SEMIPRIMARY HEREDITARY ALGEBRAS

BY

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Abstract. Let \( \Sigma \) be a semiprimary \( k \)-algebra, with radical \( M \). If \( \Sigma \) admits a splitting then \( \dim_k \Sigma/M \leq \dim_k \Sigma \). The residue algebra \( \Sigma/M^2 \) is finite (cohomological) dimensional if and only if all residue algebras are finite dimensional. If \( \dim_k \Sigma=1 \) then all residue algebras are finite dimensional.

1. Introduction. We consider the following properties of algebras over a field \( k \):

(p1): \( \dim_k \Sigma \leq 1 \).

(p2): \( \dim_k \Sigma/I \) is finite for every two sided ideal \( I \) in \( \Sigma \).

(p3): \( \dim_k \Sigma/M^2 \) is finite, where \( M \) is the (Jacobson) radical of \( \Sigma \).

(p4): \( \Sigma \) is a residue algebra of \( \Omega \), where \( \text{gl.dim} \Omega \leq 1 \), \( \dim_k \Sigma/M \) is finite and \( \Sigma/M \) is isomorphic to \( \Omega/N \), \( N \) being the (Jacobson) radical of \( \Omega \).

(p5): \( \Sigma \) is a residue algebra of \( \Omega \), where \( \dim_k \Omega \leq 1 \) and \( \Sigma/M^2 \) is isomorphic to \( \Omega/N^2 \).

For a finite dimensional \( k \)-algebra \( \Sigma \), it was proved by Eilenberg, Nagao and Nakayama in [6] that (p1) \( \Rightarrow \) (p2), while Jans and Nakayama proved in [7] the implications (p3) \( \Rightarrow \) (p4) \( \Rightarrow \) (p5). Thus for finite dimensional \( k \)-algebras one has the equivalences (p2) \( \Leftrightarrow \) (p3) \( \Leftrightarrow \) (p4) \( \Leftrightarrow \) (p5).

The purpose of this paper is to establish the implication (p1) \( \Rightarrow \) (p2), and the equivalences (p2) \( \Leftrightarrow \) (p3) \( \Leftrightarrow \) (p4) for semiprimary rings that are \( k \)-algebras. The equivalence (p4) \( \Leftrightarrow \) (p5) can be deduced in certain particular cases as for instance if \( \Sigma/M \) is a finite dimensional \( k \)-algebra. To this extent we give an example of a semiprimary ring \( \Sigma \) for which \( \dim_k \Sigma = 1 \) and \( \dim_k \Sigma/M = 1 \).

As it turns out the passage from finite dimensional \( k \)-algebras to semiprimary ones is made possible by a lemma that seems to be of some interest in its own sake, namely:

A semiprimary \( k \)-algebra \( \Sigma \) that admits a splitting \( \Sigma = \Delta + M \) [9] satisfies the inequality \( \dim_k \Delta \leq \dim_k \Sigma \), where \( \Delta \) denotes the residue algebra \( \Sigma/M \).

In [2] Auslander proved that if \( \Delta \) is a finite dimensional \( k \)-algebra and \( \dim_k \Sigma \) is finite, then \( \dim_k \Sigma = \text{gl.dim} \Sigma \). He raised the problem whether it is necessary that \( \dim_k \Delta = 0 \) (e.g. [4] and [5]). We prove the answer to be affirmative in case that \( \dim_k \Sigma/M^2 \) is finite.
2. Hereditary algebras. A $k$-algebra $\Sigma$ is said to be a semiprimary $k$-algebra if $\Sigma$ is a semiprimary ring, i.e. its (Jacobson) radical $M$ is nilpotent and the residue ring $\Sigma/M$ is a semisimple (Artinian) ring. Set $\Delta=\Sigma/M$. We say that a $k$-algebra $\Sigma$ is an hereditary $k$-algebra if $\dim_k \Sigma \leq 1$. By $\Sigma^o$ we denote the apposite ring to $\Sigma$. By $(\Sigma:k)<\infty$ we denote that the $k$-algebra $\Sigma$ is finite dimensional (as a $k$-vector space). For the rest we write $\dim$ for $\dim_k$, and $\otimes$ for $\otimes_k$. We say that $\Sigma$ admits a splitting if $\Sigma=\Delta+M$ [7], [9]. A crucial step towards our main theorem is the following lemma:

**Lemma 1.** If $\Sigma$ admits a splitting, $\Sigma=\Delta+M$, then $\dim \Delta \leq \dim \Sigma$.

**Proof.** If $\dim \Sigma=\infty$ we are done. Otherwise $\dim \Sigma$ is finite, and we may assume that $\dim \Sigma=t<\infty$. By [4] we have the equality $\text{gl.dim } \Sigma \otimes \Delta^o=\dim \Sigma=t<\infty$.

