(a^{p+1}) = 0$ for $i = 1, \ldots, p'$ [4, Lemma 1, p. 130]. Let $k^a = k^G$ where $G$ is the group generated by $\alpha$ in $A$. By Propositions 2.1, 2.2 and the above remark we have

(2.3) **Proposition.** $k^{p+1} \subset k^a$ for $\alpha \in A_0$.

The following three propositions are self evident.

(2.4) **Proposition.** Each $\alpha \in A$ has a unique representation as a product $\beta \gamma$, $\beta \in A_k$, $\gamma \in A_0$. In fact, $\beta = \alpha^{e_0}$ and thus $\gamma = (\alpha^{e_0})^{-1} \alpha$.

(2.5) **Proposition.** $k^a = k^{ae} \cap k^{(ae)^{-1} a}$.

(2.6) **Corollary.** For $G$ a subgroup of $A$, let $H$ be the group generated by $G^{ae}$ and $G$. Then $k^H = k^{ae} \cap k^{H_0}$.

We require the following rather obvious extensions of familiar results. They are proved in Jacobson [7, pp. 49–52] with the assumption that $[k:h] < \infty$.

(2.7) **Proposition** [7, Lemma, p. 50]. Let $k > m > h$ be fields such that, for some positive integer $r$, $m^p \subset h$, and $k/m$ is separable algebraic. Then $k = m \otimes_h l$ where $l$ is the separable algebraic closure of $h$ in $k$.

**Proof.** The following argument parallels the proof in Jacobson [7]. Let $\{a_i\} = S$ be a basis for $k$ over $m$. Then

$$a_i a_j = \sum b_{i,j,a} a_m, \quad b_{i,j,a} \in m,$$

the sum being over a finite subset of $S$. Also, $a_i^p a_j^p = \sum b_{i,j,a} a_m$, and $b_{i,j,a} \in h$. Now $k^p \subset l$ since $m^p \subset h$. Hence, $a_i^p \in l$ and $S^p$ spans both $h[S^p]$ over $h$ and $m[S^p]$ over $m$. It is sufficient to show that $S^p$ is a basis for $l$ over $h$ and for $k$ over $m$.

Given $b$ in $k$, $m(b) = m(b^p)$ since $b$ is separable over $m$. Now $b = \sum b_i a_i$, $b_i$ in $m$, and $b^p = \sum b_i a_i^p$. Thus $b \in m[b^p] \subset m[S^p]$, and $b$ is in the linear span of $S^p$ over $m$. Since $k$ is a separable algebraic extension of $m$, $k^p$ and $m$ are linearly disjoint over $m^p$. The set $S^p$ is linearly independent over $m^p$, and hence, over $m$. By iteration, the set $S^p$ is linearly independent over $m$. Thus $S^p$ is a basis for $k$ over $m$.

We observed above that $h[S^p] \subset l$. As above, if $b$ is in $l$, $b \in h(b^p) \subset h[S^p]$. Hence $S^p$ is a basis for $l$ over $h$.

(2.8) **Proposition.** If $k|h$ is normal and, for some integer $r$, $k^{p} \subset l$, the separable closure of $h$ in $k$, then $k = l \otimes_h m$ where $m^{p} \subset h$.

The proof of the corresponding result assuming $[k:h] < \infty$ as given in Jacobson [7, Theorem 13, p. 52] also gives (2.8) in view of (2.7).

III. The Galois correspondence. We restate the assumption made throughout this paper, namely that $k$ is a field having characteristic $p \neq 0$. 

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(3.1) Theorem. Let \( h \) be a subfield of a field \( k \) such that \( k/h \) is algebraic. The following four conditions are equivalent.

(i) \( h = k^G \) for a subgroup \( G \) of \( A \).

(ii) \( k \) is a normal modular extension of \( h \) such that \( h(k^{p+1})/h \) is separable.

(iii) There are intermediate fields \( l \) and \( m \) such that \( m^{p+1} \subset h, m/h \) is modular, \( l/h \) is normal separable and \( k = l \otimes_h m \) (that is, \( k \) is generated by \( l \) and \( m \), subfields which are linearly disjoint over \( h \)).

(iv) There are intermediate fields \( l \) and \( m \) such that \( l/h \) is normal separable, \( m \) is a tensor product over \( h \) of simple purely inseparable extension of \( h \) having degree \( \leq p^{r+1} \) and \( k = l \otimes_h m \).

