TOPOLOGIES FOR $2^X$; SET-VALUED FUNCTIONS AND THEIR GRAPHS(\(^{(1)}\))

BY

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Abstract. We consider the problem of topologizing $2^X$, the set of all closed subsets of a topological space $X$, in such a way as to make continuous functions from a space $Y$ into $2^X$ precisely those functions with closed graphs. We show there is at most one topology with this property, and if $X$ is a regular space, the existence of such a topology implies that $X$ is locally compact. We then define the compact-open topology for $2^X$, which has the desired property for locally compact Hausdorff $X$. The space $2^X$ with this topology is shown to be homeomorphic to a space of continuous functions with the well-known compact-open topology. Finally, some additional properties of this topology are discussed.

1. Introduction. Given a topological space $X$, let $2^X$ denote the set of all closed subsets of $X$ (including the empty set $\emptyset$). We will consider the problem of putting a topology on the set $2^X$ which has a natural relationship to the given topology on $X$. In particular we choose conditions for the topology on $2^X$ which will identify continuous maps from spaces $Y$ into $2^X$ in terms of the topologies of $Y$ and $X$.

Let $Y$ be a topological space, and $F: Y \to 2^X$. Define the graph of $F$ to be the set $G_F = \{(x, y) \in X \times Y | x \in F(y)\}$. We say a topology on $2^X$ is admissible if for any space $Y$ and any continuous map $F: Y \to 2^X$, $G_F$ is a closed subset of $X \times Y$. Conversely, we say a topology on $2^X$ is proper if for any space $Y$ and function $F: Y \to 2^X$, if $G_F$ is closed then $F$ is continuous. We will consider the problem of giving $2^X$ a topology which is both admissible and proper, i.e., one for which continuity of functions is equivalent to their having closed graphs. (Recall that for point-valued functions into a compact Hausdorff space $X$ it is in fact true that the properties of continuity and closed graph are equivalent.)

In §2 we consider some properties of proper and admissible topologies for $2^X$. In particular, we will show that there is at most one topology for $2^X$ which is both proper and admissible. In addition if $X$ is a regular space ($T_1$, and points and closed...
sets can be separated), the existence of a proper-admissible topology for \(2^X\) implies that \(X\) is locally compact. In §3 we define a topology for \(2^X\) (called the compact-open topology) which for Hausdorff spaces \(X\) is always proper, and is admissible if and only if \(X\) is locally compact. In §4 we will show that we can weaken the restriction on \(X\), and instead require that \(X \times Y\) be a \(k\)-space, and this will imply that \(F: Y \to 2^X\) is continuous in the compact-open topology if and only if \(G_F\) is closed. We will also describe some other properties of this topology.

2. Admissible and proper topologies. We begin by giving a characterization of an admissible topology.

**Lemma 2.1.** A topology on \(2^X\) is admissible if and only if
\[
\Omega = \{(x, A) \in X \times 2^X \mid x \in A\}
\]
is closed in \(X \times 2^X\).

**Proof.** If the topology is admissible, then \(\Omega\) is closed since it is the graph of the identity from \(2^X\) to itself, which is obviously continuous.

Suppose now that \(\Omega\) is closed. Let \(Y\) be any space and \(F: Y \to 2^X\) be continuous. Then \(G_F = (1, F)^{-1}(\Omega)\) where \((1, F): X \times Y \to X \times 2^X\) is given by \((1, F)(x, y) = (x, F(y))\) and is continuous. Hence \(G_F\) is closed.

If \(t_1\) and \(t_2\) are two topologies for the same set, and \(t_1 \subseteq t_2\), we say \(t_1\) is weaker than \(t_2\), and \(t_2\) is stronger than \(t_1\). When more than one topology is being considered at one time, we will denote by \(2^X(t)\), the space \(2^X\) with the topology \(t\).

**Lemma 2.2.** (a) A topology stronger than an admissible topology is also admissible.
(b) A topology weaker than a proper topology is also proper.
(c) Any admissible topology is stronger than any proper topology.

**Proof.** Parts (a) and (b) are clear. To prove (c), let \(t_1\) be a proper topology and \(t_2\) be an admissible topology. Then by Lemma 2.1, \(\{(x, A) \mid x \in A\}\) is closed in \(X \times 2^X(t_2)\), which implies \(1: 2^X(t_2) \to 2^X(t_1)\) is continuous. Thus \(t_1 \subseteq t_2\).