Since $\Delta$ is a semisimple ring, $M$ is a projective right $\Delta$-module. From the natural isomorphism of $M \otimes \Delta^o$ with $M \otimes_{A^o}(\Delta \otimes \Delta^o)$ it follows that $M \otimes \Delta^o$ is a projective right $\Delta \otimes \Delta^o$-module. Hence via the natural embedding of $\Delta \otimes \Delta^o$ into $\Sigma \otimes \Delta^o$, $\Sigma \otimes \Delta^o$ becomes a projective right $\Delta \otimes \Delta^o$-module. Denote by $f$ the natural embedding of $\Delta \otimes \Delta^o$ into $\Sigma \otimes \Delta^o$, and denote by $g$ the canonical epimorphism of $\Sigma \otimes \Delta^o$ onto $\Delta \otimes \Delta^o$, then $g \circ f$ is the identity map on $\Delta \otimes \Delta^o$.

For any left $\Delta \otimes \Delta^o$-module $A$ we set $A_f=(\Sigma \otimes \Delta^o) \otimes_{(\Delta \otimes \Delta^o)} A$.

For any left $\Sigma \otimes \Delta^o$-module $B$ we set $B_f=(\Delta \otimes \Delta^o) \otimes_{(\Delta \otimes \Delta^o)} B$.

There results a $\Delta \otimes \Delta^o$ isomorphism from $A$ onto $(A_f)_g$, for every left $\Delta \otimes \Delta^o$-module $A$.

Let $A$ be a left $\Delta \otimes \Delta^o$-module, and let

$$0 \rightarrow L \rightarrow P_{t-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

be an exact sequence of left $\Delta \otimes \Delta^o$-modules, where $P_0, \ldots, P_{t-1}$ are projective $\Delta \otimes \Delta^o$-modules. We claim that either $L$ is a projective $\Delta \otimes \Delta^o$-module, or else $L=0$.

Since $\Sigma \otimes \Delta^o$ is a projective right $\Delta \otimes \Delta^o$-module, there results an exact sequence of left $\Sigma \otimes \Delta^o$-modules:

$$0 \rightarrow L_f \rightarrow (P_{t-1})_f \rightarrow \cdots \rightarrow (P_0)_f \rightarrow (A_f)_g \rightarrow 0$$

where $(P_i)_f=(\Sigma \otimes \Delta^o) \otimes_{(\Delta \otimes \Delta^o)} P_i$, for $i=0, \ldots, (t-1)$. Thus $(P_0)_f, \ldots, (P_{t-1})_f$, are $\Sigma \otimes \Delta^o$-projective. Since $l.gl.dim \Sigma \otimes \Delta^o=\dim \Sigma=t$, it follows that either $L_f$ is a projective $\Sigma \otimes \Delta^o$-module, or else $L_f=0$. Hence $L$ is $\Delta \otimes \Delta^o$ isomorphic to the $\Delta \otimes \Delta^o$-module $(L_f)_g=(\Delta \otimes \Delta^o) \otimes_{(\Delta \otimes \Delta^o)} L_f$. Therefore $L=0$ or else $L$ is a projective $\Delta \otimes \Delta^o$-module.

Therefore for every left $\Delta \otimes \Delta^o$-module $A$ we have $l.p.dim_{\Delta \otimes \Delta^o} A \leq t$, thus $l.gl.dim \Delta \otimes \Delta^o \leq t$. Since by [4] we have the equality $\dim \Delta=l.gl.dim \Delta \otimes \Delta^o$ we may conclude that the inequality $\dim \Delta \leq \dim \Sigma$ holds.

Recall that all residue rings of a semiprimary ring $\Sigma$ are of finite global dimension iff $\Sigma$ is a residue ring of a semiprimary ring $\Omega$ for which $\text{gl.dim } \Omega \leq 1$, and this
is the case iff $\text{gl.dim } \Sigma/M^2$ is finite [9]. Under each of these equivalent conditions $\Sigma$ admits a splitting $\Sigma = \Delta + M$.

