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WEAKENING A THEOREM ON DIVIDED POWERS

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Abstract. We show that if a Hopf algebra has finite dimensional primitives and a primitive lies in arbitrarily long finite sequences of divided powers then it lies in an infinite sequence of divided powers.

Introduction. K. Newman has shown us a counter-example to [3, Theorem 2, p. 521]. The theorem states that if H is a cocommutative Hopf algebra over a perfect field k of characteristic $p>0$ and k is the unique simple subcoalgebra of H then a primitive element ${}^1x$ of H lies in a sequence of divided powers $1={}^0x, {}^1x, \ldots, {}^{p^{n+1}-1}x$ if and only if ${}^1x$ has coheight n, for $n=0, 1, \ldots, \infty$. (The sequence should be considered infinite if $n=\infty$.) The proof given in [3, p. 524] correctly shows that the existence of the sequence of divided powers implies that ${}^1x$ has the desired coheight for all n. The proof there also correctly shows that *for finite* n if ${}^1x$ has coheight n then the desired sequence of divided powers exists. The error seems to be the assertion $K^1 \cong H^1/(\bigcup \ker F^i)$ [3, p. 254, lines 23–24].

Newman's example shows that ${}^1x$ may have infinite coheight—so lies in arbitrarily long finite sequences of divided powers—and yet ${}^1x$ lies in no infinite sequence of divided powers.

We show here that with the further assumption that the primitives of H are finite dimensional then ${}^1x$ having infinite coheight implies that ${}^1x$ lies in an infinite sequence of divided powers. This result is important because it plays a key role in the proof of Jacobson's conjecture in [1]. We explain this in more detail at the end of the present paper.

1. Suppose C is a cocommutative coalgebra with a unique simple subcoalgebra which is one dimensional. We identify this simple subcoalgebra with k and so consider $k \subset C$. Then the primitive elements of C, $P(C) \equiv \{c \in C \mid \Delta c = 1 \otimes c + c \otimes 1\}$.

HEYNEMAN'S THEOREM. *If* $\dim P(C) < \infty$ *then* C *satisfies the minimum condition and descending chain condition for subcoalgebras.*

Proof. For a vector space U the coalgebra $Sh(U)$ is defined in [4, p. 244, p. 254]. By [4, p. 254, 12.1.1] there is a finite-dimensional space U and an injective coalgebra

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map $F: C \to Sh(U)$. In [4, p. 261] $B(U)$ is defined as the maximal cocommutative subcoalgebra of $Sh(U)$. Since C is cocommutative $\operatorname{Im} F \subset B(U)$ and we consider F as a coalgebra injection $C \xrightarrow{F} B(U)$. Since F is injective by applying F to the coalgebras in a family of subcoalgebras of C we see that it suffices to show that $B(U)$ has the minimal condition on subcoalgebras. Similarly for the descending chain condition.

By [4, p. 278, Example-Exercise] the linear dual $\operatorname{Hom}_k(B(U), k)$ is a power-series ring in dim U variables; hence, is Noetherian. By [4, p. 16, 1.4.3] if D is a subcoalgebra of $B(U)$ then $D^{\perp} = \{f \in \operatorname{Hom}_k(B(U), k) \mid f(D) = 0\}$ is an ideal in $\operatorname{Hom}_k(B(U), k)$. Thus the maximal condition and ascending chain condition for ideals in $\operatorname{Hom}_k(B(U), k)$ implies the desired conditions for $B(U)$. Q.E.D.

THEOREM 2. *Let k be a perfect field of characteristic $p > 0$ and H a cocommutative Hopf algebra over k where k is the unique simple subcoalgebra of H. Suppose ${}^1x$ is a primitive element of H.*

(i) *If $1 = {}^0x, {}^1x, {}^2x, \ldots, {}^{p^{n+1}-1}x$ is a sequence of divided powers in H then ${}^1x$ has coheight n, for $n = 0, 1, \ldots, \infty$.*

(ii) *If ${}^1x$ has coheight n then there is a sequence of divided powers in H. $1 = {}^0x, {}^1x, \ldots, {}^{p^{n+1}-1}x$, for $n = 0, 1, \ldots$.*

(iii) *If $\dim P(H) < \infty$ and ${}^1x$ has infinite coheight then there is an infinite sequence of divided powers in H, $\{{}^ix\}_{i=0}^{\infty}$.*

Proof. In [3, p. 524, proof of Theorem 2] statements 1 and 2 are correctly proved. In [3, p. 520, Theorem 1] the V map $V: H \to H$ is defined. By [3, p. 521, Theorem 1] $V^n(V) = V \cdots V(H)$ (n-times) is a sub-Hopf algebra of H, thus a subcoalgebra. Assuming $\dim P(H) < \infty$ the descending chain $H \supset V(H) \supset V^2(H) \cdots$ must stabilize say at $V^n(H)$. Thus $VV^n(H) = V^n(H)$. Since ${}^1x$ has infinite coheight it follows that ${}^1x \in V^n(H)$. Thus $1 = {}^0x, {}^1x$ may be extended to an infinite sequence of divided powers lying in $V^n(H)$ by [3, p. 522, Lemma 7]. Q.E.D.

The statement just above [1, p. 285, 3.3.4] that $(PH)_{\infty} = \{x \in PH \mid$ there is an infinite sequence of divided powers lying over $x\}$ is false (by Newman's example) unless we have $\dim PH < \infty$, so Theorem 2 (this paper) applies. Thus the proof given for [1, p. 285, 3.3,4] is incorrect unless one assumes that *both* PH and PJ are finite dimensional. The use made of 3.3,4 is [1, p. 290, 3 lines above 3.5] in the proof of 3.5,3, Jacobson's conjecture, where *both* PH and PJ are finite dimensional. (Actually it can be shown by techniques developed by Newman that 3.3,4 is correct as it stands.)

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