TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 154, February 1971

TOPOLOGICAL PROPERTIES OF ANALYTICALLY UNIFORM SPACES

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Abstract. In the first part of the article we study certain topological properties of analytically uniform spaces (AU-spaces, cf. L. Ehrenpreis, *Fourier transforms in several complex variables*, Interscience, New York, 1970). In particular we prove that AU-spaces and their duals are always nuclear. From here one can easily obtain some important properties of these spaces, such as the Fourier type representation of elements of a given AU-space, etc.

The second part is devoted to one important example of AU-space which was not investigated in the aforementioned monograph: the scale of Beurling spaces $\mathscr{D}_\omega$ and $\mathscr{D}'_\omega$. We find a simple family of majorants which define the topology of the space $\hat{\mathscr{D}}_\omega$. This shows that the spaces of Beurling distributions are AU-spaces. Moreover, it leads to some interesting consequences and new problems.

Introduction. The notion of analytically uniform space was introduced into analysis by Leon Ehrenpreis in 1960 [13] as a natural framework for his so-called fundamental principle and other related results on division problems. Later it turned out that not only the division problems for convolution equations, but also many other problems in analysis such as the quasi-analycity, gap and density theorems, balayage, etc., can be successfully studied in terms of analytically uniform spaces. These topics together with numerous concrete examples of analytically uniform spaces are treated in the recent monograph of L. Ehrenpreis [16].

The analytically uniform spaces can be viewed as the largest class of topological vector spaces which can be studied and described in terms of the Fourier transform. However, it is then natural to ask: What are the functional analysis properties of analytically uniform spaces? This question is studied in the first part of the present article. In particular, we prove that such spaces are always nuclear (cf. Theorems 1 and 2).

The second part deals with the Beurling distribution spaces $\mathscr{D}'_\omega$ considered as analytically uniform spaces. In order to show that $\mathscr{D}'_\omega$ is an analytically uniform space (cf. Corollary 1 of Theorem 3) one has to find an intrinsic description of the

Received by the editors March 16, 1970.

AMS 1969 *subject classifications*. Primary 4630, 4640.

Key words and phrases. Analytically uniform spaces, Beurling spaces of distributions.

[1] The work of the first author was supported by the Courant Institute and the Sloan Foundation; the work of the second author was partially supported by the National Research Council of Canada.

topology in the space $\hat{\mathscr{D}}_\omega$ of Fourier transforms of Beurling test functions. We give three different definitions of this topology and show their equivalence with the original one (Theorem 3). This gives a simplified and more direct way of looking on such spaces, which is especially suitable for the study of division problems in Beurling spaces. These spaces are not discussed in [16] except for the classical case when $\mathscr{D}_\omega = \mathscr{D}$ and $\mathscr{D}'_\omega = \mathscr{D}'$ are spaces of L. Schwartz [30]. However, even in this special case our method gives a definition of the topology of $\mathscr{D}$ different from the usual ones [30], [14], [16]; actually, it is based on the same idea as one theorem of B. Malgrange (cf. [26, §2. Un théorème de régularitè]).

We wish to express our thanks to Professor L. Ehrenpreis for his constant interest in our work.

1. **Nuclearity of analytically uniform spaces and their duals.** We start with some general remarks on topological vector spaces. In what follows all spaces are always assumed to be Hausdorff locally convex spaces over complex numbers. We shall call them l.c. spaces. Given a l.c. space T, we denote by T'_b the strong dual of T and by $\langle \cdot, \cdot \rangle_T$ the bilinear form defining the duality between T and T'_b. For all other terminology, notations and definitions see [25], [23], [28], [32]. We shall only recall the following two definitions which will be used in this section:

1°. T is called semi-Montel if every bounded set in T is relatively compact. Obviously, semi-Montel spaces are quasi-complete and semireflexive [23].

2°. T is called separable (boundedly separable (²)) if T contains a countable dense subset (if each bounded set in T is contained in the closure of some bounded countable subset of T).

If the space T is metrizable, then the separability of T obviously implies its b-separability. The converse is true whenever T contains a fundamental sequence of bounded sets; therefore, it holds, e.g., for duals of metrizable spaces and more generally for (DF)-spaces. On the other hand this converse is known to be false for metrizable spaces [7] and even for (F)-spaces [24](³).

The next lemma gives some simple conditions which are sufficient for b-separability:

LEMMA 1. *Each of the following conditions is sufficient for the b-separability*:

(i) *T is a strict inductive limit of a sequence of b-separable spaces T_n $(n = 1, 2, \ldots)$. In particular, if all T_n are separable (F)-spaces, then the (LF)-space T is separable and b-separable.*

(ii) *T is semi-Montel and T'_b separable. In this case the bounded subsets in T are also metrizable.*

(iii) *T'_b is nuclear.*

(iv) *T is a Fréchet-Montel space or the dual of such a space. In this case T is also separable.*

(²) Abbreviated in what follows as b-separable.

(³) These results were proved under the assumption of the continuum hypothesis.

Proof. Condition (i) is obvious. (ii) Let A be a closed bounded set in T. We can assume that A contains the origin. Consider the closed convex hull, $h(A)$, of the set A. From our hypotheses and by the Kreĭn-Milman theorem it follows that $h(A)$ is compact. However $h(A)^{00}=h(A)$ is equicontinuous and hence w^{**}-compact in $(T_b')'=T$. Therefore $h(A)$ is also w-compact in T and thus necessarily w-metrizable ([4]). However, since obviously the weak and strong topology of T induce on $h(A)$ the same topology, $h(A)$ is a compact metrizable set and therefore also separable. (iii) is a well-known property of dually nuclear spaces [28]. (iv) If T is a (FM)-space, then T is separable by a result of Dieudonné [8]; and, as a metrizable space, T is also b-separable. If T is the dual of a (FM)-space, then T_b' is a (F)-space which must be separable [8], and we use (ii).

LEMMA 2. *Let T be nuclear. Then*

(i) *If T is quasi-barrelled, then T_b' is a Montel space; in particular, every b-separable space which is a dual of a nuclear σ-quasi-barrelled space* ([5]) *is a Montel space. If T is also quasi-complete, then both T and T_b' are Montel spaces and T_b' is b-separable.*

(ii) *If T is a (F)-space or a quasi-complete (DF)-space, then T and T_b' are complete ultrabornological* ([6]) *nuclear separable and b-separable spaces.*

Proof. (i) As the dual of a nuclear space, T_b' is barrelled (cf. [17, p. 155]), and the bounded subsets in T_b' are precompact. However, since T is quasi-barrelled, T_b' is quasi-complete and therefore a Montel space. The rest follows easily. (ii) Let T be a nuclear (F)-space. Then T is obviously a complete ultrabornological reflexive space and T_b' is also nuclear [28]. Hence by (i), T_b' is a quasi-complete b-separable space. Since T_b' is a (DF)-space, it follows from this that the space T_b' must be complete (see [25, p. 405, (3) a)]) and separable. However, T_b' is also ultrabornological as the complete dual of a reflexive (F)-space. The separability of T follows again from [8]. Now let T be a (1) quasi-complete; (2) nuclear; (3) (DF)-space. Then (1) and (2) imply that (4) T is a semi-Montel space and therefore also semi-reflexive. Furthermore, T_b' is a nuclear (F)-space [28], hence also a (FM)-space; thus, by [8] T_b' is separable. This together with (4) and (ii) of Lemma 1 shows that all bounded sets in T are metrizable, and by [25, p. 402, (12) b)], the latter is a sufficient condition for T to be quasi-barrelled. Therefore by (4), the spaces T and T_b' form a reflexive pair with T_b' a nuclear space, and we are again in the preceding case.

REMARK. Several articles were recently devoted to different denumerability conditions in l.c. spaces (cf. [12], [29], [33], [34] and also a series of articles by W. Słowikowski and V. Pták [35]). Motivation for these investigations comes always

([4]) Cf. e.g., N. Bourbaki, *Espaces vectoriels topologiques*, Actualités Sci. Indust., no. 1229, Hermann, Paris, 1955, Chapter IV, §2, Proposition 3. **MR** 17, 1109.

([5]) E is called σ-quasi-barrelled provided each countable strongly bounded subset of E_b' is equicontinuous, cf. [23], [34].

([6]) Cf. [23].

from function and distribution spaces where such properties of sequential character play an important role. It would be interesting to study more systematically also the notion of b-separability. (In [27] b-separability is called Runge property.)