The splitting of $\Sigma$ is inherited by every residue ring $\Sigma_1$ of $\Sigma$, $\Sigma_1 = \Delta_1 + M_1$. Furthermore, $\Delta_1$ is (up to an isomorphism) a direct factor of $\Delta$.

We are now ready to state and prove our main theorem that establishes the equivalences (p2)$ \Leftrightarrow$ (p3)$ \Leftrightarrow$ (p4) for semiprimary $k$-algebras.

**Theorem 1.** The following are equivalent:
(a) $\dim \Delta < \infty$ and $\text{gl.dim } \Sigma/M^2 < \infty$.
(b) $\dim \Sigma/I < \infty$ for every two sided ideal $I$ in $\Sigma$.
(c) $\dim \Sigma/M^2 < \infty$.

**Proof.** (a) $\Rightarrow$ (b): From $\text{gl.dim } \Sigma/M^2 < \infty$ it follows by [9] that $\text{gl.dim } \Sigma/I < \infty$ for every two sided ideal $I$ in $\Sigma$. Set $\Delta_1 = (\Sigma/I)/(I + M/I)$ then $\Delta_1$ is a direct factor of $\Delta$, hence $\dim \Delta_1 \leq \dim \Delta$. Combining the equality $\dim \Sigma/I = 1 \cdot \text{gl.dim } \Sigma/I \otimes \Delta^2_1$ [4] with the inequality $1 \cdot \text{gl.dim } \Sigma/I \otimes \Delta^2_1 \leq 1 \cdot \text{gl.dim } \Sigma/I + \dim \Delta^2_1$ [5] it results that $\dim \Sigma/I$ is finite.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a): Since $\text{gl.dim } \Sigma/M^2 \leq \dim \Sigma/M^2$ [5], then $\text{gl.dim } \Sigma/M^2 < \infty$. Hence by [9] $\Sigma/M^2$ admits a splitting, and thus Lemma 1 implies the inequality $\dim \Delta \leq \dim \Sigma/M^2$, therefore $\dim \Delta$ is finite.

Observe that under each of the equivalent conditions in Theorem 1, $\Sigma$ is a residue of a semiprimary $k$-algebra $\Omega$ with radical $N$, such that $\Omega/N$ is isomorphic with $\Delta$, and $\text{gl.dim } \Omega \leq 1$. This is an immediate consequence of Theorem 1 applied to $\Omega(\Delta, N)$ [9]. It is worth noticing that $\Sigma$ admits a splitting, $\Sigma = \Delta + M$.

As for $\dim \Omega$, from $\dim \Omega = 1 \cdot \text{gl.dim } \Omega \otimes \Delta^\infty$ it follows that $\dim \Delta \leq \dim \Omega \leq \dim \Delta + 1$.

In the next section we will bring some examples showing that it is possible that $\dim \Omega = \dim \Delta + 1$, but it is also possible that the equality $\dim \Omega = \dim \Delta$ will hold.

Consider the case where $k$ is the center of $\Sigma$. One can easily construct examples in which $\Sigma$ is a residue ring of a semiprimary hereditary ring $\Omega$ with radical $N$, such that $\Omega/N$ is isomorphic with $\Delta$, but $\Omega$ is not a $k$-algebra, i.e., not every semiprimary hereditary ring—of which $\Sigma$ is a residue ring—is a $k$-algebra [10, Example 1].

Notice that if $\Delta$ is a finite dimensional $k$-algebra then $\dim \Delta = 0$. One verifies that if $M \neq 0$ then $\dim \Omega = 1$. Furthermore, if $\Omega$ is any semiprimary hereditary ring with radical $N$ of which $\Sigma$ is a residue ring, such that $\Omega/N$ is isomorphic with $\Delta$, then $\Omega$ admits a splitting, $\Omega = \Delta + A + N^2$. Therefore, if one insists on $\Omega/N^2$ being isomorphic to $\Sigma/M^2$ it follows that up to an isomorphism $\Omega$ is uniquely determined. This establishes the equivalence (p4)$ \Leftrightarrow$ (p5) in case $\Delta$ is a finite dimensional $k$-algebra. Also in this case we have $\dim \Omega/I = \text{gl.dim } \Omega/I$ for every two sided ideal $I$ in $\Omega$. In particular from [10] it results that $\dim \Omega/I \leq \dim \Omega/N^2$, whenever $I \subseteq N^2$. 

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We do not know if this last inequality holds without the assumption \( \dim \Delta = 0 \).

