CONDITIONS ON AN OPERATOR IMPLYING
\[ \text{Re } \sigma(T) = \sigma(\text{Re } T) \]

BY

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Abstract. It is shown that the equation of the title is valid for certain classes of not necessarily normal operators (including Toeplitz operators, and operators whose spectrum is a spectral set), and a new proof is given of C. R. Putnam's theorem that it is valid for seminormal operators.

If \( T \) is any operator (bounded linear, in Hilbert space) then \( f(\sigma(T)) = \sigma(f(T)) \) for every rational function \( f(\lambda) \) without poles in \( \sigma(T) \). On the other hand, if \( T \) is normal then for every polynomial \( f(\lambda, \lambda^*) \) one has \( \sigma(f(T)) = f(\sigma(T)) = \{f(\lambda, \lambda^*) : \lambda \in \sigma(T)\} \) (see [16] for a recent elegant proof). In the present paper, we are concerned with the polynomial \( f(\lambda, \lambda^*) = \frac{1}{2}(\lambda + \lambda^*) = \text{Re } \lambda \) for not necessarily normal operators \( T \), i.e., we consider conditions on an operator \( T \) such that

\[ \text{(*) } \quad \text{Re } \sigma(T) = \sigma(\text{Re } T). \]

This is a very humble "spectral mapping theorem", but it is a considerable achievement for a nonnormal operator. For example, the fact that \( (*) \) holds for \( T \) seminormal, which is due to C. R. Putnam [10], plays a role in his recent proof that a seminormal operator whose spectrum has zero area is normal [13]. In Theorem 1 we give a simple new proof of Putnam's theorem, and condition \( (*) \) is verified for some classes of not necessarily seminormal operators in Theorems 2–4. Specifically, we prove

**Theorem 1 (Putnam).** If \( T \) is a seminormal operator, then \( (*) \) holds.

**Theorem 2.** If \( T \) is a Toeplitz operator, then \( (*) \) holds.

**Theorem 3.** If \( T \) is an operator such that \( \sigma(T) \) is a spectral set for \( T \), then \( (*) \) holds.

**Theorem 4.** If \( T \) satisfies the growth condition \( (G_1) \) and \( \sigma(T) \) is connected, then \( (*) \) holds.

All operators contemplated in Theorems 1–4 are convexoid, i.e., \( \text{conv } \sigma(T) = \text{Cl } W(T) \). (Here \( \text{conv } \) denotes convex hull, \( \text{Cl } \) denotes closure, and \( W(T) = \{(Tx|x) : \|x\| = 1\} \) is the numerical range of \( T \).) Lemmas 4–6 give results weaker than \( (*) \) for certain convexoid operators.

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Lemma 1. If \( T \) is hyponormal then \( \text{Re } \sigma(T) \subset \sigma(\text{Re } T) \).

Proof. Let \( \lambda = \alpha + i\beta \in \sigma(T) \), the approximate point spectrum of \( T \), and let \( x_n \) be a sequence of unit vectors such that \( T x_n - \lambda x_n \to 0 \). By hyponormality, \( \| (T - \lambda I)^* x_n \| \leq \| (T - \lambda I) x_n \| \), therefore also \( T^* x_n - \lambda^* x_n \to 0 \). Then \( (\text{Re } T) x_n - \alpha x_n = \frac{1}{2} (T + T^*) x_n - \frac{1}{2} (\lambda + \lambda^*) x_n \to 0 \), thus \( \alpha \in \sigma(\text{Re } T) \).

Lemma 2. If \( T \) is seminormal then \( \sigma(T) \subset \sigma(\text{Re } T) \).

Proof. One can suppose \( T \) hyponormal. Let \( \lambda_0 \in \sigma(T) \). The vertical line \( \text{Re } \lambda = \text{Re } \lambda_0 \) exits the spectrum at a boundary point \( \nu \) of \( \sigma(T) \). Since \( \sigma(T) \subset \pi(T) \), one has \( \text{Re } \lambda_0 = \text{Re } \mu \in \sigma(\text{Re } T) \) by Lemma 1.

Lemma 3. If \( T \) is normal then \((*)\) holds.