If \( k \) satisfied one of (i) through (iv) and \( G = A^h \) then \( l = k^{G_0} \) and \( m = k^{G_0^*} \) where \( l \) and \( m \) are given by (iii) or (iv) above.

Proof. By Corollary 2.6 we may assume that \( G \subset G \) and hence that \( k^G = k^{G_0} \cap k^{G_0^*} \). We first prove that \( k^{G_0} \) is normal over \( k^G \) and is the separable closure of \( k^G \) in \( k \). The modularity of \( k/k^{G_0} \) will follow essentially from certain results due to Sweedler.

(3.2) Lemma. \( \sigma(k^{G_0}) = k^{G_0} \) for \( \sigma \in G^c \).

Proof. Suppose to the contrary that for some \( \sigma \) in \( G^c \) and in \( k^{G_0} \), \( \sigma(a) \not= b \) in \( k^{G_0} \). Choose \( a \in G_0 \) for which \( \sigma(b) \neq b \). Then \( \beta = \sigma^{-1} \alpha \sigma^c \) is in \( G_0 \), whereas \( \beta(a) \not= a \), which is a contradiction. Thus, \( \sigma(k^{G_0}) \subset k^{G_0} \) and \( \sigma^{-1}(k^{G_0}) \subset k^{G_0} \). Hence \( \sigma(k^{G_0}) = k^{G_0} \).

(3.3) Lemma. \( k \) is a normal extension of \( k^{G_0} \).

Proof. By Lemma 3.2, the restriction to \( k^{G_0} \) of \( \alpha \) in \( G^c \) is an automorphism. Let \( K \) be the group of all such automorphisms on \( k^{G_0} \). Since \( k^G = k^{G_0} \cap k^{G_0^*} \), and \( G_k = G^\nu \), the subfield of \( k^{G_0} \) invariant under \( K \) is \( k^G \). Thus \( k^{G_0}/k^G \) is normal separable. By Proposition 2.3 \( k^{p+1} \subset k^{G_0} \) from which we conclude that \( k^{G_0} \) is the separable closure of \( k^G \) in \( k \).

To complete the proof, let \( a \) be in \( k \) and let \( f(X) \) be its minimum polynomial over \( k^G \). Then for some positive integer \( e \), \( f(X) = g(X^e) \) and \( g(X) \) is a separable polynomial. Since \( g(X) \) has \( a^e \) in \( k^{G_0} \) as root, \( g(X) \) splits over \( k^{G_0} \). Let \( g(X) = (X-b_1) \cdots (X-b_r) \) with \( a^e = b_1 \). Given \( b_i \), there is an \( \alpha \) in \( G^c \) such that \( \alpha(b_i) = b_i \) or \( \alpha(a)^{p^e} = b_i \) and evidently each \( b_i \) has a \( p^e \)th root in \( k \). It follows that \( f(X) \) splits in \( k \) which proves the lemma.

The field \( k^{G_0} \) is the field of constants of the group \( \Delta^{-1}(G_0) \) of higher derivations and thus, by a theorem of Sweedler, \( k/k^{G_0} \) is modular [8, Theorem 1, p. 403]. We conclude that \( k/k^G \) is modular from the following.

(3.4) Proposition. Let \( k \) be a normal extension of a field \( h \) with the property that for some positive integer \( r \), \( h(k^{p^r}) \) is separable over \( h \). There is a unique extension \( k' \) of \( k \), modular over \( l \), the separable closure of \( h \) in \( k \). Moreover, \( k' \) is modular over \( h \).

Proof. This result with the added assumption \( [k:h] < \infty \) is due to Sweedler [9,
Corollary 7, p. 206]. Sweedler's proof with obvious modifications and using (2.8) of this paper where needed suffices to prove the above.

If $k/h$ satisfies (ii) then, by Proposition 2.8, the conditions of (iii) are fulfilled, the modularity of $m/h$ following from a result of Sweedler [8, Lemma 5 (3), p. 407]. The equivalence of (iii) and (iv) is also due to Sweedler [8, Theorem 1, p. 403].

We show that (iv) implies (i) as follows. Let $G_1$ be the group of extensions to $A$ of the automorphism group of $l$ over $h$.

(3.5) Proposition. $k^{G_1} = m$.