Our next aim is to prove the implication \((p1) \Rightarrow (p2)\) for semiprimary \(k\)-algebras. Recall that the validity of this implication for finite dimensional \(k\)-algebras is based on the equality \( \dim \Sigma = \mathrm{gl.dim} \Sigma \), which is a consequence of \( \dim \Delta = 0 \) under these circumstances (e.g. [6]). For semiprimary \(k\)-algebras we have by [5] the inequality \( \mathrm{gl.dim} \Sigma \leq \dim \Sigma \). Furthermore, if \( \mathrm{gl.dim} \Sigma/M^2 \) is finite then by [9] \( \Sigma \) admits a splitting \( \Sigma = \Delta + M \). We proceed with a sequence of corollaries to get the desired implication.

**Corollary 1.** If \( \Sigma \) admits a splitting, \( \Sigma = \Delta + M \), then \( \dim \Delta \leq \dim \Sigma/I \) for every two sided ideal \( I \) in \( \Sigma \) that is contained in the radical.

**Proof.** Since the splitting is inherited by all residue rings of \( \Sigma \), and since \( I \subseteq M \) implies that \( (\Sigma/I)/(I+M/I) \) is isomorphic with \( \Delta \), then we have applying Lemma 1: \( \dim \Delta \leq \dim \Sigma/I \).

**Corollary 2.** If \( M^2 = 0 \) then \( \dim \Delta \leq \dim \Sigma \).

**Proof.** If \( \dim \Sigma = \infty \) we are done. Otherwise \( \dim \Sigma \) is finite, hence \( \mathrm{gl.dim} \Sigma \) is finite. Therefore \( \Sigma \) admits a splitting, \( \Sigma = \Delta + M \), and the result follows from Lemma 1.

**Corollary 3.** If \( \dim \Sigma = 1 \) then \( \dim \Delta \leq 1 \).

**Proof.** The proof is an immediate consequence of Lemma 1 which is applicable in this case, since \( \mathrm{gl.dim} \Sigma \leq \dim \Sigma = 1 \) implies the splitting of \( \Sigma \).

It seems interesting to notice that one can prove that \( \dim \Delta \) is finite by observing that \( \Delta \otimes \Delta^* \) is a residue ring of the hereditary ring \( \Sigma \otimes \Delta^* \) by the nilpotent two sided ideal \( M \otimes \Delta^* \) (e.g. [6]).

As a consequence there results the implication \((p1) \Rightarrow (p2)\).

**Corollary 4.** If \( \dim \Sigma = 1 \) then \( \dim \Sigma/I \) is finite for every two sided ideal \( I \) in \( \Sigma \).

**Proof.** This is an immediate consequence of Theorem 1 since by Corollary 3 \( \dim \Delta \leq 1 \), and since \( \mathrm{gl.dim} \Sigma \leq \dim \Sigma \) implies that \( \mathrm{gl.dim} \Sigma/M^2 \) is finite.

3. **Examples.** In this section we will bring some examples of \(k\)-algebras, all the residue algebras of which have finite cohomological dimension. We will be mainly concerned with the inequalities \( \dim \Delta \leq \dim \Sigma \leq \dim \Delta + \mathrm{gl.dim} \Sigma \), and with the equality \( \dim \Sigma = \mathrm{gl.dim} \Sigma \) without \((\Delta:k)\) being finite.

Let \( k(x_1, \ldots, x_n, y_1, \ldots, y_m) \) be the field of rational functions in \( n+m \) variables over the field \( k \). We will identify \( k(k(x_1, \ldots, x_n), k(y_1, \ldots, y_m)) \) with its natural embedding in \( k(x_1, \ldots, x_n, y_1, \ldots, y_m) \).

**Example 1.** Let \( \Sigma \) be the \( k \)-subalgebra of the \( 2 \times 2 \) matrix algebra over the field of rational functions in one variable over a field \( k \), \( k(x) \). A matrix \( \sigma \) belongs to \( \Sigma \) iff \( \sigma \) is of the form

\[
\begin{pmatrix}
a & 0 \\
b & c
\end{pmatrix},
\]

where \( a \) is an element of \( k \), and \( b, c \) are elements of \( k(x) \).
Obviously $\Sigma$ is a left Artinian hereditary ring with radical $M$ of square zero, and $\dim \Delta = 1$. We claim that $\dim \Sigma = 1$. It will suffice to show that $\text{l.gl.dim } \Sigma \otimes k(x) = 1$. Identify $\Sigma \otimes k(x)$ with a subring of the $2 \times 2$ matrix algebra over $k(x) \otimes k(x)$, namely: $\sigma'$ belongs to $\Sigma \otimes k(x)$ iff $\sigma'$ is of the form:

$$\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}$$

where $\alpha$ belongs to $k \otimes k(x)$ (which is isomorphic to $k(x)$), and $\beta$, $\gamma$ belong to $k(x) \otimes k(x)$.