Proof. This is immediate from the continuous functional calculus; see also the remarks at the beginning of the paper. Alternatively, see Theorem 3.

Proof of Theorem 1. One can suppose \( T \) hyponormal. In view of Lemma 2, it suffices to show that \( \sigma(\text{Re } T) \subset \sigma(T) \). Write \( T = H + iJ \), \( H \) and \( J \) selfadjoint, and let \( D = T^* T - T T^* \); \( D \geq 0 \) by hypothesis, and one has

\[
HJ - JH = -\frac{1}{2} i D.
\]

Changing Hilbert space \([2]\), one can suppose \( \sigma(H) = \pi_0(H) \) (the point spectrum of \( H \)).

Assuming \( \alpha \in \sigma(H) \), it is to be shown that \( \alpha \in \sigma(\text{Re } T) \). Let \( \mathcal{M} = N(H - \alpha I) \), the null space of \( H - \alpha I \); since \( \alpha \) is an eigenvalue, \( \mathcal{M} \neq \{0\} \). We show that \( \mathcal{M} \) is invariant under \( J \). Let \( x \in \mathcal{M} \), i.e., \( (H - \alpha I) x = 0 \). By (1),

\[
(H - \alpha I) J - J (H - \alpha I) = -\frac{1}{2} i D,
\]

therefore

\[
-\frac{1}{2} i (D x | x) = (J x | (H - \alpha I) x) - (J (H - \alpha I) x | x) = 0,
\]

thus \( 0 = (D x | x) = \| D^{1/2} x \|^2 \), \( D^{1/2} x = 0 \), \( D x = 0 \). Then (1) yields

\[
0 = H(J x) - J(H x) = H(J x) - \alpha(J x),
\]

thus \( J x \in \mathcal{M} \).

Let \( J_1 = J | \mathcal{M} \). Of course \( \mathcal{M} \) is also invariant under \( H \), with \( H | \mathcal{M} = \alpha I \). Therefore \( \mathcal{M} \) is invariant under \( T = H + iJ \), and \( T | \mathcal{M} = \alpha I + i J_1 \); since \( J_1 \) is selfadjoint, clearly \( T | \mathcal{M} \) is normal; since \( T \) is hyponormal, it follows that \( \mathcal{M} \) reduces \( T \) \([1, p. 160, Example 9]\). Writing \( T_1 = T | \mathcal{M} \) and \( T_2 = T | \mathcal{M}^\perp \), we have \( T = T_1 \oplus T_2 \), \( \sigma(T) = \sigma(T_1) \cup \sigma(T_2) \), therefore

\[
\text{Re } \sigma(T) = \text{Re } \sigma(T_1) \cup \text{Re } \sigma(T_2).
\]

Since \( T_1 \) is normal, by Lemma 3 we have

\[
\text{Re } \sigma(T_1) = \sigma(\text{Re } T_1) = \sigma(\alpha I) = \{ \alpha \},
\]

therefore \( \alpha \in \text{Re } \sigma(T) \) by (2).
Remark. The proof of Theorem 1 (and of Lemma 8 below) uses the fact that
the representation \( T \rightarrow T^* \) of [2] preserves spectrum; this fact is not proved explicitly
in [2], but it follows at once from the observation that \( T \) is invertible iff \( T^*T \) and
\( TT^* \) are invertible iff \( T^*T \geq cI \) and \( TT^* \geq cI \) for some \( c > 0 \). (The underlying general
principle is that if \( b \in B \subset \mathcal{A} \), where \( \mathcal{A} \) is a \( \mathcal{C}^* \)-algebra with unity and \( B \) is a closed
\( \mathcal{C}^* \)-subalgebra containing the unity, then \( \sigma_a(b) = \sigma_a(b) \) [6, p. 8].)

Lemma 4. Let \( T \) be any convexoid operator. If \([\alpha_0, \beta_0]\) is the smallest interval
containing \( \text{Re} \sigma(T) \), then \( \alpha_0, \beta_0 \in \sigma(\text{Re} T) \).

