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A GENERALIZATION OF FEIT'S THEOREM

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Abstract. This paper is part of a doctoral thesis at Harvard University. The title of the thesis is *Finite linear groups in six variables.*

Using the methods of this paper, I believe that I can prove that if p is a prime greater than five with $p \equiv -1 \pmod 4$, and G is a finite group with faithful complex representation of degree smaller than $4p/3$ for $p>7$ and degree smaller than 9 for $p=7$, then G has a normal p-subgroup of index in G divisible at most by p^2. These methods are particularly effective when there is nontrivial intersection of p-Sylow subgroups. In fact, if the current work people are doing on the trivial intersection case can be extended, it should be possible to show that, for p a prime and G a finite group with a faithful complex representation of degree less than $3(p-1)/2$, G has a normal p-subgroup of index in G divisible at most by p^2. (It may be possible to show that the index is divisible at most by p if the representation is primitive and has degree unequal to p.)

Introduction. In this paper all representations are assumed to be over C, the complex numbers. We use standard mathematical notation without comment. If X is a representation of the group G on the vector space V, we call a subspace U of V a homogeneous space for G if U is invariant for G and U is maximal with the property that the irreducible constituents of the representation of G on U are equivalent. The representation X is called quasiprimitive if it is irreducible, and for all normal subgroups N of G, $X|N$ has just the homogeneous subspace V. For $x \in G$ and $\gamma \in C$, $C_V(\gamma^{-1}x) = \langle v \mid v \in V, X(x)v = \gamma v\rangle$ and for $H \subset G$, $C_V(H) = \bigcap_{h \in H} C_V(h)$. The term i_Π is defined in the following theorem. This theorem generalizes the theorem in [5]. Equality is allowed by C.

THEOREM. *Let Π be a set of primes and let X be a faithful representation on the vector space V of the finite group G of degree n over the complex numbers. Define $i_\Pi(G) = |G|_\Pi / |O_\Pi(G)|$. Assume that $p \geqq n+1$, $p \geqq 7$ for all $p \in \Pi$. Assume that G has a Π-Sylow subgroup, H. Then either*:

I. *$i_\Pi(G)$ is not composite.*

II. *X is imprimitive or reducible on the spaces V_1 and V_2 of dimension $n/2$ where $V = V_1 \oplus V_2$. Also, $n+1 = p \in \Pi$ for some p. There exists a normal subgroup M of*

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G of index 1 or 2 having the V_i as invariant, irreducible subspaces. There exist subgroups T_i of M with $M=Z(M)(T_1\times T_2)$; $C_V(T_i)=V_i$; and

$$\begin{aligned} T_i &\simeq PSL(2,p), \quad p\equiv -1 \pmod 4,\\ &\simeq SL(2,p), \quad p\equiv 1 \pmod 4, \qquad \textit{for } i=1,2. \end{aligned}$$

The theorem generalizes [5] when $n+1\in\Pi$. When $n+1\notin\Pi$, an abelian Π-Sylow subgroup of G was guaranteed by Blichfeldt. When $n+1\in\Pi$, the existence of a Π-Sylow subgroup must be assumed (such a group is abelian by Lemma 7). For example $SL(2,13)$ has a representation of degree 6, but has no subgroup of order (7)(13). When $n+1\notin\Pi$, our proof uses only Lemmas 1 and 2 and furnishes a short proof of Feit's Theorem. Furthermore, when for $p\in\Pi$, $p\geqq 2n+1$, we may use our proof to prove that $i_\Pi(G)=1$ or $p=2n+1\in\Pi$ for some p and $G/Z(G)\simeq PSL(2,p)$, the result of [6], by induction on n. Here, only Lemma 1 is needed. Also, (E) and (H) follow immediately from the stronger induction hypothesis that $i_\Pi(G_0)=1$ when G_0 has a faithful representation of degree $n_0<n$. Then only steps (A), (B), (C), (F), (I), and (J) are needed to complete the proof, since [2] can be applied if $|G|_\Pi$ is prime.