Let $J$ be a left ideal in $\Sigma \otimes k(x)$, then one readily verifies that $J$ is of one of the following two types:

- **Type 1.** Every element in $J$ is of the form

$$\begin{pmatrix} 0 & 0 \\ \beta & \gamma \end{pmatrix}$$

- **Type 2.** $J$ is a direct sum of two subideals $J_1$ and $J_2$ where every element of $J_1$ is of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}$$

and a matrix $\sigma'$ belongs to $J_2$ iff it is of the form

$$\begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix}$$

Since $\dim k(x) = 1$ it follows from [3, Theorem 5.4, p. 14] that in either case $J$ is a projective left $\Sigma \otimes k(x)$-module. Hence by [3, Theorem 5.4, p. 14] it follows that $\text{l.gl.dim } \Sigma \otimes k(x) = 1$.

A similar treatment, using the fact that $k(x_1, \ldots, x_n) \otimes k(x_1, \ldots, x_n)$ is a Noetherian ring—where $k(x_1, \ldots, x_n)$ is the field of rational functions in $n$ variables over $k$—gives:

**Example 1*.** Let $\Sigma$ be the $k$-subalgebra of the $2 \times 2$ matrix algebra over $k(x_1, \ldots, x_n)$, $\Sigma$ is a left Artinian ring with radical $M$ of square zero, $\dim \Delta = n$ [5], and $\text{gl.dim } \Sigma = 1$. Finally, by the remark made above, we have by checking $\dim \Sigma$ via $\text{l.gl.dim } \Sigma \otimes k(x_1, \ldots, x_n)$ that $\dim \Sigma = n$.

**Example 2.** By taking successive rings of triangular matrices of the ring $\Sigma$ that was constructed in Example 1 (1*) we obtain a left Artinian ring $\Lambda = T_{n_1}(\cdots T_{n_t}(\Sigma) \cdots)$. By [5] it follows that

$$\dim \Lambda = \text{gl.dim } \Lambda = t+1 \quad (\dim \Lambda = t+n, \text{gl.dim } \Lambda = t+1).$$

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Furthermore, if $N$ is the radical of $\Lambda$ then $\dim \Lambda/N = \dim \Delta$, since $\Lambda/N$ is isomorphic to a direct product of $n_1 \cdots n_t$ copies of $\Delta$.

Summarizing we have:

**Proposition 1.** For every pair of positive integers $n, s$ there exists a $k$-algebra $\Sigma$ for which $\text{gl.dim } \Sigma = s$, $\dim \Delta = n$, and $\dim \Delta < \text{gl.dim } \Sigma + \dim \Delta$.

Taking $n = 1$ there will result a $k$-algebra $\Sigma$ for which $\dim \Sigma = \text{gl.dim } \Sigma < \infty$, such that $\dim \Delta = 1$.

**Example 3.** Let $\Sigma$ be the $k$-subalgebra of the $2 \times 2$ matrix algebra over $k(x, y)$—the field of rational functions in two variables over the field $k$. A matrix $\sigma$ belongs to $\Sigma$ iff $\sigma$ is of the form

\[
\begin{pmatrix}
a & 0 \\
b & c
\end{pmatrix},
\]

where $a$ belongs to $k(y)$, $b$ belongs to $k(x, y)$, and $c$ belongs to $k(x)$. $\Sigma$ is a semiprimary ring with radical of square zero. Obviously $\dim \Delta = 1$, $\text{gl.dim } \Sigma = 1$, and it is an easy exercise to check that $\dim \Sigma = 2$.

**Example 3*. Take $\Sigma$ to be the $k$-subalgebra of the $2 \times 2$ matrix algebra over $k(x_1, \ldots, x_n, y_1, \ldots, y_n)$—the field of rational functions in $2n$ variables over the field $k$. A matrix $\sigma$ belongs to $\Sigma$ iff $\sigma$ is of the form

\[
\begin{pmatrix}
a & 0 \\
b & c
\end{pmatrix},
\]

where $a$ belongs to $k(y_1, \ldots, y_n)$, $b$ belongs to $k(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and $c$ belongs to $k(x_1, \ldots, x_n)$.