From Lemma 2.2(c) we obtain the uniqueness of a topology which is both proper and admissible.

**Theorem 2.3.** For any topological space \(X\), \(2^X\) can have at most one topology which is both proper and admissible. Such a topology is necessarily the strongest proper topology on \(2^X\) and the weakest admissible topology on \(2^X\). (By "strongest" or "weakest" we mean "containing" or "contained in" all other such topologies.)

We are thus naturally led to consider the question of when \(2^X\) can have a strongest proper topology or a weakest admissible topology. Such topologies must necessarily be unique. The first question is easily settled.

**Proposition 2.4.** For any \(X\), \(2^X\) has a strongest proper topology.

**Proof.** Let \(\{t_a\}\) be the family of all proper topologies. Let \(t\) be the topology which has the set \(\bigcup_a t_a\) as a subbasis. Since this is stronger than any proper topology, we need only show that it is proper.
Let \( Y \) be any space, and \( F: Y \to 2^X \) be such that \( G_F \) is closed. Since any subbasic \( V \in t \) is an open set for some proper topology, \( F^{-1}(V) \) must be open and hence \( F \) is continuous.

On the question of a weakest admissible topology, we first state the following proposition whose proof is clear.

**Proposition 2.5.** If \( 2^X \) has a weakest admissible topology, it must be the intersection of all admissible topologies.

We now prove the main result of this section.

**Theorem 2.6.** If \( X \) is a regular space and \( 2^X \) has a weakest admissible topology then \( X \) is locally compact.

**Proof.** Let \( t \) be the weakest admissible topology on \( 2^X \). Let \( x \in X \). We will show that \( x \) has a relatively compact neighborhood.

Since \( t \) is admissible, there is an open set \( V \subset X \) and a \( \mathcal{U} \in t \) such that \( x \in V \), \( \emptyset \in \mathcal{U} \) and \( V \times \mathcal{U} \subset \Omega \), the complement in \( X \times 2^X \) of the set \( \Omega \) defined in Lemma 2.1. We will show that \( V \), the closure of \( V \), is compact.

Let \( \mathcal{A} \) be an open cover of \( V \), and let \( \mathcal{A}^* = \mathcal{A} \cup \{X - V\} \), an open cover of \( X \). Let \( \{F_a\} \) be the collection of all closed subsets of \( X \) each of which is contained in some member of \( \mathcal{A}^* \), and let the sets \( \{A \in 2^X \mid A \subset X - F_a\} \) be a subbasis for a topology \( t' \) on \( 2^X \). We will show now that \( t' \) is admissible.

Suppose \( y \in X \), \( B \in 2^X \) and \( y \notin B \). There is a \( A \in \mathcal{A} \) such that \( y \in A \). Since \( X \) is regular, there is an open set \( W \subset X \) such that \( y \in W \subset W \subset A \cap A^* \).

Thus, \( W \subset A^* \) implies \( W \in \{F_a\} \), and \( W \subset B \) implies that \( B \subset X - W \). Hence, \( X \times 2^X(t') \) is an open set in \( X \times 2^X(t') \), \( (y, B) \in Z \) and \( Z \subset \Omega \), showing that \( \Omega \) is closed in \( X \times 2^X(t') \), and hence that \( t' \) is admissible.

Since \( t \) is the weakest admissible topology, we must have \( t \subset t' \) and thus \( \mathcal{U} \in t' \). From this it follows that there exist \( F_1, \ldots, F_n \) in \( \{F_a\} \) such that

\[
\emptyset \in \bigcap_{i=1}^{n} \{A \mid A \subset X - F_i\} = \left\{A \mid A \subset X - \bigcup_{i=1}^{n} F_i\right\} \subset \mathcal{U}.
\]

We show that \( V \subset \bigcup_{i=1}^{n} F_i \). Suppose, to the contrary, that there is a \( y \in V - \bigcup F_i \). We have then that \( \{y\} \in \{A \mid A \subset X - \bigcup F_i\} \subset \mathcal{U} \) and therefore \( (y, \{y\}) \in V \times \mathcal{U} \subset \Omega \). However, this is impossible since \( y \in \{y\} \), and thus we have \( V \subset \bigcup F_i \). It follows that also \( \overline{V} \subset \bigcup F_i \).