DEFINITION 1. Given a l.c. space W, we shall call W an analytically uniform space (AU-space) provided

(i) W is the strong dual of some l.c. space U; and

(ii) There exists a continuous analytic embedding ω of the n-dimensional complex space $\boldsymbol{C}^n$ onto a set whose span is a dense subspace of W. Thus, in particular, for each $S \in U$, the function $\langle S, \omega(z)\rangle_U$ is an entire function in $\boldsymbol{C}^n$; we shall denote this function by $\hat{S}(z)$.

(iii) There exists a family $K=\{k\}$ of positive continuous functions $k(z)$ defined on the whole space $\boldsymbol{C}^n$ and such that for each $S \in U$ and $k \in K$ we have $\hat{S}(z)=\mathcal{O}(k(z))$, and if we define

$$p_k(S) = \sup_z |\hat{S}(z)|/k(z), \tag{1}$$

then the sets

$$V(k, \varepsilon) = \{S \in U : p_k(S) \leqq \varepsilon\},$$

for all $k \in K$ and ε positive, form a fundamental system of neighborhoods of the origin in the space U. The family K is called an *analytic uniform structure* (AU-structure) for the space W.

REMARKS. (a) In his original definition [15] Ehrenpreis assumes the pair (U, W) to be reflexive. Then the space U is just the strong dual of W (7). Although this definition is more natural, Definition 1 above covers a more general class of AU-spaces (cf. the example at the end of this section).

(b) Let us notice that condition (ii) implies that the space *W is always separable.*

(c) We shall use the following terminology: the entire function $\hat{S}(z)$, for $S \in U$, will be called the Fourier transform of the element S. The functions $\hat{S}$ form a vector space $\boldsymbol{U}$. If we equip the space $\boldsymbol{U}$ with the locally convex topology defined by the norms $\tilde{p}_k(\hat{S})=p_k(S)$ (cf. (1)), then the algebraic isomorphism $S \leftrightarrow \hat{S}$ becomes an isomorphism of the l.c. spaces U and $\boldsymbol{U}$. For concrete examples of AU-spaces this terminology agrees with the usual one: the spaces U and W which occur in practice are always some distribution or function spaces and the mapping ω is the exponential mapping $\omega: z \mapsto \exp(i\langle x, z\rangle)$, $z \in \boldsymbol{C}^n$, and the entire function $\hat{S}(z)$ is then the usual Fourier transform of the distribution S: $\hat{S}(z)=S(\exp(i\langle x, z\rangle))$.

(d) Given an AU-space W, there can be different AU-structures defining the same AU-space W. Actually, it will be our objective in the second part of this article to find *suitable* AU-structures for some important special examples of AU-spaces.

(7) On the other hand, we shall see later that the space U is almost always semireflexive. Thus U could be called the AU-dual of the space W.

Conditions (i)–(iii) suffice already for a Fourier-type representation of elements of the space W (cf. [15, Theorem 1]) which we shall discuss later. However, in order to prove the basic analytic theorems for AU-spaces such as the so-called fundamental principle (cf. [13], [16]), Ehrenpreis imposes certain additional conditions on the space U. These conditions can be divided into two groups: 1° conditions (iv)–(vii) listed below which represent only very mild requirements of a general functional analysis character and which do not essentially restrict the class of AU-spaces; and 2° certain conditions of purely analytic character (see conditions (e) and (d) in [16, Chapter IV]) which are necessary for the analytic part of the proofs in [16]. These conditions already restrict essentially the class of AU-spaces. Ehrenpreis calls such spaces product localizable AU-spaces ($PLAU$-spaces). Since in this section we are interested in properties of a possibly large class of AU-spaces, only some of the conditions (iv)–(vii) below will be always assumed. On the other hand it would be interesting to find some functional analysis interpretation of the conditions of second type. It would probably lead to simpler proofs of some important theorems in [16] and provide an additional insight into the whole theory. In our view these conditions are related to the validity of the open mapping theorem in certain spaces derived from the $PLAU$-spaces (cf. Problem 4.2 in [16, Chapter IV]).

Let U and W be the l.c. spaces of Definition 1:

(iv) Any entire function F in $\boldsymbol{C}^n$ such that $F(z)=\mathcal{O}(k(z))$ for all $k \in K$ is of the form $F=\hat{S}$ for some S in U.

(v) For any $N>0$, if we replace the AU-structure $K=\{k\}$ by the family $K_N=\{k_N\}_{k\in K}$ where

$$k_N(z) = \max_{|z-z'|\leqq N} \{k(z')(1+|z'|)^N\}, \tag{2}$$

then K_N is again an AU-structure for W defining the same topology in the space U.

(vi) We assume that in addition to the analytic uniform structure $K=\{k\}$, there is also given a family $M=\{m\}$ called a bounded analytic uniform structure (BAU-structure) of the space W which describes the bounded sets in the space U. The family M is defined as the system of all positive continuous functions $m(z)$ on $\boldsymbol{C}^n$ such that $m(z)=\mathcal{O}(k(z))$ for all $k \in K$. The sets

$$A_m = \left\{S \in U : \sup_{z\in \boldsymbol{C}^n} \frac{|\hat{S}(z)|}{m(z)} < \text{const.}\right\}$$

are obviously bounded subsets in U; and, we require that these sets form a fundamental system of bounded sets in the space U.

We can also express condition (iv) by saying that the family K is "complete", i.e. the majorants $k(z)$ completely determine which entire functions $\hat{S}(z)$ are elements of the space U by restricting their growth. However, condition (iv) also implies another completeness:

PROPOSITION 1. *Each space U satisfying condition* (iv) *is a complete l.c. space.*

Proof. Indeed, the topology $\mathscr{T}$ of the space U is obviously finer than the topology $\mathscr{T}_c$ of the uniform convergence of functions $\hat{S}(z)$, $S \in U$, on compact subsets in $\boldsymbol{C}^n$. Hence, by (iv), the topology $\mathscr{T}$ has a fundamental system of neighborhoods which are complete in the topology $\mathscr{T}_c$. The completeness of $(U, \mathscr{T})$ then follows.

Condition (v) is quite natural when W is a function or a distribution space: it essentially assures the continuity of differentiation. (Let us note that condition (v) also enables us to replace the symbol "$\mathcal{O}$" in (iii) by "o" ([8]).) Once differentiation is a continuous operation in U, we suspect U is a nuclear space. We shall show that it is actually so. We begin with the following simple fact:

LEMMA 3. *Let $H(z)$ be an entire function in $\boldsymbol{C}^n$ and $l(z)$ any positive continuous majorant of H in $\boldsymbol{C}^n$, i.e. $H(z)=\mathcal{O}(l(z))$. Set*

$$\tilde{l}(z) = \sup_{z'-z\in\Delta_1} \{l(z')(1+|z'|)^{2n+\alpha}\}$$

where $\Delta_1=\{\zeta \in \boldsymbol{C}^n : \max_j |\zeta_j| \leqq 1\}$ and α is any fixed positive number. Then

$$\sup_{z\in\boldsymbol{C}^n} \frac{|H(z)|}{\tilde{l}(z)} \leqq \frac{1}{\pi^n}\int_{\boldsymbol{C}^n} \frac{|H(\zeta)|\cdot|d\zeta|}{l(\zeta)(1+|\zeta|)^{2n+\alpha}}, \tag{3}$$

where $|d\zeta|$ is the Lebesgue measure in $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$.

Proof. By the mean value property of harmonic functions applied to the polydisc $\Delta_r=\Delta(z_1; r)\times\cdots\times\Delta(z_n; r)$ and the function H we obtain

$$H(z) = \frac{1}{\pi^n}\int_{\Delta_1} H(z+\zeta)|d\zeta|. \tag{4}$$

Multiplying the integrand in (4) by the function $\tilde{l}(z)/l(z')\times(1+|z'|)^{2n+\alpha}$, which is $\geqq 1$ in the polydisc Δ_1, we get

$$|H(z)| \leqq \frac{1}{\pi^n}\int_{\Delta_1} |H(z+\zeta)|\cdot|d\zeta| \leqq \frac{\tilde{l}(z)}{\pi^n}\int_{\boldsymbol{C}^n} \frac{|H(\zeta)|\cdot|d\zeta|}{l(\zeta)(1+|\zeta|)^{2n+\alpha}}. \tag{5}$$

Thus the lemma is proved.

By $\mathscr{C}(X)$ ($\mathscr{C}_b(X)$) we shall denote the Banach space of all continuous (bounded continuous) functions on a compact (locally compact) space X.