Proof. Changing Hilbert space, one can suppose that \( \text{W}(T) \) is closed and that
\( \pi(T) = \pi_T(T) \) [5]. Choose \( \lambda_0 \in \sigma(T) \) with \( \text{Re} \lambda_0 = \alpha_0 \). Clearly \( \lambda_0 \) is a boundary point
of \( \sigma(T) \) (by the extremality of \( \alpha_0 \)), therefore \( \lambda_0 \in \partial \sigma(T) \subseteq \pi(T) = \pi_T(T) \), hence also
\( \lambda_0 \in \text{W}(T) \). On the other hand, since \( T \) is convexoid, \( \text{W}(T) \) is closed and \( \text{Re} \sigma(T) \geq \alpha_0 \),
we have
\[
\text{Re} \text{W}(T) = \text{Re} \text{conv} \sigma(T) \geq \alpha_0;
\]
thus \( \lambda_0 \) is clearly a boundary point of \( \text{W}(T) \), therefore it is a normal eigenvalue of
\( T \), i.e., \( N(T - \lambda_0 I) = N(T^* - \lambda_0^* I) \) [8, Satz 2]. Choose \( u \neq 0 \) with \( Tu = \lambda_0 u \); then also
\( T^*u = \lambda_0^* u \), therefore \( (\text{Re} T)u = \alpha_0 u \), thus \( \alpha_0 \in \sigma(\text{Re} T) \). Similarly \( \beta_0 \in \sigma(\text{Re} T) \).

Lemma 5. Suppose \( T \) is convexoid and \( \sigma(T) \) is connected. If \([\alpha, \beta]\) is the smallest
interval containing \( \sigma(\text{Re} T) \), then \( \sigma(\text{Re} T) \subseteq [\alpha, \beta] \subseteq \sigma(T) \).

Proof. Since \( \sigma(T) \) is connected, \( \text{Re} \sigma(T) \) is an interval, thus it will suffice to show
that \( \alpha, \beta \in \sigma(T) \). Assume to the contrary, e.g., that \( \alpha \notin \text{Re} \sigma(T) \). Thus, if \( L \) is the
vertical line \( \text{Re} \lambda = \alpha \), then \( L \) is disjoint from \( \sigma(T) \). Since \( \sigma(T) \) is connected, it must
lie strictly to one side of \( L \). Suppose, e.g., that it lies to the right. Then there exists
\( \epsilon > 0 \) such that \( \sigma(T) \supseteq \alpha + \epsilon \). Since \( T \) is convexoid, it follows that \( \text{Re} \text{W}(T) \supseteq \alpha + \epsilon \)
thus \( \text{W}(T) \supseteq \alpha + \epsilon \), i.e., \( T^* (\alpha + \epsilon) I \), hence \( \sigma(\text{Re} T) \supseteq \alpha + \epsilon \); in particular,
\( \alpha \geq \alpha + \epsilon \), contrary to \( \epsilon > 0 \). The proof with “left” in place of “right” is similar, with
\( \alpha + \epsilon \) replaced by \( \alpha - \epsilon \) and the inequalities reversed.

Lemma 6. If \( T \) is a convexoid operator such that both \( \sigma(T) \) and \( \sigma(\text{Re} T) \) are con-
nected, then \( \ast \) holds.

Proof. Say \( \sigma(\text{Re} T) = [\alpha, \beta] \) and \( \sigma(T) = [\alpha, \beta] \). By Lemma 5, \([\alpha, \beta] \subseteq [\alpha_0, \beta_0] \); by
Lemma 4, \( \alpha_0, \beta_0 \in [\alpha, \beta] \) and so \([\alpha_0, \beta_0] \subseteq [\alpha, \beta] \).

Proof of Theorem 2. If \( T \) is a Toeplitz operator, then \( T \) is convexoid [7, Problem
196, Corollary 4] and \( \sigma(T) \) is connected by a theorem of H. Widom [17]; since \( \text{Re} T \)
is also a Toeplitz operator, \( \sigma(\text{Re} T) \) is connected too (cf. [7, Problem 199]), thus
Lemma 6 is applicable.

Lemma 7. If \( T \) is a spectral set for \( T \), then \( \sigma(\text{Re} T) \subset \text{Re} \sigma(T) \).