To complete the proof, for \( i = 1, \ldots, n \) let \( V_i \in \mathcal{A}^* \) be such that \( F_i \subset V_i \). Therefore \( \overline{V} \subset \bigcup_{i=1}^{n} V_i \), and we have produced a finite subcover of \( \mathcal{A}^* \). We also have a finite subcover of \( \mathcal{A} \) since we can omit \( X - \overline{V} \) if it occurs among the \( V_i \).

**Corollary 2.7.** Let \( X \) be a regular space. If the intersection of all admissible topologies for \( 2^X \) is admissible, then \( X \) is locally compact.
Corollary 2.8. Let $X$ be a regular space. If $2^X$ has a topology which is both proper and admissible, then $X$ is locally compact.

In the next section we will show that converses of Theorem 2.6 and the corollaries are true. That is, we will show that if $X$ is a locally compact Hausdorff space, then it has a proper-admissible topology, and hence a weakest admissible topology. (It is clearly regular.)

The problem remains as to whether the hypothesis on $X$ can be weakened, say, to Hausdorff. Theorem 2.6 says that if $X$ is a regular space which is not locally compact (e.g. $X = \mathbb{Q}$, the space of rationals in the real line), then $2^X$ has no weakest admissible topology (and thus no proper-admissible topology). An open question is whether a nonregular Hausdorff space can have a weakest admissible topology.

Before proceeding to the next section, we will study an interesting consequence of the proof of Theorem 2.6. If $X$ is regular, and $\mathcal{A}$ is a cover of $X$ by open sets, then $\mathcal{A}$ gives rise to an admissible topology, $t_\mathcal{A}$ on $2^X$ as follows. Let $\{F_a\}$ denote the set of closed subsets of $X$ each of which is contained in some member of $\mathcal{A}$. Let $t_\mathcal{A}$ be the topology on $2^X$ whose subbasis consists of all sets of the form

$$\{A \in 2^X \mid A \subset X - F_a\}.$$

Proposition 2.9. (a) Let $X$ be a regular space. For each open cover $\mathcal{A}$ of $X$, $t_\mathcal{A}$ is an admissible topology on $2^X$.

(b) If $\mathcal{A}$ and $\mathcal{B}$ are open covers, and $\mathcal{A}$ is a refinement of $\mathcal{B}$, then $t_\mathcal{A} \leq t_\mathcal{B}$.

(c) If $\mathcal{A}$ is an open cover which is closed under finite unions, then the sets

$$\{A \in 2^X \mid A \subset X - F_a\}$$

form a basis for $t_\mathcal{A}$.

(d) Let $X$ be a normal space. If $\mathcal{B}$ is an open cover, and $\mathcal{A}$ is an open cover consisting of all elements of $\mathcal{B}$ and finite unions of elements of $\mathcal{B}$, then $t_\mathcal{A} = t_\mathcal{B}$ and hence a basis for $t_\mathcal{B}$ is formed by the sets given in (c).

Proof. The proof of (a) is contained in the proof of Theorem 2.6. Part (b) follows from the fact that if $\mathcal{A}$ is a refinement of $\mathcal{B}$ then we have $\{F_a\} \subseteq \{F_b\}$ where these are the respective collections used in the construction of $t_\mathcal{A}$ and $t_\mathcal{B}$. Part (c) follows from the fact that

$$\bigcap_{i=1}^{n} \{A \mid A \subset X - F_i\} = \{A \mid A \subset X - \bigcup_{i=1}^{n} F_i\}.$$

For part (d), since $\mathcal{B}$ is a refinement of $\mathcal{A}$, we have $t_\mathcal{A} \leq t_\mathcal{B}$. That $t_\mathcal{B} \subseteq t_\mathcal{A}$ follow from the fact that in a normal space if $F$ is closed and $F \subseteq A_1 \cup \cdots \cup A_n$, where the $A_i$ are open, then there exist closed sets $F_i$ such that $F_i \subseteq A_i$ and $F = F_1 \cup \cdots \cup F_n$.