LEMMA 1. *If G is a finite group and Π is a set of primes, define $i_\Pi(G)=|G|_\Pi/|O_\Pi(G)|$. Then if H is a homomorphic image or a subgroup of G, $i_\Pi(H)\mid i_\Pi(G)$. Furthermore, if K and L are finite groups, $i_\Pi(K\times L)\mid(i_\Pi(K))(i_\Pi(L))$. Finally, $i_\Pi(G)=i_\Pi(G/Z(G))$.*

Proof. If α is a homomorphism from G onto H with kernel K, then $\alpha(O_\Pi(G))\subset O_\Pi(H)$ and

$$|H|_\Pi/|\alpha(O_\Pi(G))| = [|G|_\Pi/|K|_\Pi]/[|O_\Pi(G)|/|K\cap O_\Pi(G)|]$$

which divides $|G|_\Pi/|O_\Pi(G)|$. If $H\subset G$ and β is the natural homomorphism from G to $G/O_\Pi(G)$, then $H\cap O_\Pi(G)\subset O_\Pi(H)$ and $H/H\cap O_\Pi(G)\simeq\beta(H)\subset G/O_\Pi(G)$. The middle statement of Lemma 1 follows from $O_\Pi(K)\times O_\Pi(L)\subset O_\Pi(K\times L)$. We have already shown that $i_\Pi(G/Z(G))\mid i_\Pi(G)$. Let γ be the natural homomorphism of G into $G/Z(G)$. Let $M=\gamma^{-1}(O_\Pi(G/Z(G)))$. Then $Z(G)=[Z(G)]_\Pi\times[Z(G)]_{\Pi'}$, where $[Z(G)]_{\Pi'}$ is characteristic in $Z(G)$ and is a normal Π'-Sylow of M. By Schur-Zassenhaus, there exists N, a Π-Sylow subgroup of M. As $M=N\times[Z(G)]_{\Pi'}$, N is characteristic in M which is normal in G. Therefore, $N\subset O_\Pi(G)$. As $|G|_\Pi/|N|=|G/M|_\Pi=|G/Z(G)|_\Pi/|O_\Pi(G/Z(G))|$, this concludes the proof of Lemma 1.

LEMMA 2. *Let X be a faithful irreducible representation of a finite group G which affords the character χ. Let $p\geqq 5$ be a prime. Let $H=(O^{p'}(G))'$ and let P be a p-Sylow subgroup of H. Assume that $i_p(G)=p$ and $n=\chi(1)<p-1$. Then*

(i) *X is primitive, $n\geqq(p-1)/2$, $p\parallel |H|$, $i_p(H)=p$ and $O^{p'}(G)\subseteq HZ(G)$.*

(ii) *$X|H$ is irreducible.*

(iii) *$\chi|P$ is the sum of distinct linear characters. The principal character of P is contained in this sum if and only if $n>(p-1)/2$.*

(iv) *If x is a p-element of G then either $X(x)$ is scalar or has distinct eigenvalues.*

Proof. By [6], $i_p(G)\neq 1$ implies that $n\geqq(p-1)/2$. If X is imprimitive on the spaces $V_1,\ldots,V_m$, let K be the subgroup of G fixing the V_i. As dim $V_i=n/m<(p-1)/2$, the constituent $X_i(K)$ of $X(K)$ acting on V_i satisfies $i_p(X_i(K))=1$ for $i=1,\ldots,m$. As $|G/K|\;|m!$ and $O_p(K)$ is characteristic in K which is normal in G, $O_p(K)$ is a normal p-Sylow subgroup of G, a contradiction.

Using Blichfeldt's method of replacing a generator $X(x)$ by

$$Y(y) = [\det \chi(x)]^{-1/n}X(x),$$

one may find a finite group L with a unimodular representation Y of degree n with $Y(L)(ZGL(n,C))=X(G)(ZGL(n,C))$. Then Y is primitive. Furthermore,

$$L/Z(L) \simeq Y(L)(ZGL(n,C))/ZGL(n,C) = X(G)(ZGL(n,C))/ZGL(n,C) \simeq G/Z(G).$$