The following criterion for nuclearity of a l.c. space E is due to A. Pietsch [28]:

(P) The space E is nuclear if and only if for each continuous seminorm p on E there exists a neighborhood V of the origin in E and a positive Radon measure on the w^*-compact set V^0 such that for each $z \in E$,

$$p(z) \leqq \int_{V^0} |\langle z, x\rangle_E|\, d\mu(x). \tag{6}$$

([8]) More exactly, there exists an AU-structure $K=\{k\}$ for which (iii) holds with "o".

Here we are using the following notation: for each $f \in \mathscr{C}(V^0)$ we write symbolically

$$\begin{aligned} \langle f, \mu \rangle_{\mathscr{C}(V^0)} &= \int_{V^0} \langle f, \delta(x) \rangle_{\mathscr{C}(V^0)} \, d\mu(x) \\ &= \int_{V^0} f(x) \, d\mu(x), \end{aligned} \tag{7}$$

where $\delta(x)$ is the natural embedding $V^0 \to \mathscr{C}'(V^0)$. If β is the natural open embedding of E into its bidual E'', then each $z \in E$ defines a continuous function $\tilde{z} = \beta(z)|_{V^0}$ on V^0. The integrand in (7) is just the absolute value of this function and (6) can be written simply as $p(z) \leqq \langle |\tilde{z}|, \mu \rangle_{\mathscr{C}(V^0)}$.

THEOREM 1. *If W is an analytically uniform space such that the corresponding space U satisfies* (v), *then U is nuclear.*

Proof. We shall prove that U is nuclear. Denote by $\boldsymbol{W}$ the space $\boldsymbol{U}'$. If $K = \{k\}$ is an AU-structure for $\boldsymbol{W}$ so is the family $K_{2n+1} = \{k'\}$ according to our hypothesis. Take any $k' \in K_{2n+1}$; then

$$k'(z) = \sup_{z - z' \in \Delta_{2n+1}} \{k(z')(1 + |z'|)^{2n+1}\}$$

for some $k \in K$. Hence by Lemma 3,

$$p_{k'}(H) \leqq \frac{1}{\pi^n} \int_{\boldsymbol{C}^n} \frac{|H(w)| \cdot |dw|}{k(w)(1 + |w|)^{2n+1}}. \tag{8}$$

The vector space $\boldsymbol{U}_k$ of all functions of the form H/k for some $H \in \boldsymbol{U}$ is a subspace of $\mathscr{C}_b(\boldsymbol{C}^n)$ and the corresponding (algebraic) isomorphism $\kappa: \boldsymbol{U} \to \boldsymbol{U}_k$ is a continuous embedding of $\boldsymbol{U}$ into $\mathscr{C}_b(\boldsymbol{C}^n)$. The restriction of any evaluation functional $\Delta(\lambda) \in \mathscr{C}_b'(\boldsymbol{C}^n)$ $(\lambda \in \boldsymbol{C}^n)$ on the subspace $\boldsymbol{U}_k$ generates a continuous linear form $\Delta^*(\lambda)$ on $\boldsymbol{U}$, $\Delta^*(\lambda) = \Delta(\lambda)|_{\boldsymbol{U}_k} \circ \kappa$ and for each $H \in \boldsymbol{U}$, $\langle H, \Delta^*(\lambda) \rangle_{\boldsymbol{U}} = H(\lambda)/k(\lambda)$.

Let us set $V = \{H \in \boldsymbol{U} : p_k(H) \leqq 1\}$. Then the mapping $\Delta^*: \lambda \mapsto \Delta^*(\lambda)$ maps w^*-continuously $\boldsymbol{C}^n$ into the w^*-compact subset V^0 of $\boldsymbol{W}$. Let $a \mapsto \delta(a)$ be the natural embedding of V^0 into $\mathscr{C}'(V^0)$.

Integrating this mapping over $\boldsymbol{C}^n$ with respect to the measure $d\rho(\lambda) = \pi^{-n}(1 + |\lambda|)^{2n+1}|d\lambda|$, we again obtain a positive Radon measure μ on V^0, $\mu = \int_{\boldsymbol{C}^n} \delta(\Delta^*(\lambda)) \, d\rho(\lambda)$, since obviously $\|\mu\|_{\mathscr{C}'(V^0)} \leqq \|\rho\|_{\mathscr{C}_b'(\boldsymbol{C}^n)}$.

Now take an arbitrary $H \in \boldsymbol{U}$ and let $|\tilde{H}|$ be defined as the absolute value of the function $\tilde{H}$ generated by H (cf. the text preceding Theorem 1). Then, for each $\lambda \in \boldsymbol{C}^n$, $|\tilde{H}|(\delta(\Delta^*(\lambda))) = |\langle \Delta^*(\lambda), \beta(H) \rangle_{\boldsymbol{W}}| = |H(\lambda)|/k(\lambda)$ and thus by (8) we get

$$p_{k'}(H) \leqq \int_{\boldsymbol{C}^n} \frac{|H(w)| \, d\rho(w)}{k(w)} = \langle |\tilde{H}|, \mu \rangle_{\mathscr{C}(V^0)}. \tag{9}$$

Therefore condition (P) is verified and this proves the theorem.

If E is a l.c. space and A a bounded closed absolutely convex subset in E, we shall denote by $E(A)$ the normed space defined as follows: as a set, E is given by

$E(A)=\bigcup_{\lambda\geqq 0}\lambda A$; $E(A)$ is a vector subspace of E and the norm on $E(A)$ is given by the unit ball A. The unit ball in the dual space $E'(A)$ will be denoted by $S(A)$. It follows from the theory of nuclear spaces that the next condition is sufficient for the nuclearity of E' (cf. [28, Proposition 4.1.6]):

(P′) Let $\mathfrak{M}(E)$ be the system of all closed absolutely convex bounded subsets in E. Then for each $A\in\mathfrak{M}(E)$ there exists a $B\in\mathfrak{M}(E)$ such that (i) $A\subset\tau B$ for some $\tau>0$; and (ii) for some positive Radon measure μ on the w^*-compact subset $S(A)$ of $E'(A)$ we have

$$\|z\|_{E(B)}\leqq\int_{S(A)}|\langle z,x\rangle|\,d\mu(x) \tag{10}$$

for each $z\in E(A)$.

THEOREM 2. *If W is an analytically uniform space such that conditions* (v) *and* (vi) *are satisfied, then W is nuclear.*

Proof. It is obviously sufficient to check condition (P′) only for the sets A of the form $A=A_m$ for some $m\in M$ (cf. (vi)). If $m\in M$, then the functions

$$m'(z)=\sup_{|z-z'|\leqq 2n+1}\{m(z')(1+|z'|)^{2n+1}\}$$

satisfies the conditions $m'(z)=\mathcal{O}(k'(z))$ for all $k'\in K_{2n+1}$ (cf. (v)). Hence the set $B=\{S\in U:|\hat{S}(z)|\leqq m'(z)$ for all $z\in \boldsymbol{C}^n\}$ is bounded in U and by Lemma 3 we have for all $H\in U(A)$

$$p_B(H)\leqq\int_{\boldsymbol{C}^n}\frac{|H(\lambda)|\,d\rho(\lambda)}{m(\lambda)}. \tag{11}$$

Now, following the same lines as in the proof of Theorem 1, one constructs a positive Radon measure μ on the set $S(A)$ such that the integral in (11) becomes the scalar product $\langle|\tilde{H}|,\mu\rangle_{\mathscr{C}(S(A))}$.

Given an AU-space W together with the corresponding space U and the AU-structure $K=\{k\}$, the following norms can be considered on the space $\boldsymbol{U}$ (or equivalently on U): for each $k\in K$ we set

$$\begin{aligned}\|\hat{S}\|_1^{(k)}&=\int_{\boldsymbol{C}^n}\frac{|\hat{S}(z)|}{k(z)}\,d\rho(z),\\ \|\hat{S}\|_2^{(k)}&=\left(\int_{\boldsymbol{C}^n}\frac{|\hat{S}(z)|^2}{k^2(z)}\,d\rho(z)\right)^{1/2},\end{aligned} \tag{12}$$

where S is an arbitrary element of U and $d\rho$ the measure in (9). Lemma 3 combined with the Schwarz inequality gives immediately

PROPOSITION 2. *If W satisfies condition* (v), *all three systems of norms*

(i) $\{\tilde{p}_k\}_{k\in K}$;

(ii) $\{\|\cdot\|_1^{(k)}\}_{k\in K}$;

(iii) $\{\|\cdot\|_2^{(k)}\}_{k\in K}$

are equivalent, i.e. define the same topology on the space $\boldsymbol{U}$. *In particular, this topology can also be defined by the scalar products*

$$[\hat{S}_1, \hat{S}_2]_k = \int_{C^n} \frac{\hat{S}_1(z)\cdot\overline{\hat{S}_2(z)}}{k^2(z)}\, d\rho(z). \tag{13}$$

Hence by Proposition 2, we could define AU-spaces by means of L^2-norms instead of sup-norms. This answers Question 3 of [15, p. 74].