$\Sigma$ is a semiprimary ring, and $\text{gl.dim } \Sigma = 1$. Again by straightforward computations it follows that $\dim \Sigma = n + 1$, and from [5] $\dim \Delta = n$.

**Example 4.** By taking successive rings of triangular matrices of the ring $\Sigma$ that was constructed in Example 3 (3*) we obtain a semiprimary ring $\Lambda = T_{n_1} \cdots T_{n_t}(\Sigma) \cdots$ with radical $N$. From [5] it follows that $\text{gl.dim } \Lambda = t + 1$, and $\dim \Lambda = t + 2$ (dim $\Lambda = t + n + 1$). Furthermore, since $\Lambda/N$ is the direct product of $n_1 \cdots n_t$ copies of $\Delta$, $\dim \Lambda/N = 1$ (dim $\Lambda/N = n$).

Summarizing we obtain:

**Proposition 2.** For every pair of positive integers $n, s$ there exists a $k$-algebra $\Sigma$ for which $\text{gl.dim } \Sigma = s$, $\dim \Delta = n$, and $\dim \Sigma = \text{gl.dim } \Sigma + \dim \Delta$.

Notice that in all our examples, $k$ is the center of each of the constructed rings.

4. Applications. In [2] Auslander proved that if $\dim \Sigma < \infty$ and if $(\Delta : k) < \infty$ then $\dim \Sigma = \text{gl.dim } \Sigma$. He raised the problem whether $\dim \Sigma$ is necessarily zero. In [4] Eilenberg proved that if $\dim \Sigma < \infty$ and $(\Sigma : k) < \infty$ then $\dim \Delta = 0$ and $\dim \Sigma = \text{gl.dim } \Sigma$. In §3 we saw that it is possible to have $\dim \Sigma < \infty$ and $\dim \Sigma = \text{gl.dim } \Sigma$ without $(\Sigma : k)$ nor $(\Delta : k)$ being finite. Furthermore, $\dim \Sigma = \text{gl.dim } \Sigma$ may hold without $\dim \Delta$ being zero. Still we have:
Proposition 3. If $M^2 = 0$, if $(\Delta : k) < \infty$, and if $\dim \Sigma$ is finite then $\dim \Delta = 0$.

Proof. By Corollary 2 $\dim \Delta \leq \dim \Sigma < \infty$. Since $(\Delta : k) < \infty$ we now have $\dim \Delta = 0$.

In this respect it is worth stating an immediate consequence of Theorem 1, that turns out to be just an affirmative answer to the problem raised by Auslander in a particular case.

Corollary 5. If $(\Delta : k) < \infty$ then the following are equivalent:

(a) $\dim \Delta = 0$ and $\text{gl.dim } \Sigma/M^2 < \infty$,

(b) $\dim \Sigma/M^2 < \infty$.

Under each of these equivalent conditions $\dim \Sigma/I = \text{gl.dim } \Sigma/I < \infty$ for every two sided ideal $I$ in $\Sigma$.

Let $\Sigma_1 (\Sigma_2)$ be a semiprimary $k$-algebra with radical $M_1 (M_2)$, and set $\Delta_i = \Sigma_i/M_i$ for $i = 1, 2$. Assuming that $(\Delta_i : k) < \infty$, and $\dim \Sigma_i/M^2 < \infty$ for $i = 1, 2$ it follows that $\dim \Delta_i = 0$ for $i = 1, 2$. Denote $\Delta = \Delta_1 \otimes \Delta_2$, and $N = M_1 \otimes \Delta_2 + \Delta_1 \otimes M_2$ then it readily follows that $\dim \Omega \leq 1$, where $\Omega = \Omega(\Delta, N)$ [9]. Furthermore, $\Sigma_1 \otimes \Sigma_2$ is a residue $k$-algebra of $\Omega$, and $(\Delta : k) < \infty$. We therefore have:

Theorem 2. The class of semiprimary $k$-algebras $\mathcal{C}$ is closed under tensor products. A semiprimary $k$-algebra $\Sigma$ belongs to $\mathcal{C}$ iff $\dim \Sigma/M^2 < \infty$ and $(\Delta : k) < \infty$.

Notice that this theorem is no longer valid if we replace $\dim \Sigma/M^2 < \infty$ by $\text{gl.dim } \Sigma/M^2 < \infty$.

References