We remark here that if $X$ is any Hausdorff space, and if we take the open cover $\mathcal{A} = \{X\}$, then the collection $\{F_a\}$ consists of all closed subsets of $X$ and $t_\mathcal{A}$ is the well-known upper semifinite topology. See Michael [7, p. 179] and Ponomarev [9] (where it is called the $\kappa$ topology). Notice that for any open cover $\mathcal{B}$, $t_\mathcal{B} \leq t_\mathcal{A}$. Also, since for a regular space the weakest admissible topology $t$ is weaker than $t_\mathcal{A}$, this topology $t$ will not be Hausdorff.
In the next section we shall consider a topology for \( \mathbb{R}^2 \) which for locally compact Hausdorff \( X \) can be described as that derived from the cover of \( X \) by relatively compact open sets. It therefore must be admissible. In fact, we will show it is also proper, and thus is the weakest admissible topology.

3. Compact-open topology: \( \mathbb{R}^2 \) as a function space. Let \( X \) be a topological space, and let \( \{C_a\} \) be the set of all compact subsets of \( X \). We define the compact-open topology on \( \mathbb{R}^2 \) to be the topology with the basis consisting of sets of the form \( \{A \in \mathbb{R}^2 \mid A \subseteq X - C_a\} \).

**Theorem 3.1.** Let \( X \) be a Hausdorff space. Then the compact-open topology on \( \mathbb{R}^2 \) is admissible if and only if \( X \) is locally compact.

**Proof.** Suppose \( X \) is locally compact and let \( \mathcal{F} \) be the open cover of \( X \) consisting of all relatively compact open sets. It follows that the compact-open topology is the topology \( \tau_\mathcal{F} \), and since \( X \) is regular, Proposition 2.9(a) shows that \( \tau_\mathcal{F} \) is admissible.

Now suppose the compact-open topology is admissible and let \( U \subseteq X \) be open and let \( x \in U \). We will find a relatively compact open set \( V \subseteq X \) such that \( x \in V \subset V \subset U \).

Since \( \Omega \) is closed in \( X \times \mathbb{R}^2 \), we can find an open set \( V \subseteq X \) and a basic open set \( \mathcal{W} = \{A \in \mathbb{R}^2 \mid A \subseteq X - C\} \), \( C \) compact, so that

\[(x, X-U) \in V \times \mathcal{W} \subset \Omega.\]

It follows that \( V \subseteq C \) and thus we have

\[x \in V \subseteq V \subseteq C \subseteq U,\]

and the proof is complete.

We will also show that for any space \( X \), the compact-open topology on \( \mathbb{R}^2 \) is proper. Before doing this, however, we will discuss a perhaps more familiar space which is homeomorphic to \( \mathbb{R}^2 \) with the compact-open topology.

We let \( \mathcal{S} \) denote the space having two points, 0 and 1, for which the open sets are \( \emptyset, \mathcal{S}, \) and \( \{0\} \). (This is known as the Sierpinski space, see [2, p. 63].) Consider the set of all continuous functions from \( X \) into \( \mathcal{S} \), denoted \( \mathcal{S}^X \). For each \( f \in \mathcal{S}^X \), there corresponds a closed set \( \Gamma(f) \in \mathbb{R}^2 \) given by \( \Gamma(f) = f^{-1}(\{1\}) \). Similarly, for each closed set \( A \in \mathbb{R}^2 \), we can define \( \Phi(A) \in \mathcal{S}^X \) by

\[\Phi(A)(x) = 1 \quad \text{if} \ x \in A,\]

\[\quad = 0 \quad \text{if} \ x \notin A.\]

Clearly, \( \Gamma(\Phi(A)) = A \) and \( \Phi(\Gamma(f)) = f \). Thus we have a bijection of the set \( \mathcal{S}^X \) onto the set \( \mathbb{R}^2 \). Now consider \( \mathbb{R}^2 \) as a topological space with the compact-open topology defined earlier, and consider \( \mathcal{S}^X \) with the compact-open topology as defined for spaces of continuous functions (see, e.g., [2, p. 257]).
Lemma 3.2. If \( S^X \) and \( 2^X \) have the respective compact-open topologies, the bijection \( \Gamma: S^X \to 2^X \) is a homeomorphism.

Proof. We need only verify that \( \Gamma((f \mid f(C) \subseteq \{0\})) = \{A \mid A \subseteq X - C\} \) for any compact \( C \subseteq X \). But

\[
\Gamma((f \mid f(C) \subseteq \{0\})) = \{A \mid \Phi(A)(C) \subseteq \{0\}\} = \{A \mid A \cap C = \emptyset\},
\]

which concludes the proof.