Let T be any element in $\boldsymbol{W}$. Then the mapping $T \mapsto \hat{T}$ given by the formula $\langle \hat{S}, T\rangle_U = \langle S, \hat{T}\rangle_U$ defines a natural isomorphism between l.c. spaces $\boldsymbol{W}$ and W. However, by Proposition 2 there exists a function $k \in K$ such that T defines a bounded linear functional on the pre-Hilbert space $(\boldsymbol{U}, [\cdot, \cdot]_k)$. Denote by $\boldsymbol{U}_{(k)}$ the completion of the latter space. The mapping $H \mapsto \tilde{H} = H(z)/(k(z)(1+|z|)^{n+1/2})$ is an isomorphism of $\boldsymbol{U}_{(k)}$ onto a closed subspace $\tilde{\boldsymbol{U}}_{(k)}$ of $L^2(\boldsymbol{C}^n)$. If $\tilde{T}$ is the image of T in this isomorphism, then $\tilde{T}$ can be extended to the functional $\tilde{T}^\sim$ defined on the whole space $L^2(\boldsymbol{C}^n)$. (For instance, we can extend $\tilde{T}$ by setting $\tilde{T}^\sim = 0$ on the orthogonal complement of $\tilde{\boldsymbol{U}}_{(k)}$ in $L^2(\boldsymbol{C}^n)$.) In particular, we have $\langle H, T\rangle_U = \langle \tilde{H}, \tilde{T}^\sim\rangle_{L^2(\boldsymbol{C}^n)}$. Let $F_T(z)$ be the function in $L^2(\boldsymbol{C}^n)$ representing the functional $\tilde{T}^\sim$, i.e.

$$\langle G, \tilde{T}^\sim\rangle_{L^2(\boldsymbol{C}^n)} = \int_{C^n} G(z) \times \overline{F_T(z)}\, |dz|$$

for all $G \in L^2(\boldsymbol{C}^n)$. Applying this representation to the elements $G \in L^2(\boldsymbol{C}^n)$ of the form

$$G(z) = \widetilde{\hat{S}(z)}, \qquad S \in U,$$

we obtain the following

COROLLARY 1. *For any* $\hat{T} \in W$ *there exists a majorant* $k \in K$ *and a function* $F_T(z) \in L^2(\boldsymbol{C}^n)$ *such that* $\hat{T}$ *has the Fourier-type representation*

$$\hat{T} = \int_{C^n} \omega(z)\overline{F_T(z)}\, \frac{|dz|}{k(z)(1+|z|)^{n+1/2}} \tag{14}$$

where the integral is to be understood in the functional sense.

REMARKS. (1) The corollary is a more concrete version of Theorem 1 established in [15] under a slightly more general assumption, namely that for each $S \in U$ and each $k \in K$ one has $\hat{S}(z) = o(k(z))$ which follows immediately from condition (v).

(2) It is clear from the proof of Theorem 1 that the integral in (14) can always be given a measure-theoretic meaning; in some important special cases this integral exists even in the Lebesgue-Stieltjes sense (cf. [15], [16]). Since in all concrete examples of AU-spaces we always have $\omega(z) = \exp(i\langle z, x\rangle)$, Corollary 1 really gives a Fourier-type representation of each element in W. The importance of such a representation in applications is discussed in detail in [16].

(3) Starting from formula (14), it would be possible to define a "carrier" of

any element in the space W (similarly as it is done in [27]), because the integration in (14) is carried out only over the set supp F_T whose size depends upon how "large" is the subspace $\tilde{U}_{(k)}$ in the space $L^2(C^n)$. This is evidently related to the notion of a sufficient set (cf. [16] and [1]).

(4) Let us note that if $\hat{T}$ varies in (14) so that $p_k(\hat{T})$ remains bounded, then the L^2-norms of the corresponding functions F_T are also bounded.

Lemmas 1 and 2 combined with Theorems 1 and 2 imply different properties of AU-spaces. We shall explicitly mention only the following:

COROLLARY 2. *Assume that W is an AU-space such that the corresponding space U is quasi-barrelled* (9) *and conditions* (iv), (v) *and* (vi) *are satisfied. Then U and W are both complete nuclear reflexive b-separable spaces. Moreover, if one of the spaces is a (DF)-space, then U and W are also bornological spaces.*

Finally, let us quote the last condition of nonanalytic character which Ehrenpreis imposes on all AU-spaces occurring in his theory (cf. [16, Chapter IV]):

(vii) ... we assume, in addition, that the AU-structure K can be chosen to consist of products of functions of single variables, that is, each $k \in K$ is of the form $k_1(z_1)k_2(z_2)\cdots k_n(z_n)$, where the functions k_j are continuous and positive.

This condition is obviously a requirement on the tensor structure of the space W. In connection with (vii) we can ask the following question:

($\otimes$) For $i=1, 2$, let W_i be a given AU-space with respect to the space U_i and the AU-structure K_i. Let $\omega_i: C^{n_i} \to W_i$ be the corresponding analytic embedding. Consider the algebraic tensor products $U=U_1 \otimes U_2$, $W=W_1 \otimes W_2$ and the family $K=\{k_1(z_1)\cdot k_2(z_2)\}_{k_1\in K_1; k_2\in K_2}$ of majorants defined on the space $C^n = C^{n_1+n_2}$, and put $\omega(z_1, z_2)=\omega_1(z_1) \otimes \omega_2(z_2)$. Let χ be the topology on U given by the majorants from K and $\tilde{U}_\chi$ the completion of U in this topology. Which topology has one to consider on the space W in order that W (or rather the completion of W in this topology) be an AU-space with respect to the space U and the family K?

Denote by ε, ι the coarsest (respectively finest) l.c. topology compatible with the tensor structure of the space U. Let π be the projective topology on U. Then $\varepsilon \prec \pi \prec \iota$. Let S be any element in $U_1 \otimes U_2$ and $S=\sum_{r=1}^m S_r^{(1)} \otimes S_r^{(2)}$ an arbitrary tensor representation of S. Then the number

$$\hat{S}(z_1, z_2) = \sum_{r=1}^{m} S_r^{(1)}(z_1)S_r^{(2)}(z_2), \tag{15}$$

for $(z_1, z_2) \in C^n$, does not depend on this particular representation; therefore the topology χ is defined by the seminorms

$$p_{k_1k_2}(S) = \sup_{C^n} \frac{|\hat{S}(z_1, z_2)|}{k_1(z_1)\cdot k_2(z_2)}; \qquad k_1 \in K_1,\ k_2 \in K_2. \tag{16}$$

(9) Or only: σ-quasi-barrelled and its dual b-separable.

Similarly, the number

$$\varepsilon_{T_1T_2}(S) = \sum_{r=1}^{m} \langle S_r^{(1)}, T_1\rangle_{U_1} \cdot \langle S_r^{(2)}, T_2\rangle_{U_2}, \tag{17}$$

where $T_1 \in W_1$, $T_2 \in W_2$, is independent of any particular tensor representation of S. The topology ε is defined by the seminorms

$$q_{k_1k_2}(S) = \sup_{T_1\in V_1; T_2\in V_2} |\varepsilon_{T_1T_2}(S)| \tag{18}$$

for all $k_1 \in K_1$, $k_2 \in K_2$ (cf. [28]), where we denoted by V_i the polar (in W_i) of the set $\{S \in U_i : p_{k_i}(S) \leqq 1\}$.

LEMMA 4. *The topology χ is compatible with the tensor structure of the space U. More precisely, the following relations hold*:

$$\varepsilon \prec \chi \prec \pi. \tag{19}$$

Proof. That χ is coarser than π follows immediately from the (obvious) continuity of the canonical bilinear mapping $U_1 \times U_2 \to U_\chi$. Now let $q_{k_1k_2}$ be any seminorm defining the topology ε. By Proposition 2 and remarks following Corollary 1, there exist functions k_1', k_2' such that in the Fourier representation (14) associated with k_i'

$$C_i = \sup\{\|F_{T_i}\|_{L^2} : T_i \in V_i\} < \infty, \qquad i = 1, 2.$$

And using (14) we easily obtain, with constant $C > 0$,

$$q_{k_1k_2}(S) \leqq C \cdot C_1C_2 \cdot p_{k_1'k_2'}(S)$$

and this proves our lemma.