Proof. Clearly $m \subseteq k^{G_1}$. If $a \in k^{G_1}$ then, since $h(k^{p+1}) \subseteq l$, $a^{p+1}$ is in $l \cap k^{G_1} = h$. Hence $a$ is in $m$ and we have $k^{G_1} = m$.

Let $\mathcal{H}_1$ represent the group of all rank $p'$ higher derivations of $m$ into $m$ which are trivial on $h$. Each $d \in \mathcal{H}_1$ has a unique extension to $k$ since $k/m$ is separable algebraic [4, Theorem 3]. Then $\mathcal{H} = \{ d \mid d$ is an extension to $k$ of an element of $\mathcal{H}_1 \}$ is a group of rank $p'$ higher derivations on $k$ with the property $l \subseteq k^{\mathcal{H}}$. Let $G$ be the subgroup of $A$ generated by $G_1$ and $\Delta \mathcal{H}$. By Corollary 2.6, $k^G = k^{G_1} \cap k^{G_0} = m \cap k^{\mathcal{H}} = h$.

The last sentence of the theorem remains to be proved. In establishing that (i) implies (ii) it was shown that $k^{G_0} = l$. The proof of (3.5) gives $m = k^{G_0}$.

(3.6) Definition. A subgroup $G$ of $A$ is Galois if $G = A^h$ for a subfield $h$ of $k$ such that $k/h$ is algebraic.

(3.7) Definition. A subfield $h$ of $k$ is Galois if (i) $k/h$ is algebraic and (ii) $h = k^G$ for a subgroup $G$ of $A$.

Theorem 3.1 identifies those subfields of $h$ which are Galois. The Krull infinite Galois theory asserts that a subgroup $G$ of $A_k$ is Galois if and only if $G^c$ is compact in the finite topology [7, Example 5, p. 151]. R. L. Davis has characterized those subgroups of $\mathcal{H}$ having the form $\mathcal{H}^h$ and hence, via $\Delta$, those subgroups of $A_0$ which are Galois, with the assumption, however, that $[k:h] < \infty$ [1], [2].

The following result reduces the question of when a subgroup of $A$ is Galois to subgroups of $A_0$ and $A_k$.

(3.8) Theorem. A subgroup $G$ of $A$ is Galois if and only if (i) $G^{ce} \subseteq G$ and (ii) $G^{ce}$ and $G^0$ are Galois.

Proof. The necessity of (i) is clear (see the first sentence of the proof of Theorem 3.1). If $G$ is Galois then in the notation of Theorem 3.1 $G \supseteq A^l$ and $G \supseteq A^m$ where $k = l \otimes k^{G_0} \supseteq m$. For $\alpha$ in $A^l$, $\alpha^c$ is an automorphism on $k$ which is the identity on $k^{p+1}$, hence is the identity on $k$. Thus, $\alpha$ is in $A_0$ or $A^l \subseteq G_0$. By the last sentence of Theorem 3.1 $G_0 \subseteq A^l$ and we have $G_0 = A^l$. Also, by Theorem 3.1 $G_k \subseteq G^m$. Conversely, if $\alpha$ is in $A^m$ then, letting $\beta = \alpha^{ce^{-1}}$, we have $k/k^\beta$ is separable algebraic since by (2.5) $m \subseteq k^\beta$. However, $k^\beta$ is the field of constants of a finite higher derivation and hence $k = k^\beta$, $\alpha \in A_k$ or $A^m \subseteq A_k \cap G = G_k$. Hence $G^{ce} = G_k = A^m$ and $G^{ce}$ is Galois.
Suppose, conversely, that $G$ satisfies (i) and (ii). We first show that $k/k^G$ is algebraic. Lemma 3.2 applies since its proof does not require $k/k^G$ to be algebraic. By Lemma 3.2 $G^e|k_{e^0}$ is a group of automorphisms with the property that $k^e$ is algebraic over its subfield of invariants which is $k^0$. Since $k^{e+1} \subset k^e$ we have $k/k^G$ algebraic. Let $h=k^0$ and $H=A^h$. Then $G \leq H$, $G^e \subset H^e$ and $G^0 \subset H^0$. By Theorem 3.1 $k=m \otimes_k l$, $m=k^e\otimes k^h$ and $l=k^0\otimes k^h$. But $k^0$ and $k^e$ are the fields of invariants of $G^e$ and $H^e$ respectively and, since $G^e$ is Galois, $G^e \supset H^e$ or $G^e = H^e$. Similarly, $G_0 = H_0$ and $G=G^e\otimes G_0 = H^e\otimes H_0 = H$.