Thus the compact-open topology on \( 2^X \) is in fact the compact-open topology on a space of continuous functions. We will use this connection to show that the compact-open topology on \( 2^X \) is proper.

Suppose \( \Omega \) is any space. For every map \( \alpha: X \times Y \to \mathcal{S} \), where \( \alpha \) is continuous on \( X \) for each fixed \( y \in Y \), there is an associated map \( \alpha: Y \to S^X \) given by \( [\alpha(y)](x) = \alpha(x, y) \). Conversely, given \( \alpha: Y \to S^X \), we can define \( \alpha: X \times Y \to \mathcal{S} \). Recall that a topology on the function space \( S^X \) is said to be proper if for any space \( Y \) and function \( \alpha: X \times Y \to \mathcal{S} \), the continuity of \( \alpha \) (jointly in \( x \) and \( y \)) implies the continuity of \( \alpha \). Similarly, a topology on \( S^X \) is said to be admissible if for any space \( Y \) and function \( \alpha: Y \to S^X \), the continuity of \( \alpha \) implies that of \( \alpha \). Equivalently, \( S^X \) has an admissible topology if and only if the evaluation map \( \omega: S^X \times X \to \mathcal{S} \), given by \( \omega(f, x) = f(x) \), is continuous. See [2, p. 274].

Since the sets \( S^X \) and \( 2^X \) are in a one-to-one correspondence, a topology on one can be viewed as a topology on the other, making \( \Gamma \) a homeomorphism. We now show that the various definitions of admissible and proper are compatible.

Lemma 3.3. A topology on the set of subsets \( 2^X \) is admissible [proper] if and only if as a topology on the function space \( S^X \) it is admissible [proper].

Proof. The proof follows from the observation that \( \alpha: X \times Y \to \mathcal{S} \) is continuous if and only if

\[
\alpha^{-1}(1) = \{(x, y) \in X \times Y \mid [\alpha(y)](x) = 1\}
\]

\[
= \{(x, y) \mid x \in \Gamma(\alpha(y))\}
\]

\[
= G_{\Gamma \ast \alpha}
\]

is closed.

We remark here that the continuity of the evaluation map \( \omega \) is equivalent to the closure of the set \( \Omega \) defined in §2.

Corollary 3.4. The compact-open topology on \( 2^X \) is always proper.

Proof. This follows by Lemmas 3.2 and 3.3 and the fact that the compact-open topology on \( S^X \) is always proper (see [2, Theorem 3.1, p. 261]).

Corollary 3.5. Let \( X \) be a Hausdorff space. Then the compact-open topology on \( S^X \) is admissible if and only if \( X \) is locally compact.

The next two results provide converses to Theorem 2.6 and its corollaries.
Theorem 3.6. If $X$ is a locally compact Hausdorff space, then the compact-open topology on $2^X$ is both proper and admissible, and hence is the weakest admissible topology on $2^X$.

Corollary 3.7. If $X$ is a locally compact Hausdorff space, then the intersection of all admissible topologies for $2^X$ is admissible.

In the next section, we will exploit further the function space structure of $2^X$.

4. Compact-open topology: Further properties. Each space of subsets and function space considered in this section will be assumed to have the respective compact-open topology.

We now change the original problem somewhat. Given spaces $X$ and $Y$ we ask under what conditions can we assert that, for any function $F: Y \to 2^X$, the continuity of $F$ (in the compact-open topology) is equivalent to its graph $G_F$ being closed.

To give an answer to that question, we must recall the definition of a $k$-space. A Hausdorff space $Z$ is said to be a $k$-space if $U \subseteq X$ is open in $X$ if and only if for every compact subset $C$ of $Z$, $U \cap C$ is open in $C$. It is known (see [2, p. 248]) that all locally compact and all first countable Hausdorff spaces (and hence all metric spaces) are $k$-spaces. Also if both $X$ and $Y$ are first countable Hausdorff spaces, or one is a locally compact Hausdorff space and the other is a $k$-space then $X \times Y$ is a $k$-space (see [2, p. 263]). This is important because of the following result.