Using Grothendieck's theory of tensor products (cf. [19, Chapter II, §4] and [32]) we obtain from Lemma 4 the following statement which answers question ($\otimes$):

PROPOSITION 3. *Let U_1, U_2 be such that conditions* (iv) *and* (v) *hold and let one of these spaces be quasi-barrelled* ([10]).

Then: $\varepsilon = \chi = \pi$, *i.e. χ is the "right" topology on the space U; the completion $\tilde{U}_\chi$ of U is nuclear; and the space $\tilde{W}_\iota = W_1 \hat{\otimes}_\iota W_2$ is the AU-space with respect to the space $\tilde{U}_\chi$ and the AU-structure $\{k\}$. If both spaces U_i are metrizable* (*or both are* (*DF*)*-spaces*), *then $\tilde{W}_\iota = \tilde{W}_\pi$.*

EXAMPLE. Part of the difficulties above are due to the fact that we are not asking U to be reflexive, as the following example shows:

Let $|\cdot|$ be any norm in $\boldsymbol{R}^n$. For each integer $l > 0$ set $B_l = \{x : |x| \leqq l\}$. Denote by $\mathscr{D}_l^s$, for $s = 0, 1, \ldots$, the space $\{f \in C_0^s(\boldsymbol{R}^n) : \operatorname{supp} f \subseteq B_l\}$ with the natural topology.

([10]) For these statements only, it would be sufficient to assume that one of the spaces U_i satisfies condition (v).

Finally we define $\mathscr{D}^s = \lim\operatorname{ind}_l \mathscr{D}_l^s$ and $\mathscr{D}_F = \lim\operatorname{proj}_s \mathscr{D}^s$. Then $\mathscr{D}_F$ coincides as a set with $C_0^\infty(\boldsymbol{R}^n)$. If $\mathscr{D}$ denotes the space $C_0^\infty(\boldsymbol{R}^n)$ with the standard Schwartz topology [30], then obviously the identical mapping $\mathscr{D} \to \mathscr{D}_F$ is continuous.

Now, let A be any bounded set in $\mathscr{D}_F$. Then A is a bounded set in any $\mathscr{D}^s$. However, all spaces $\mathscr{D}^s$, and in particular the space $\mathscr{D}^0$, are strict inductive limits of Banach spaces. Therefore, the set A must be contained in some $\mathscr{D}_l^0$, $l>0$. This means that all $f \in A$ have the support contained in B_l. Since A is also bounded in $\mathscr{D}^s$, $s>0$, we immediately see that A is bounded in $\mathscr{D}$. Thus the bounded sets in $\mathscr{D}$ and $\mathscr{D}_F$ coincide and the space $(\mathscr{D}_F)'_b$ has the relative topology of the space $\mathscr{D}'$. This space, denoted by $\mathscr{D}'_F$, is usually called the space of distributions of finite order (cf. [23], [32]). Since $\mathscr{D}'_F$ is dense in $\mathscr{D}'$, we have $\mathscr{D} = (\mathscr{D}')'_b = (\mathscr{D}'_F)'_b$. The nuclearity of the space $\mathscr{D}_F$ follows from the general theory of nuclear spaces [28]. However the nuclearity of $\mathscr{D}_F$ and $\mathscr{D}'_F$ also follows from Theorems 1 and 2 above, because it can be shown that the space $\mathscr{D}'_F$ is an AU-space (i.e. $U = \mathscr{D}_F$, $W = \mathscr{D}'_F$) satisfying conditions (i)–(vii) (cf. [16, Chapter V]).

We can recapitulate the properties of $\mathscr{D}_F$ and $\mathscr{D}'_F$ as follows: The space $\mathscr{D}_F$ is nuclear, complete (cf. Proposition 1 above), semireflexive, but not reflexive, thus not quasi-barrelled, and therefore not Montel space; moreover $\mathscr{D}_F$ is also not bornological (because of the inequality $\mathscr{D}'_F \neq \mathscr{D}'$). The space $\mathscr{D}'_F$ is nuclear, barrelled (as the strong dual of a nuclear space) and neither quasi-complete nor Montel nor semireflexive.

Finally, let us notice that the space $\mathscr{D}'_F$ is an AU-space in the sense of Definition 1 above, but not in the sense of the original definition in [15] where the reflexivity of the pair (W, U) was assumed.

2. **Beurling spaces.** In this section we shall study the Beurling spaces of test functions and distributions. These spaces represent a very interesting generalization of the classical Schwartz spaces $\mathscr{D}$, $\mathscr{D}'$ (cf. [4]). Actually the latter spaces become in the Beurling scale a special, but in a well-defined sense, extreme case. Another interesting property of Beurling spaces is their relationship to Denjoy-Carleman classes [4].

These spaces were introduced by Arne Beurling in 1961 [3]. A systematic study of Beurling spaces was later published by G. Björck [4] who carried out in the frame of such distributions much of the theory of partial differential equations following the program of Hörmander's monograph [21]. A regularity theorem for solutions to elliptic equations was proved by O. John [20]. Other problems concerning Beurling distributions are studied in [5], [9].

In this section we want to find an AU-structure for the Beurling spaces, i.e. to exhibit majorants describing the neighborhoods of the origin in these spaces. (For an analogous description in the case of $\mathscr{D}$, cf. our note [2].)

As a by-product which at the same time gives an intuitive idea of the proof we obtain a decomposition of any entire function of the form $\hat{\varphi}$, where φ is a Beurling

test function, into a sum $\hat{\varphi}=\sum \hat{\varphi}_k$, where the summands $\hat{\varphi}_k$ satisfy sharper bounds than the function $\hat{\varphi}$.

To begin with, let us recall the definitions of spaces $\mathscr{D}_\omega$ and $\mathscr{D}'_\omega$: Let $\omega(\xi)$ be a real-valued function defined on $\boldsymbol{R}^n$ and such that

$$(\alpha) \qquad 0 = \omega(0) = \lim_{x\to 0} \omega(x) \leqq \omega(\xi+\eta) \leqq \omega(\xi)+\omega(\eta), \qquad (\forall \xi, \eta \in \boldsymbol{R}^n),$$

$$(\beta) \qquad \int_{\boldsymbol{R}^n} \frac{\omega(\xi)\, d\xi}{(1+|\xi|)^{n+1}} < \infty,$$

for some constants $a>0$, b real,

$$(\gamma) \qquad b+a\cdot\log(1+|\xi|) \leqq \omega(\xi) \qquad (\forall \xi \in \boldsymbol{R}^n).$$

Let $\{K_s\}_{s\geqq 1}$ be a fixed sequence of balls $K_s=\{|x|\leqq R_s\}$ with $R_s \nearrow +\infty$. Then we define

$$(20) \qquad \begin{aligned} \mathscr{D}_\omega(K_s) = \Big\{\varphi \in L^1(\boldsymbol{R}^n) : \operatorname{supp} \varphi \subset K_s, \\ \|\varphi\|_\lambda^{(\omega)} = \sup_\xi (|\hat{\varphi}(\xi)| e^{\lambda\omega(\xi)}) < \infty \ (\forall \lambda > 0)\Big\} \end{aligned}$$

and we set

$$(21) \qquad \mathscr{D}_\omega = \liminf_{s\to\infty} \mathscr{D}_\omega(K_s).$$

Then $\mathscr{D}_\omega$ is obviously an (LF)-space and its topology will be denoted by $\mathscr{T}$.

REMARK 1. If $\omega(\xi)$ is as above, we define $\tilde{\omega}(\xi)=1+(\rho * \omega)(\xi)$, where $\rho(\xi)$ is a positive C^∞ function with a sufficiently small support and $\int_{\boldsymbol{R}^n} \rho(t)\, dt=1$. Then we have (i) $\tilde{\omega} \in C^\infty$; (ii) $|\tilde{\omega}(\xi)-\omega(\xi)| \leqq 2$ $(\forall \xi)$; (iii) $1 \leqq \tilde{\omega}(\xi+\xi') \leqq \tilde{\omega}(\xi)+\tilde{\omega}(\xi')$; (iv) $|\partial\tilde{\omega}(\xi)/\partial\xi_j| \leqq M(1+|\xi|)^{n+1}$ for some $M>0$ and all $j=1,\ldots,n$ and $\xi \in \boldsymbol{R}^n$. Obviously $\tilde{\omega}$ also satisfies conditions (β), (γ) and the norms $\|\varphi\|_\lambda^{(\tilde{\omega})}$ and $\|\varphi\|_\lambda^{(\omega)}$ are equivalent. Therefore, replacing the function ω by $\tilde{\omega}$, if necessary, we shall always assume in what follows that ω has the properties (i)–(iv).