(3.9) **Definition.** Given subgroups $H_1$ and $H_2$ of a group $H$, we say $H_1$ is $H_2$ invariant if for $\alpha$ in $H_2$, $\alpha^{-1} H_1 \alpha \subset H_1$.

We consider the following question. When are two subgroups $H$ of $A_k$ and $K$ of $A_0$ compatible in the sense that there is a group $G$ in $A$ for which $G_k = H$ and $G_0 = K$? Such a group will exist if and only if $KH$ is such a group and, since $K$ must be an invariant subgroup of $G$, $K$ and $H$ will be compatible if and only if $K$ is $H$ invariant.

Let $\mathcal{G}$ be the set of groups of automorphisms on $k$, $\mathcal{D}$ the set of groups of rank $p'$ higher derivations on $k$.

(3.10) **Definition.** A pair $(H, \mathcal{H})$ in $\mathcal{G} \times \mathcal{D}$ is compatible if there is a subgroup $G$ of $A$ such that $G^e = H$ and $G_0 = \Delta(\mathcal{H})$. A pair $(H, \mathcal{H})$ is Galois if it is compatible and $H^e \Delta(\mathcal{H})$ is Galois.

Given $(H, \mathcal{H})$ in $\mathcal{G} \times \mathcal{D}$, $\mathcal{H}$ is invariant under $H$ if given $\sigma \in H$ and $d = \{d^{(i)}\}$ in $\mathcal{H}$ then $\sigma^{-1} d \sigma = \{\sigma^{-1} d^{(i)} \sigma\}$ is in $\mathcal{H}$. We sum up these remarks with

(3.11) **Proposition.** A pair $(H, \mathcal{H})$ in $\mathcal{G} \times \mathcal{D}$ is compatible if and only if $\mathcal{H}$ is $H$ invariant. A compatible pair $(H, \mathcal{H})$ is Galois if and only if $H^e$ and $\Delta(\mathcal{H})$ are Galois.

**Proof.** The last sentence follows from Theorem 3.8. The rest is a translation of the condition for compatibility of $H^e$ and $\Delta(\mathcal{H})$ stated above.

**IV. The subgroup subfield correspondence.** In this section we consider some of the implications of invariance. Specifically, let $H \subset G$ be Galois subgroups of $A$. We consider the consequences for $k^H/k^G$ of invariance of $H_0$ in $G_k$ and of $H_k$ in $G_k$. The objective of this section is the identification of conditions on $H$ relative to $G$ equivalent to $k^H/k^G$ being Galois. Theorem 4.2 is a partial result in this direction. The discussion following Corollary 4.4 indicates that invariance conditions alone will be insufficient to determine whether or not $k^H/k^G$ is Galois.

(4.1) **Theorem.** Let $G$ be a Galois subgroup of $A$. Then $G_k$ is $G_0$ invariant if and only if $G_k$ is $G \{1\}$.

**Proof.** If $G_k$ is $G_0$ invariant then $G_k$ is invariant in $G$. Hence, since $G_k \cap G_0 = \{1\}$, $G$ is the direct product of $G_k$ and $G_0$ which means that for $d \in \Delta^{-1}(G_0)$ and $\alpha$ in $G^e_k$, $ad^{(i)} = d^{(i)} \alpha$, $i = 1, \ldots, p'$. Using the freedom available in constructing $d$ in $\Delta^{-1}(G_0)$ we will exhibit $d$ and $\alpha$ in $G_k$ which do not commute assuming that $G_k$ and $G_0$ are nontrivial. By Theorem 3.1(iv), $m$ is a tensor product over $h=k^G$ of
purely inseparable extensions \( h(x_i) \) where \( p^n, \) the degree of \( x_i \) over \( h, \) is such that \( n_i \leq r + 1. \) It follows that the set \( S = \{x_i^{p^n}\} \) is a \( p \)-independent subset of \( h \) and hence of \( l \) since separable extensions preserve \( p \)-independence. Let \( S \cup S_1 \) be a \( p \)-basis for \( l. \) Then \( \{x_i\} \cup S_1 \) is a \( p \)-basis for \( k. \) We now use the fact that a higher derivation \( d \) is determined by its action on a \( p \)-basis which action may be arbitrarily prescribed for each map of \( d \) [4, Theorem 1, p. 131]. We defined \( d \) by the requirement \( d^{(i)}(s) = 0 \) for \( s \in \{x_i\} \cup S_1 \) and \( i < p^n. \) For \( x_i \in \{x_i\} \) we let \( d^{(i)}(x_i) = a \in l, \ a \neq m, \) and let \( d^{(i)} \) map every other element of \( \{x_i\} \cup S_1 \) into \( 0. \) Clearly \( d \) is trivial on \( S \cup S_1 \) and hence \( \Delta(d) \) is in \( G_0 \) since \( G_0 \) is Galois. However, \( a(a) \neq a \) for some \( a \in G_k \) and \( a(d^{(i)}(x_i)) = a(a) \neq a = d^{(i)}(x_i) = d^{(i)}(a(x_i)). \) Thus, if \( G_k \) is \( G_0 \) invariant, either \( G_0 \) or \( G_k \) must be the trivial group.