Theorem 4.1. If $X$ is locally compact Hausdorff or if $X \times Y$ is a $k$-space, then the continuity of $F: Y \to 2^X$ is equivalent to its graph $G_F$ being closed.

Proof. For the case where $X$ is locally compact Hausdorff, this is Theorem 3.6. Since the compact-open topology is always proper, we always have that closed graph implies continuity. Thus we need only prove that if $X \times Y$ is a $k$-space, then continuity of $F$ implies $G_F$ is closed. This follows from [2, Corollary 3.2, p. 261] and the observation made in the proof of Lemma 3.3.

Theorem 4.1 tells us that, with the right conditions on $X$ or $X \times Y$, any continuous function $F: Y \to 2^X$ gives rise to a closed subset of $X \times Y$. Conversely, any closed subset $A \subseteq X \times Y$ gives rise to a continuous function $F: Y \to 2^X$, defined by $F(y) = \{x \mid (x, y) \in A\}$, having the property that $G_F = A$. Thus we have a bijection $G$ between the set of continuous functions from $Y$ into $2^X$, denoted $(2^X)^Y$, and the set of closed subsets of $X \times Y$, $2^{X \times Y}$, defined by $G(F) = G_F$.

Theorem 4.2. Let $X$ be a locally compact Hausdorff space, or let $X \times Y$ be a $k$-space. Then the function $G: (2^X)^Y \to 2^{X \times Y}$, which assigns to each continuous $F$ its graph $G_F$, is defined and is a homeomorphism.

Proof. This follows from Lemma 3.2 and [2, Theorem 5.3, p. 265].
Corollary 4.3. If $X \times Y$ is a k-space, then $(2^X)^Y$ is homeomorphic to $(2^Y)^X$. In fact the homeomorphism is given by mapping the function $F: Y \to 2^X$ to the function $F^* : X \to 2^Y$ given by $F^*(x) = \{ y \mid x \in F(y) \}$.

Corollary 4.4. Let $F : Y \to 2^X$ and let $F^*(x) = \{ y \mid x \in F(y) \}$ for all $x \in X$. If $X$ is locally compact Hausdorff, then $F$ is continuous implies $F^* : X \to 2^Y$ is defined and is continuous. If $X \times Y$ is a k-space, then $F$ is continuous if and only if $F^*$ is defined and is continuous. (Note that by $F^*$ is defined we mean $F^*(x)$ is closed for all $x \in X$.)

Thus, under the proper conditions, the study of continuous closed-set valued functions from $Y$ to $X$ is equivalent to the study of closed subsets of $X \times Y$. Conversely, consideration of closed subsets of $X \times Y$ leads to the study of continuous closed-set valued functions from one factor into the other.

The remaining results of this section describe some further continuity properties of the compact-open topology on $2^X$. The following is obvious.

Proposition 4.5. Let $X$ be a Hausdorff space. Then
(a) the map $i : X \to 2^X$ given by $i(x) = \{ x \}$ is continuous, and
(b) if $f : Y \to X$ is continuous, the map $F : Y \to 2^X$ given by $F(y) = \{ f(y) \}$ is continuous.

Theorem 4.6. Let $X$ and $Y$ be any spaces. Then the map $H : Y^X \times 2^Y \to 2^X$ given by $H(f, B) = f^{-1}(B)$ is continuous in $f$ for fixed $B$ and in $B$ for fixed $f$. If $Y$ is a locally compact Hausdorff space, then $H$ is continuous jointly in $f$ and $B$.

Proof. The theorem is true for the map $T : Y^X \times 2^Y \to 2^X$ given by $T(f, g) = g \circ f$ [2, p. 259]. Noting that if $g \in 2^Y$ is such that $B = g^{-1}(1)$, then $(g \circ f)^{-1}(1) = f^{-1}(g^{-1}(1)) = f^{-1}(B)$, the proof follows from Lemma 3.2.

We will denote by $H_B : Y^X \to 2^X$ the map $H_B(f) = H(f, B)$ and by $H_f : 2^Y \to 2^X$ the map $H(f, B) = H(f, B)$.

Corollary 4.7. Let $X$ be a Hausdorff space. Let $f : Y \to X$ be continuous. Then the map $f^{-1} : X \to 2^Y$ is continuous.

Proof. The map $H_f \circ i : X \to 2^X \to 2^Y$ is continuous, and $H_f \circ i(x) = H_f(\{ x \}) = f^{-1}(x)$.