REMARK 2. If K is a compact set, we define the supporting function H_K of K by the relation

$$H_K(\eta) = \max_{x\in K} \langle x, \eta\rangle.$$

It was proved in [4] that on each $\mathscr{D}_\omega(K_s)$ the system of seminorms $\|\cdot\|_\lambda^{(\omega)}$ $(\lambda>0)$ is equivalent to the system

$$(22) \qquad \Big\{\varphi \to |||\varphi|||_\lambda^{(\omega)} = \sup_{\zeta\in C^n} (|\hat{\varphi}(\xi)| \exp(\lambda\omega(\zeta)-H(\eta)-|\eta|/4)\Big\}_{\lambda>0}$$

where, as usual, $\zeta=\xi+i\eta$.

REMARK 3. As it is shown in [4, Theorem 1.3.18], condition (γ) implies that $\mathscr{D}_\omega \subset \mathscr{D}$ and $\mathscr{D}_\omega(\Omega)$ is dense in $\mathscr{D}(\Omega)$ for each open $\Omega\subset\boldsymbol{R}^n$. Thus, in particular, for each $\varphi \in \mathscr{D}_\omega$, $\hat{\varphi}(\zeta)$ is an entire function and $\mathscr{D}' \subset \mathscr{D}'_\omega$.

In addition to the topology $\mathscr{T}$ we shall also consider the following three topologies on the space $\mathscr{D}_\omega$:

Topology $\mathscr{T}_\omega$. Given positive constants C, λ and two sequences of positive numbers $r_k \nearrow +\infty$ and $a_k \nearrow +\infty$ $(k=0, 1, \ldots)$, $a_0=0$, we define

$$\Lambda_k = \{\zeta \in \boldsymbol{C}^n : a_k\omega(\xi) \leqq |\eta| \leqq a_{k+1}\omega(\xi)\}, \tag{23}$$

and

$$\begin{aligned} &\mathscr{U}(C, \lambda, r_k, a_k) \\ &\quad = \Big\{\varphi \in \mathscr{D}_\omega : \sup_{\zeta\in\Lambda_k} (|\hat{\varphi}(\zeta)| \exp(\lambda\omega(\xi)-r_k|\eta|) \leqq C \text{ for } k = 0, 1, \ldots\Big\} \end{aligned} \tag{24}$$

The system of all such sets $\mathscr{U}(C, \lambda, r_k, a_k)$ defines a locally convex topology $\mathscr{T}_\omega$. Indeed, these sets are obviously closed and convex. Thus we only have to check that they are absorbing. Let $\varphi \in \mathscr{D}_\omega$. Then, by Remark 2, for some A and for any $\sigma > 0$, there exists a constant $C_\sigma > 0$ such that

$$|\hat{\varphi}(\zeta)| \leqq C_\sigma \exp(-\sigma\omega(\xi)+A|\eta|).$$

Furthermore, for k_0 large enough, we have $r_{k_0} \geqq A$. Thus, for $\sigma \geqq \lambda$, the function $(C\cdot C_\sigma^{-1})\varphi$ satisfies the inequalities defining the set $\mathscr{U}(C, \lambda, r_k, a_k)$ for $k \geqq k_0$. For the remaining strips Λ_k we have $|\eta| \leqq a_{k_0}\cdot\omega(\xi)$ and therefore by choosing $\sigma \geqq \lambda + A\cdot a_{k_0}$, we obtain

$$|\hat{\varphi}(\zeta)| \leqq C_\sigma \exp(-\sigma\omega(\xi)+A|\eta|) \leqq C_\sigma \exp(-\lambda\omega(\xi)).$$

Hence $(C\cdot C_\sigma^{-1})\varphi \in \mathscr{U}(C, \lambda, r_k, a_k)$.

Topology $\mathscr{T}_k$. Let $H_s \nearrow +\infty$ be any strictly concave sequence of positive numbers, i.e. $\{H_s\}_{s=1}$ is dominated by any linear function of s. Choose a positive μ and a sequence ε_s tending to zero so fast that the series

$$k(\zeta) = \sum_{s=1}^{+\infty} \varepsilon_s \exp[-(s+\mu)\omega(\xi)+H_s|\eta|] \tag{25}$$

is convergent for all $\zeta \in \boldsymbol{C}^n$. Any such function is obviously positive and continuous in the whole space $\boldsymbol{C}^n$. We set

$$\mathscr{V}(k) = \{\varphi \in \mathscr{D}_\omega : |\hat{\varphi}(\zeta)| \leqq k(\zeta)\ (\forall\zeta \in \boldsymbol{C}^n)\}. \tag{26}$$

The topology $\mathscr{T}_k$ is defined by all such sets $\mathscr{V}(k)$.

Topology $\mathscr{T}_{k,\Sigma}$. This topology is defined by the following fundamental system of neighborhoods of the origin:

$$\begin{aligned} \mathscr{W}_\Sigma(k) = \{&\varphi \in \mathscr{D}_\omega\text{: there exists a positive integer } N_\varphi \text{ such that} \\ &\varphi = \textstyle\sum_{s=1}^{N_\varphi} \varphi_s, \text{ where } \varphi_s \in \mathscr{D}_\omega \text{ and} \\ &|\hat{\varphi}_s(\zeta)| \leqq \varepsilon_s \exp[-(s+\mu)\omega(\xi)+H_s|\eta|],\ s = 1, 2, \ldots, N_\varphi,\ \zeta \in \boldsymbol{C}^n\}, \end{aligned} \tag{27}$$

where k is any function defined in (25).

THEOREM 3. *All four topologies* $\mathscr{T}$, $\mathscr{T}_\omega$, $\mathscr{T}_k$ *and* $\mathscr{T}_{k,\Sigma}$ *defined above on the space* $\mathscr{D}_\omega$ *coincide.*

The proof of Theorem 3 will follow from Propositions 4, 5 and 6.

PROPOSITION 4. *The topologies $\mathscr{T}$ and $\mathscr{T}_{k,\Sigma}$ coincide.*

Proof. The inclusion $\mathscr{T}_{k,\Sigma} \prec \mathscr{T}$ follows from the fact that the sets $\mathscr{W}_\Sigma(k)$ are convex and absorb bounded sets in the topology $\mathscr{T}$, which, as an (LF)-topology, is bornologic. In order to prove the opposite relation, $\mathscr{T}_{k,\Sigma} \succ \mathscr{T}$, consider a convex neighborhood $\mathscr{X}$ of the origin in the topology $\mathscr{T}$. Then for some $\delta_s > 0$ and positive integers $\lambda'_s \nearrow +\infty$ we have

$$\mathscr{X} \cap \mathscr{D}_\omega(K_s) \supseteq \{\varphi : |||\varphi|||_{\lambda'_s} \leqq \delta_s\}.$$

Now we are going to define a new sequence of integers $\lambda_s \geqq \lambda'_s$ as follows: Set $\lambda_1 = \lambda'_1$, $\lambda_2 = \lambda'_2$ and let p_1 be the segment with endpoints $[0, 0]$ and $[\lambda_2, R_1]$ in the plane $[\lambda, R]$. Next, consider a halfray originating at $[\lambda_2, R_1]$ whose slope is half the slope of p_1. Let A be a point on this halfray whose second coordinate is R_2. Then if we write $A = [\nu, R_2]$, we define λ_3 as the integral part of the number $1 + \max\{\lambda'_3, \nu\}$, and p_2 as the segment with endpoints $[\lambda_2, R_1]$ and $[\lambda_3, R_2]$. Repeating this procedure countably often, we obtain a broken line $p_1 \cup p_2 \cup p_3 \cup \cdots$ whose equation $R = \psi(\lambda)$ in the $[\lambda, R]$-plane is such that the function ψ is concave and dominated by any linear function of λ. Now we set $\mu = \lambda_1$, $H_j = \psi(j)$ and we choose $\varepsilon_j \leqq \delta_j 2^{-j}$ so small that the corresponding function $k(\zeta)$ is well defined. Let φ be any function in $\mathscr{W}_\Sigma(k)$, i.e. for some N we can write $\varphi = \sum_{j=1}^N \varphi_j = \sum_{j=1}^N (2^j \varphi_j)/2^j$ with

$$\sup_\zeta \{|\hat{\varphi}_j(\zeta)| \exp[(j+\mu)\omega(\xi) - H_j|\eta|]\} \leqq \varepsilon_j.$$

Then for $\lambda_s \leqq j < \lambda_{s+1}$ we obtain $\operatorname{supp} \varphi_j \subseteq K_s$ and $|||\varphi|||_{\lambda_s} \leqq \delta_s$. By the choice of μ, the same holds for $j = 1, \ldots, \lambda_1$.