(4.2) Theorem. Let \( H \subset G \) be Galois subgroups of \( A. \) Let \( k = l \otimes_k m \) as in Theorem 3.1. Then \( k^H = l_1 \otimes_{k^0} m_1 \) with \( l_1 \subset l \) and \( m_1 \subset m \) if and only if \( H_0 \) is \( G_k \) invariant. Moreover, if \( k^H = l_1 \otimes_{k^0} m_1, \) then \( k^{H_0} = l_1 \otimes_{k^0} m_1 \) and \( k^{H_k} = l_1 \otimes_{k^0} m. \)

Proof. Assume \( k^H = l_1 \otimes_{k^0} m_1. \) Then \( k^{H_0} \supseteq l \otimes m_1 \) and \( k^{H_k} \supseteq l_1 \otimes m. \) But, \( k = k^{H_0} \otimes_{k^H} k^{H_k} = (l \otimes m_1) \otimes_{k^H} (l_1 \otimes m). \) This in turn means that \( k^{H_0} = l \otimes m_1 \) and \( k^{H_k} = l_1 \otimes m. \) If \( a \in G_k \) then \( a(l) \subset l \) and \( a|m \) is the identity. Hence \( a(k^{H_0}) = k^{H_0}, \) from which it follows that if \( d \) is in \( \Delta^{-1}(H_0) \) then \( a^{-1} \alpha \) is in \( \Delta^{-1}(H_0) \) or \( H_0 \) is \( G_k \) invariant.

Conversely, assume that \( H_0 \) is \( G_k \) invariant and let \( M = G_k H_0. \) By Theorem 3.8 \( G_k \) and \( H_0 \) are Galois. Since \( M^c = G_k^c H_0^c = G_k \) and \( M_0 = H_0 m, \) \( M \) is also Galois by Theorem 3.7.

(4.3) Lemma. \( k^{H_0} = m_1 \otimes_{k^0} l \) where \( m_1 = k^{H_0} \cap k^{G_k} \) and \( k^{H_k} = m \otimes_{k^0} l_1 \) where \( l_1 = k^{H_k} \cap k^{G_0}. \)

Proof of lemma. By the proof of Theorem 3.1 \( k^{H_0} = k^{H_0} \) is the separable closure of \( k^H = k^{H_0} \cap k^{G_k} = m_1 \) in \( k. \) However, \( l \otimes m_1 \) is the separable closure of \( m_1 \) in \( k. \) We prove this as follows. Let \( \{x_i\} \) be a basis for \( l \otimes m_1 \) over \( m_1 \) and \( \{x_i, y_j\} \) a basis over \( m_1 \) for the separable closure of \( m_1 \) in \( k. \) In general, if a field \( k' \) is a separable algebraic extension of a field \( h' \) and \( U \) is a linear basis for \( k'/h' \) then \( U^p = \{u^p \mid u \in U\} \) is also a basis. Hence \( \{x^{p+1}_i, y^{p+1}_j\} \) is a basis for the separable closure. However, \( \{y^{p+1}_j\} \subset l \) and \( \{x^{p+1}_i\} \) spans \( l \) over \( m_1. \) Thus \( \{y_j\} \) is empty and \( l \otimes_{k^0} m_1 \) is the separable closure of \( m_1 \) in \( k, \) and \( k^{H_0} = l \otimes m_1. \)