We remark here that if $f : Y \to X$ and if $F : Y \to 2^X$ is given by $F(y) = \{ f(y) \}$, then $F^* : X \to 2^Y$ is given by $F^*(x) = \{ y \mid x \in \{ f(y) \} \} = f^{-1}(x)$, provided $F^*$ is defined, i.e., provided $f^{-1}(x)$ is closed for all $x \in X$. Thus by Corollary 4.4 we get that if $X \times Y$ is a k-space then $f^{-1}$ is defined and is continuous if and only if $F$ is continuous. Notice that $F^*$ acts in some sense as an inverse of $F$.

Theorem 4.8. Let $X$ be a compact Hausdorff space and $Y$ a k-space. Then $f : Y \to X$ is continuous if and only if $f^{-1} : X \to 2^Y$ is defined and is continuous.

Proof. By Corollary 4.7 and the preceding discussion, all we need prove is that if $F(y) = \{ f(y) \}$ is continuous then $f$ is continuous.
Let \( U \subset X \) be open. Since \( X \) is compact Hausdorff, \( \mathcal{V} = \{ A \in 2^X \mid A \subset U \} \) is open in \( 2^X \), hence the continuity of \( F \) implies \( F^{-1}(\mathcal{V}) \) is open in \( Y \). But

\[
F^{-1}(\mathcal{V}) = \{ y \in Y \mid f(y) \in U \} = f^{-1}(U).
\]

**Proposition 4.9.** The map

\[
\Sigma: 2^X \times 2^X \to 2^X,
\]

given by \( \Sigma(A, B) = A \cup B \), is continuous.

**Proof.** Notice that

\[
\{(A, B) \in 2^X \times 2^X \mid A \cup B \subset U\} = \{A \in 2^X \mid A \subset U\} \times \{B \in 2^X \mid B \subset U\}.
\]

**Corollary 4.10.** If \( F_i: Y \to 2^X \) is continuous for \( i = 1, 2 \), then \( F_1 \cup F_2: Y \to 2^X \), given by \( (F_1 \cup F_2)(y) = F_1(y) \cup F_2(y) \), is continuous.

**Proof.** Let \( F_1 \times F_2: Y \to 2^X \times 2^X \) be the product map, then \( F_1 \cap F_2 = \Sigma \circ (F_1 \times F_2) \).

**Lemma 4.11.** Let \( F_\alpha: Y \to 2^X \) and let \( \bigcap F_\alpha: Y \to 2^X \) be given by \( (\bigcap F_\alpha)(y) = \bigcap F_\alpha(y) \). Then \( G \cap F_\alpha = \bigcap G F_\alpha \).

**Proof.**

\[
G \cap F_\alpha = \{(x, y) \mid x \in \bigcap F_\alpha(y)\}
\]

\[
= \{(x, y) \mid x \in F_\alpha(y) \text{ for all } \alpha\}
\]

\[
= \bigcap \{(x, y) \mid x \in F_\alpha(y)\}
\]

\[
= \bigcap G F_\alpha.
\]

**Proposition 4.12.** Let \( X \) be a locally compact Hausdorff space or let \( X \times Y \) be a \( k \)-space. Then if \( F_\alpha: Y \to 2^X \) is continuous for all \( \alpha \), then \( \bigcap F_\alpha \) is continuous.

**Proof.** By Theorem 4.1, \( G F_\alpha \) is closed for all \( \alpha \). From Lemma 4.11, we conclude that \( G \cap F_\alpha \) is closed, hence that \( \bigcap F_\alpha \) is continuous.

The last few results suggest some applications of the continuity properties of the compact-open topology.

**Proposition 4.13.** Let \( X \) be a locally compact topological group. Then the map \( C: 2^X \times X \to 2^X \), given by \( C(A, x) = xA \), is continuous.

**Proof.** \( X \) is necessarily Hausdorff (see [8, p. 27]). We need only show \( G \) is closed. But

\[
G = \{(A, x, y) \mid y \in xA\}
\]

\[
= \{(A, x, y) \mid x^{-1}y \in A\}
\]

\[
= (1, m)^{-1}(\Omega),
\]

where \( (1, m): 2^X \times X \times X \to 2^X \times X \) is given by \( (1, m)(A, x, y) = (A, x^{-1}y) \) and is continuous. Thus \( G \) is closed.
Corollary 4.14. Multiplication by an element of a locally compact topological group $X$ is a homeomorphism of $2^X$ onto itself.