PROPOSITION 5. *The topologies $\mathscr{T}_\omega$ and $\mathscr{T}_k$ are the same.*

Proof. First we shall prove that $\mathscr{T}_\omega \prec \mathscr{T}_k$. Therefore, given a neighborhood $\mathscr{U}(C, \lambda, r_k, a_k)$, we have to find a function $k(\zeta)$ so that $\mathscr{V}(k) \subset \mathscr{U}(C, \lambda, r_k, a_k)$.

By induction we shall construct a differentiable, even, convex function $p(t)$ defined on $\boldsymbol{R}$ as follows: Set $s_0 = \alpha_0 = 0$, $p(0) = 0$ and find an integer $s_1 > 0$ so that (i) $p(r_1) = s_1$; (ii) if $[\alpha_1, -1]$ is the normal vector to the graph of the function p at the point $[r_1, s_1]$, then $\alpha_1 > a_1$ and for some q_1, $\alpha_1 = a_{q_1}$. Assume that we have already found the integers $s_1 < s_2 < \cdots < s_m$ and the function p defined on the interval $[-r_m, r_m]$ so that

$$p(r_k) = s_k, \qquad k = 1, \ldots, m, \tag{28}$$

if $[\alpha_k, -1]$ is the normal vector to the graph of p at the point $[r_k, s_k]$, then for some q_k

$$\alpha_k = a_{q_k} > a_k. \tag{29}$$

Let us note that (28) implies that for $B_k=\{x : |x| \leqq r_k\}$ we have $B_k=\{x : p(|x|) \leqq s_k\}$ and by convexity $\alpha_1<\alpha_2<\cdots<\alpha_m$. Now it is obvious that by taking s_{m+1} sufficiently large, the function p can be extended as a convex function to the interval $[-r_{m+1}, r_{m+1}]$ and so that (28) and (29) hold for $k=1, 2, \ldots, m+1$.

Now we define H_j by

$$p(H_s) = s, \qquad s = 1, 2, \ldots. \tag{30}$$

Thus by (28), $H_{s_l}=r_l$, and by (29) the sequence H_s is strictly concave. We choose

$$\mu = \lambda+a_1 \tag{31}$$

and $\varepsilon_s \searrow 0$ so that $k(\zeta)$ is well defined and

$$\sum_{s=1}^{\infty} \varepsilon_s < \frac{C}{2}. \tag{32}$$

Take $\zeta \in \boldsymbol{C}^n$ arbitrary and write $\zeta=\xi+i\eta=\xi+i\theta\omega(\xi)$. Then for some integers l, q

$$\alpha_l \leqq |\theta| < \alpha_{l+1} \quad \text{and} \quad a_q \leqq |\theta| < a_{q+1}; \tag{33}$$

thus $\zeta \in \Lambda_q$. Assume first $\theta \neq 0$. We shall prove that

$$k(\zeta) \leqq C \exp(-\lambda\omega(\xi)+r_{l+1}|\eta|). \tag{34}$$

However, if we write $k(\zeta)=\sum_{s=1}^{s_l}+\sum_{s=s_l+1}^{+\infty}$, then obviously

$$\sum_{s=1}^{s_l} \cdots \leqq \frac{C}{2} \exp(-\lambda\omega(\xi)+r_l|\eta|). \tag{35}$$

Because of the symmetry of the function $p(|x|)$ there exists a point x, $r_l \leqq |x| < r_{l+1}$ and such that $[\theta, -1]$ is the normal vector to the graph of $p(|x|)$ at the point $[x, p(|x|)]$. Thus, by convexity of p, the scalar product

$$\{[\tilde{x}, p(|\tilde{x}|)]-[x, p(|x|)]\} \circ [\theta, -1] \leqq 0, \tag{36}$$

for any $\tilde{x} \in \boldsymbol{R}^n$. Again by the symmetry of the function p, $x=|x|\cdot\theta/|\theta|$, and we obtain from (36) that for $\tilde{x}=H_s\cdot\theta/|\theta|$, $p(|\tilde{x}|)=s$ $(s>s_l)$,

$$(H_s-|x|)\cdot|\theta| \leqq s-p(|x|) \leqq s-s_l;$$

whence

$$\begin{aligned} &\sum_{s=s_l+1}^{\infty} \varepsilon_s \exp\{(\lambda-s-\mu)\omega(\xi) + (H_s-r_{l+1})|\eta|\} \\ &\qquad \leqq \sum \varepsilon_s \exp\{-s\omega(\xi)+(H_s-|x|)|\theta|\omega(\xi)\} \\ &\qquad \leqq \left(\sum \varepsilon_s\right) \exp(-s_l) < \frac{C}{2}. \end{aligned} \tag{37}$$

Therefore by (35) and (37), inequality (34) is verified.

If $l \geqq 1$, then by (29) and (33), $l < q_l \leqq q$ so that (34) implies

$$k(\zeta) \leqq C \exp(-\lambda\omega(\xi) + r_q|\eta|) \qquad (\zeta \in \Lambda_q). \tag{38}$$

If $l=0$ and $q \geqq 1$, (38) still holds. Finally, if $l=q=0$, then $|\eta| \leqq a_1\omega(\xi)$ and by (31) and (32) we obtain the even sharper inequality

$$k(\zeta) \leqq Ce^{-\lambda\omega(\xi)} \qquad (\zeta \in \Lambda_0). \tag{39}$$

From (38) and (39) we obtain $\mathscr{V}(k) \subset \mathscr{U}(C, \lambda, r_k, a_k)$.

Now we have to prove $\mathscr{T}_k \prec \mathscr{T}_\omega$, i.e. given $k(\zeta)$ as in (25) we have to find $\mathscr{U}(C, \lambda, r_j, a_j)$ contained in $\mathscr{V}(k)$. However, for this it is sufficient to choose $C = \min(1, \varepsilon_1)$, $\lambda = \mu + 1$, $r_0 = H_1/2$, $r_s = H_s$ for $s \geqq 1$ and a sequence $a_s \nearrow +\infty$ such that

$$a_s \geqq (s - \log \varepsilon_{s+1})/(H_{s+1} - H_s)$$

and this proves Proposition 5.

PROPOSITION 6. *The topologies $\mathscr{T}$ and $\mathscr{T}_\omega$ coincide.*

Proof. Obviously we have $\mathscr{T} \succ \mathscr{T}_\omega$ since $\mathscr{T}$ is barrelled.

For the sake of simplicity, we shall limit ourselves to the case $\boldsymbol{R}^n = \boldsymbol{R}$ in the proof of the opposite inclusion $\mathscr{T} \prec \mathscr{T}_\omega$. Let $\mathscr{Z}$ be a closed convex neighborhood of zero in the topology $\mathscr{T}$. Thus for some $\varepsilon_s \searrow 0$ and $\lambda_s \nearrow +\infty$,

$$\mathscr{Z} \cap \mathscr{D}_\omega(B_s) \supseteq \left\{\varphi \in \mathscr{D}_\omega(B_s) : \sup_{\xi \in \boldsymbol{R}^n} (|\hat{\varphi}(\xi)| \exp(\lambda_s\omega(\xi))) \leqq \varepsilon_s\right\}$$

where B_s denotes the interval $[-s, s]$. We want to find $\mathscr{U}(C, \lambda, r_k, a_k) \subset \mathscr{Z}$.

According to [4] we can find a partition of unity $\{\alpha_n\}$ $(n = \ldots, -1, 0, 1, \ldots)$ in the space $\mathscr{D}_\omega$ such that $\operatorname{supp} \alpha_{-n} \subseteq I_{-n} = [-n-2, -n]$ $(n = 1, 2, \ldots)$, $\operatorname{supp} \alpha_0 \subseteq I_0 = [-2, 2]$, $\operatorname{supp} \alpha_n \subseteq I_n = [n, n+2]$ $(n = 1, 2, \ldots)$. By (22) there exist constants $C_{\sigma,n}$ such that

$$|\hat{\alpha}_n(\zeta)| \leqq C_{\sigma,n} \exp(-\sigma\omega(\xi) + H_n(\eta) + |\eta|/4), \tag{40}$$

where $H_n(\eta)$ is the supporting function of the interval I_n.

Take a $\varphi \in \mathscr{U}(C, \lambda, r_k, a_k)$ where C, λ, r_k, a_k will be determined later. Then $\varphi = \sum \alpha_n\varphi$ where the sum is actually finite and $\operatorname{supp}(\alpha_n\varphi) \subseteq B_{n+2}$.