Proof. By hypothesis, \( f(T) \) is normaloid for all rational functions \( f \) without poles
in \( \sigma(T) \) [3]. In particular, \( T - \lambda M \) is normaloid for all complex \( \lambda \), therefore \( T \) is
convexoid [14], [8, Satz 9, (i)]. {Better yet, \((T - \lambda I)^{-1}\) is normaloid for all \(\lambda \notin \sigma(T)\), i.e., \(T\) satisfies condition \((G_1)\).}

Let \(\alpha \in \sigma(\text{Re } T)\) and assume to the contrary that \(\alpha \notin \text{Re } \sigma(T)\). Then, if \(L\) is the vertical line \(\text{Re } \lambda = \alpha\), \(L\) is disjoint from \(\sigma(T)\).

Suppose first that \(\sigma(T)\) lies to one side—say the left—of \(L\). There exists \(\varepsilon > 0\) such that \(\sigma_\varepsilon(\sigma(T)) \subseteq \alpha - \varepsilon\); since \(T\) is convexoid, the argument in Lemma 5 yields the contradiction \(\alpha \leq \alpha - \varepsilon\). Similarly if \(\sigma(T)\) lies to the right of \(L\).

Suppose finally that \(\sigma(T)\) is disconnected by \(L\). Write \(\sigma(T)=X_1 \cup X_2\), where \(X_1\) lies to the left, and \(X_2\) to the right, of \(L\). Since \(\sigma(T)\) is a spectral set for \(T\), by a theorem of J. P. Williams [18, Theorem 4] there is an orthogonal decomposition \(T = T_1 \oplus T_2\) with \(\sigma(T_k) = X_k\) a spectral set for \(T_k\). Then \(\text{Re } T = \text{Re } T_1 \oplus \text{Re } T_2\), thus \(\alpha \in \sigma(\text{Re } T) = \sigma(\text{Re } T_1) \cup \sigma(\text{Re } T_2)\). Say \(\alpha \in \sigma(\text{Re } T_k)\); then the application of the preceding paragraph to \(T_k\) yields a contradiction.

The following lemma is a special case of a theorem of Putnam [12, Theorem 4], with a considerably simpler proof:

**Lemma 8.** If \(T\) satisfies condition \((G_1)\), then \(\text{Re } \sigma(T) \subseteq \sigma(\text{Re } T)\).

**Proof.** We can suppose \(\sigma(T) = \pi(\tau(T)) [2]\), and therefore \(\partial \sigma(T) \subseteq \pi(T) = \pi(\tau(T))\).

Suppose \(\alpha_0 \in \sigma(\text{Re } T)\). Thus, if \(L\) is the vertical line \(\text{Re } \lambda = \alpha_0\), the assumption is that \(L\) intersects \(\sigma(T)\). Let \(\lambda_0\) be a point where \(L\) exits the spectrum; then \(\lambda_0 \in \partial \sigma(T)\) and \(\text{Re } \lambda_0 = \alpha_0\).

It will suffice to construct a sequence of unit vectors \(x_n\) such that \((T - \lambda_0 I)x_n \to 0\) and \((T^* - \lambda_0^* I)x_n \to 0\), for this will imply \(\lambda_0 \in \sigma(\text{Re } T)\) as in the proof of Lemma 1. To this end, we construct a sequence \(\lambda_n\) of normal eigenvalues of \(T\) such that \(\lambda_n \to \lambda_0\), as follows.

For \(n = 1, 2, 3, \ldots\), let \(D_n = \{\lambda : |\lambda - \lambda_0| \leq 1/n\}\). Since \(\lambda_0 \in \partial \sigma(T)\), \(D_n\) contains a point \(\mu_n\) of the resolvent set of \(T\) such that \(|\mu_n - \lambda_0| < 1/2n\). Clearly \(\text{dist } \mu_n(T)\) is assumed in \(D_n\); say \(\lambda_n \in \sigma(T)\) with dist \((\mu_n, \sigma(T)) = |\mu_n - \lambda_n|\). Thus \(\lambda_n \in \sigma(T)\) lies on the circumference of a closed disc (centered at \(\mu_n\)) whose interior contains no point of \(\sigma(T)\); since \(T\) satisfies \((G_1)\), it follows that \(N(T - \lambda_n I) = N(T^* - \lambda_n^* I)\) [4, Lemma 2]. {The definition of “semibare point” in [4] is unnecessarily restrictive; there is no harm in other points of \(\sigma(T)\) lying on the circumference of the disc (in any case, a smaller disc will shake them off).} Obviously \(\lambda_n \in \partial \sigma(T)\), so \(\lambda_n\) is an eigenvalue; thus \(\lambda_n\) is a normal eigenvalue. Let \(x_n\) be a unit vector with \((T - \lambda_0 I)x_n = 0\), \((T^* - \lambda_0^* I)x_n = 0\). Then