Considering the remaining equality, we clearly have \( k^{H_k} \supseteq k^{G_k} = m \) and hence \( k^{H_k} \supseteq m \otimes l_1. \) To establish equality, let \( \{x_i\} \) be a basis for \( l_1 \otimes k^0, \) \( \{x_i, y_j\} \) for \( l \otimes k^0 \) and \( \{z_j\} \) for \( m \otimes k^0. \) Assume that \( \sum C_{\rho,\sigma} z_{\rho,\sigma} y_{\sigma} \) is in \( k^{H_k} \) and not in \( m \otimes l_1 \) where \( C_{\rho,\sigma} \) is in \( k^0. \) Then, since \( k^{p+1} \subset k^{G_k}, \) we have \( \sum C_{\rho,\sigma} z_{\rho,\sigma} y_{\sigma} \) is in \( k^{H_k} \cap k^{G_k} = l_1. \) But this denies the independence of \( \{x^{p+1}_i, y^{p+1}_j\} \) over \( k^0, \) since \( z_{\rho,\sigma} \) is in \( k^0. \) It follows that \( k^{H_k} = m \otimes l_1 \) and the proof of the lemma is complete. Now \( k^H = k^{H_k} \cap k^{H_0} = (m \otimes l_1) \cap (m_1 \otimes l) = m_1 \otimes l_1. \)
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(4.4) COROLLARY. If \( H \) is an invariant subgroup of \( G \) then \( k^H/k^G \) is normal.

Proof. Under the given assumption \( H_0 \) is \( G_k \) invariant and \( H_k \) is \( G_k \) invariant. Thus, \( H_k | I_1 \) is an invariant subgroup of \( G_k | I_1 \) having \( I_1 \) as its invariant field. It follows that \( I_1/k^G \) is normal and hence that \( k^H/k^G \) is normal.

It follows from Theorem 4.1 that the converse of the above corollary is not true. If \( G = G_k G_0 \) and neither \( G_k \) nor \( G_0 \) is trivial then \( k^{G_k}/k^G \) is Galois whereas \( G_k \) is not an invariant subgroup of \( G(2) \). The following example illustrates that the purely inseparable theory exhibits the same behavior.

Let \( k_1 \) be a field obtained by extending a perfect field by indeterminates \( x_1 \) and \( x_2 \). Then \( \{x_1, x_2\} \) is a \( p \) basis for \( k_0 \). Let \( y_1 = x_1^{p^{-2}}, y_2 = x_2^{p^{-2}}, \) and \( k = k_0(y_1, y_2) = k_0(y_1) \otimes_{k_0} k_1(y_2) \). Consider the groups \( \mathcal{H} \subset \mathcal{H} \) of rank \( p \) higher derivations having \( k_0(y_1) \) and \( k_0 \) as field of constants respectively.

(4.5) \( \mathcal{H} \) is not invariant in \( \mathcal{H} \).

Proof. Let \( d = \{d^{(0)}, \ldots, d^{(p)}\} \) and \( h = \{h^{(0)}, \ldots, h^{(p)}\} \) be higher derivations. Then \( h^{-1} = \{h^{(0)} - h^{(1)} - h^{(2)} + [h^{(1)}, \ldots]\} \) [5, Relation (3), p. 53] and \( h dh^{-1} = \{d^{(0)}, d^{(1)}, d^{(2)} + [h^{(1)}, d^{(1)}], \ldots\} \), where \( [h^{(1)}, d^{(1)}] = h^{(1)} d^{(1)} - d^{(1)} h^{(1)} \). We choose \( h \in \mathcal{H} \) such that \( h^{(1)}(y_1) = y_2 \) and \( d \in \mathcal{H} \) such that \( d^{(1)}(y_2) \neq 0 \) (for justification see proof of Theorem 4.1). Then \( d^{(2)} + [h^{(1)} d^{(1)}](y_1) = -d^{(1)} h^{(1)}(y_1) \neq 0 \) and \( h dh^{-1} \) is not in \( \mathcal{H} \).

Let \( G = \Delta(\mathcal{H}) \) and \( H = \Delta(\mathcal{H}) \). Then \( k^G = k_0, k^H = k_0(y) \) and \( k^H/k^G \) is normal modular, whereas \( H \) is not an invariant subgroup of \( G \).

REFERENCES


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