In fact, Corollary 4.14 also follows from a more general result. Recall the definition $H_f(B) = f^{-1}(B)$.

Proposition 4.15. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be continuous, then $H_{g \circ f} = H_f \circ H_g$. In particular, if $f$ is a homeomorphism of $X$ onto $Y$ then $H_f$ is a homeomorphism of $2^Y$ onto $2^X$.

Now consider the space $2^{X \times X}$ as the space of all closed binary relations $R$ on $X$. We define the maps

$$M: 2^{X \times X} \rightarrow 2^X$$

by $M(R) = \{x \in X \mid \{x\} \times X \subseteq R\}$, and

$$E: 2^{X \times X} \times X \rightarrow 2^X$$

by $E(R, x) = \{y \in X \mid (x, y) \in R, (y, x) \in R\}$. It is easy to show that $M(R)$ and $E(R, x)$ are always closed sets, thus $M$ and $E$ are defined. The map $M$ takes $R$ to its set of “maximum” elements, while the map $E$ takes $R$ and $x$ to the “equivalence class” of $x$ with respect to $R$.

Theorem 4.16. If $X$ is locally compact, then both $M$ and $E$ are continuous.

Proof. Let $\Omega_2 = \{(R, x, y) \mid (x, y) \in R\}$, by hypothesis a closed set in $2^{X \times X} \times X \times X$. Then

$$G_M = \{(R, x) \mid x \in M(R)\}$$

$$= \{(R, x) \mid \{x\} \times X \subseteq R\}$$

$$= \{(R, x) \mid (R, x, y) \in \Omega_2, \text{ for all } y \in X\}$$

$$= (2^X \times X) - p(\Omega_2),$$

where $p$ is the projection of $2^{X \times X} \times X \times X$ onto the first two coordinates. Hence $G_M$ is closed.

Also,

$$G_E = \{(R, x, y) \mid y \in E(R, x)\}$$

$$= \{(R, x, y) \mid (x, y) \in R, (y, x) \in R\}$$

$$= \Omega_2 \cap \Omega_2'$$

where $\Omega_2' = \{(R, x, y) \mid (y, x) \in R\}$ is also closed. Thus $G_E$ is closed.

5. Discussion. If we had viewed the space $2^X$ as a function space earlier, then Lemma 2.2, Theorem 2.3 and Proposition 2.4 could be seen as following from known results about function spaces [2, p. 275]. However, it seemed simpler to directly indicate their proofs, which follow the proofs in [2].
The proof of Theorem 2.6 follows Arens’ proof of the fact that if $I^X$ has a weakest (he calls it strongest) admissible topology where $I$ is the unit interval and $X$ is completely regular, then $X$ is locally compact [1, Theorem 3]. Note that what we have essentially proved is that if the function space $\mathcal{S}^X$ has a weakest admissible topology where $X$ is regular, then $X$ is locally compact.

Notice that the compact-open topology coincides with the upper-semifinite topology (defined at the end of §2 or see [7, p. 179]) when the space $X$ is compact Hausdorff.

We note here that if $X$ is a locally compact Hausdorff space with a countable base, then by [2, Theorem 5.2, p. 65], $2^X$ will have a countable base. In addition, for any space $X$, $2^X$ with the compact-open topology is trivially compact since $2^X$ itself must belong to any open cover.

The compact-open topology for $2^X$ appears as half of the generating set for Fell’s $H$-topology [4]. It was later isolated by Effros, who called it the global topology [3, p. 931]. Effros describes convergence in this topology when $X$ is a locally compact Hausdorff space. It is interesting to note that this description is equivalent to Arens’ description of convergence in $Y^X$ for locally compact Hausdorff $X$ [1, Theorem 4] when one consider $2^X$ as the space $\mathcal{S}^X$. See also [5].

Finally, an equivalent version of the compact-open topology (for spaces of open subsets) was considered by Kannai [6] in connection with some problems in Mathematical Economics. Theorem 2.6 and Theorem 3.1 would seem to indicate that Kannai’s methods will not work in spaces which are not locally compact.

References

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