\[
(T - \lambda_0 I)x_n = (T - \lambda_n I)x_n + (\lambda_n - \lambda_0)x_n = (\lambda_n - \lambda_0)x_n,
\]

therefore \(\| (T - \lambda_0 I)x_n \| = |\lambda_n - \lambda_0| \leq 1/n\); thus \((T - \lambda_0 I)x_n \to 0\), and similarly \((T^* - \lambda_0^* I)x_n \to 0\).

**Proof of Theorem 3.** If \(\sigma(T)\) is a spectral set for \(T\), then \(\sigma(\text{Re } T) \subseteq \sigma(\text{Re } T)\) by Lemma 7; as observed in the proof, \(T\) satisfies \((G_1)\), therefore \(\sigma(\text{Re } T) \subseteq \sigma(\text{Re } T)\) by Lemma 8.
Proof of Theorem 4. Assume $T$ satisfies $(G_1)$ and $\sigma(T)$ is connected. By Lemma 8, $\text{Re } \sigma(T) \subseteq \sigma(\text{Re } T)$. Let $[\alpha, \beta]$ be the smallest interval containing $\sigma(\text{Re } T)$; since $T$ is convexoid [9], [15], it follows from Lemma 5 that $[\alpha, \beta] \subseteq \text{Re } \sigma(T)$. Thus

$\sigma(\text{Re } T) \subseteq [\alpha, \beta] \subseteq \text{Re } \sigma(T) \subseteq \sigma(\text{Re } T)$.  

If $T$ satisfies the hypotheses of one of the above theorems, then so does $\lambda T + \mu I$ for all complex numbers $\lambda$ and $\mu$. We have remarked that the operators in Theorems 1–4 are all convexoid. In a sense, this is not an accident: If $T$ is an operator such that $\lambda T + \mu I$ satisfies $(\ast)$ for all complex $\lambda$, $\mu$ (equivalently, $\lambda T$ satisfies $(\ast)$ for all $|\lambda| = 1$), then $T$ is convexoid; the details of the proof are similar to those in Lemmas 5 and 7.

On the other hand, a convexoid operator need not satisfy $(\ast)$. For example, consider the $5 \times 5$ matrix $T = A \oplus B$, where

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is a $3 \times 3$ diagonal matrix whose eigenvalues lie off the imaginary axis and are the vertices of a triangle that contains the disc $D_{1/2} = \{ \lambda : |\lambda| \leq 1/2 \}$. Since $W(A) = D_{1/2}$ (cf. [7, Problem 166]) and $W(B) = \text{conv} \{ \lambda_1, \lambda_2, \lambda_3 \} \supseteq D_{1/2}$, it is easy to see that $T$ is convexoid (cf. [7, p. 113]). One has $\sigma(T) = \sigma(A) \cup \sigma(B) = \{0\} \cup \{\lambda_1, \lambda_2, \lambda_3\}$, and in particular $0 \in \text{Re } \sigma(T)$. On the other hand, $\text{Re } T = \text{Re } A \oplus \text{Re } B$, therefore $\sigma(\text{Re } T) = \sigma(\text{Re } A) \cup \sigma(\text{Re } B) = \{-1, 1\} \cup \text{Re } \sigma(B)$ and so $0 \notin \sigma(\text{Re } T)$. The dimension of the example is optimal, since a convexoid operator on a space of dimension less than 5 is necessarily normal [8, Satz 9, (ii)].

REFERENCES


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