Let us estimate the norm $\|\varphi\alpha_0\|_{\lambda_2}^{(\omega)}$. By the subadditivity of ω we obtain

$$\begin{aligned} |\widehat{\varphi\alpha_0}(\xi)| &\leqq \int |\hat{\varphi}(\xi - t)| \cdot |\hat{\alpha}_0(t)|\, dt \\ &\leqq C \cdot C_{\sigma,0} \int \exp(-\lambda\omega(\xi - t) - \sigma\omega(t))\, dt \qquad (41) \\ &\leqq C \cdot C_{\sigma,0} \int \exp[(\sigma - \lambda)\omega(\xi - t) - \sigma\omega(\xi)]\, dt. \end{aligned}$$

Thus for $\sigma=\lambda_2$, C small and $\lambda=\lambda_2+2/a$ where a is the constant in (γ), we get from (41)

$$|\widehat{\varphi\alpha_0}(\xi)| \leqq \varepsilon_2 \exp(-\lambda_2\omega(\xi))/3. \tag{42}$$

Therefore $\varphi\alpha_0 \in \mathscr{Z}/3$.

Now take $n \geqq 1$. Then

$$\widehat{\varphi\alpha_n}(\xi) = \int_{-\infty}^{+\infty} \hat{\varphi}(\xi-t)\hat{\alpha}_n(t)\,dt = \int_{\Gamma_\xi} \hat{\varphi}(\xi-\tau)\hat{\alpha}_n(\tau)\,d\tau \tag{43}$$

where we set $\Gamma_\xi=\{\tau=t+i\eta : \eta=-a_n\omega(\xi-t) \text{ for } t\in \boldsymbol{R}\}$. Here we can change the integration path from $\boldsymbol{R}$ to Γ_ξ. Indeed, since $\varphi\in\mathscr{D}_\omega$, we can find A such that for each $\rho>0$ there is a constant C_ρ such that

$$\left|\int_{-a_n\omega(\xi-t)}^{0} \hat{\varphi}(\xi-t-i\eta)\hat{\alpha}_n(t+i\eta)\,d\eta\right| \leqq C_\rho a_n\omega(\xi-t)\exp(-\rho\omega(t))\exp\{\omega(\xi-t)[-\rho+Aa_n+(n+2)a_n]\}.$$

Thus for $\rho=A\cdot a_n+(n+2)a_n+1$ the last term is $\leqq C_\rho a_n\omega(\xi-t)e^{-\rho\omega(t)}e^{-\omega(\xi-t)}$ which tends to zero for $|t|\to+\infty$.

Now by (43) we have

$$|\widehat{\varphi\alpha_n}(\xi)| \leqq C\cdot C_{\sigma,n}\int \exp\{\omega(\xi-t)[-\lambda+r_na_n-(n-a_n1/4)]-\sigma\omega(t)\} \cdot(1+a_n\omega'(t))\,dt. \tag{44}$$

Here we have used inequality (40) and the relation $H_n(-a_n\omega(\xi-t))=-n\cdot a_n\omega(\xi-t)$. However by Remark 1, $|\omega'(t)|\leqq M(1+|t|)^2$ so that by (44) and the subadditivity of ω we obtain

$$|\widehat{\varphi\alpha_n}(\xi)| \leqq C\cdot C_{\sigma,n}e^{-\sigma\omega(\xi)}\int \exp\{\omega(\xi-t)[-\lambda+r_n\cdot a_n+\sigma-na_n+a_n/4]\} \cdot\{1+a_nM(1+|t|)^2\}\,dt. \tag{45}$$

Now we choose $\sigma=\lambda_{n+2}$, $r_n=n-3/4$, $a_n\geqq 4\lambda_{n+2}+16/a$, where a is the constant appearing in (γ). Thus for some constant B we have

$$|\widehat{\varphi\alpha_n}(\xi)| \leqq B\cdot a_ne^{-a_n/4}\exp(-\lambda_{n+2}\omega(\xi)) \leqq (1/3\cdot 2^n)\cdot\varepsilon_{n+2}\exp(-\lambda_{n+2}\omega(\xi)), \tag{46}$$

if a_n is chosen large enough. Therefore $2^n\varphi\alpha_n\in\mathscr{Z}/3$. If $n<0$, then by deforming the path to $\eta=a_{-n}\omega(\xi-t)$ and with the same choice of constants a_n, r_n we obtain $2^{|n|}\varphi\alpha_n\in\mathscr{Z}/3$. Finally we get

$$\varphi = \sum_{-\infty}^{\infty}\varphi\alpha_n = \sum_{-\infty}^{\infty}\frac{1}{2^{|n|}}(2^{|n|}\varphi\alpha_n)\in\mathscr{Z},$$

and our proposition is proved.

REMARK 4. For $n>1$, the proof goes along the same lines.

Since trivially $\mathscr{T}_k\prec\mathscr{T}_{k,\Sigma}$, Propositions 4, 5 and 6 imply Theorem 3.

DEFINITION 2. The space $\mathscr{D}'_\omega$ dual to the space $\mathscr{D}_\omega$ is called the space of Beurling distributions corresponding to the function ω.

COROLLARY 1. *The space $\mathscr{D}'_\omega$ is an AU-space with the AU-structure given by the family $\{k\}$ where the functions k are defined as in* (25).

The reader has probably noticed that we proved more inclusions between the above topologies than were necessary for the proof of Theorem 3. On the other hand we were unable to prove directly the inclusions $\mathscr{T}_{k,\Sigma} \prec \mathscr{T}_\omega$, $\mathscr{T} \prec \mathscr{T}_k$, $\mathscr{T}_{k,\Sigma} \prec \mathscr{T}_k$. Such a direct proof of the last inclusion would presumably give a more precise form to the following consequence of Theorem 3:

COROLLARY 2. *Given the function*

$$k(\zeta) = \sum_{s=1}^{\infty} \varepsilon_s \exp\left[-(s+\mu)\omega(\xi)+H_s|\eta|\right]$$

as above, there exists

$$\tilde{k}(\zeta) = \sum_{s=1}^{\infty} \tilde{\varepsilon}_s \exp\left[-(s+\tilde{\mu})\omega(\xi)+\tilde{H}_s|\eta|\right]$$

such that if $\varphi \in \mathscr{D}_\omega$ and $|\hat{\varphi}(\zeta)| \leqq \tilde{k}(\zeta)$ for all $\zeta \in \boldsymbol{C}^n$, then $\hat{\varphi}$ can be written as

$$\hat{\varphi} = \sum_{s=1}^{N} \hat{\varphi}_s, \qquad N = N(\varphi),$$

where $\varphi_s \in \mathscr{D}_\omega$ and

$$|\hat{\varphi}_s(\zeta)| \leqq \varepsilon_s \exp\left[-(s+\mu)\omega(\xi)+H_s|\eta|\right].$$

This corollary represents an analog of a lemma due to Mac Intyre (cf. [6, p. 80]). A similar idea is used by B. A. Taylor [31] to find AU-structures in certain AU- and (DF)-spaces of entire functions. However Taylor uses the technique of L^2-estimates of the $\bar{\partial}$-operator (cf. [22]). In our case this does not seem to work.

REMARK 5. It can be shown that the topology $\mathscr{T}_k$ is particularly suitable for the study of division problems in the space $\mathscr{D}'_\omega$ (cf. [11]).

REMARK 6. The idea of introducing convex functions p into estimates of Fourier transforms goes back to B. Malgrange and L. Schwartz [26]. Proofs of Propositions 4 and 5 generalize easily to the case when, instead of taking $\mathscr{D}_\omega = \mathscr{D}_\omega(\boldsymbol{R}^n)$, one considers $\mathscr{D}_\omega(\Omega)$, where Ω is an arbitrary open convex set in $\boldsymbol{R}^n$. The proof of Proposition 6 in this more general case seems to be technically very involved. One of the reasons is the difficulty of giving an intrinsic characterization of elements of the space $\mathscr{D}_\omega$, i.e. a characterization of elements in $\mathscr{D}_\omega$ without using the Fourier transform. A characterization in terms of the theory of approximations was recently given by G. Björck [5]. However it would be very interesting and undoubtedly difficult to find a characterization of the Beurling test functions in terms of conditions imposed on the derivatives of such functions. In the case $\mathscr{D}_\omega = \mathscr{D}$, where such characterization is well known, we have carried out the proof of the

equality $\mathscr{T}=\mathscr{T}_k$ in a slightly different way [10]. Further properties of Beurling spaces are investigated in another paper which we are preparing.

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