By Lemma 1, $i_p(L)=i_p(G)=p$. As $n<p-1$, $O_p(L)$ is abelian by Lemma 7. As $O_p(L)$ is normal in L, $Y(L)$ permutes the homogeneous spaces of $Y|O_p(L)$ (the sums of spaces on which identical constituents of $Y|O_p(L)$ act). Therefore, all constituents of $Y|O_p(L)$ are identical and $Y(O_p(L))$ consists of scalars of the form αI_n. Then α is a pth root of unity for some t and $\alpha^n=\det(\alpha I_n)=1$. As $n<p-1$, $\alpha=1$ and $O_p(L)=\langle 1\rangle$. Therefore, $p\parallel |L|$. Then Lemma 3.1 of [5] applies to L and implies that $(O^{p'}(L))'=O^{p'}(L)$. If x is a p-element in G, then there exists $z\in ZGL(n,C)$ with $zX(x)=Y(y)$ for some y in L. As $\langle X(x), Y(y)\rangle\subset\langle z, X(x)\rangle$, an abelian subgroup, $z=Y(y)^{-1}X(x)$ is of finite order. Then $[z]_pX(x)=[zX(x)]_p=[Y(y)]_p$ is a power of $Y(y)$, and replacing z by $[z]_p$, we may take y to be a p-element. This and the symmetric argument show that $X(O^{p'}(G))ZGL(n,C)=Y(O^{p'}(L))ZGL(n,C)$. Then $X(H)=(X(O^{p'}(G)))'=(Y(O^{p'}(L)))'=Y(O^{p'}(L))$. Therefore, $p\parallel |H|$, $i_p(H)=p$, $X(O^{p'}(G))\subset Y(O^{p'}(L))ZGL(n,C)=X(H)ZGL(n,C)$, and $O^{p'}(G)\subset HZ(G)$. By [6], irreducible constituents $X_i(H)$ of $X|H$ with $i_p(X_i(H))=p$ have degree $\geqq(p-1)/2$. As $n<p-1$, there is at most one such constituent. By Lemma 1, there is at least one such constituent. If W is the space on which this constituent acts and $x\in G$, then $H=xHx^{-1}$ has xW as an irreducible invariant space for some constituent U of $X|xHx^{-1}$ and $i_p(U(xHx^{-1}))=p$. Therefore, $xW=W$ and by irreducibility of X, dim $W=n$ and $X|H$ is irreducible. The statement in Lemma 2 about $\chi|P$ follows from Lemma 3.1 of [5] applied to $Y(O^{p'}(L))=X(H)$. The final statement of Lemma 2 follows from our previous step where for x a p-element in G there exist $y\in L$ and $z\in ZGL(n,C)$ with $[z]_pX(x)=[Y(y)]_p$, which is I_n or has distinct eigenvalues.

The remaining lemmas are needed in the proof of our theorem only in the case where we have a proper generalization of Feit's Theorem ($n+1\in\Pi$). Some of the proofs of these lemmas require Feit's Theorem.

LEMMA 3. *Let X be a faithful, irreducible representation of a finite group G of degree $(p-1)/2$ for p, a prime greater or equal to 5. Suppose G does not have a normal p-Sylow subgroup. Then $G=G'Z(G)$ where $G'\simeq PSL(2,p)$ if $(p-1)/2$ is odd* *and* $G' \simeq SL(2, p)$ *if* $(p-1)/2$ *is even. There are exactly two distinct irreducible representations of* G' *of degree* $(p-1)/2$.

Proof. As in the proof of Lemma 2, there exists a finite group L with a faithful, unimodular $(p-1)/2$-dimensional representation Y with $Y(L)ZGL(n, C) = X(G)ZGL(n, C)$ and the following properties: $i_p(L) = p$, $p \parallel |L|$. By [2], $L/Z(L) \simeq PSL(2, p)$. Then $(X(G))' = (Y(L))' \simeq PSL(2, p)$ or $SL(2, p)$ by [11]. Furthermore,

$$(X(G))'ZGL(n, C) = (Y(L))'ZGL(n, C) = Y(L)ZGL(n, C) = X(G)ZGL(n, C).$$

Therefore, $G = G'Z(G)$. The remainder of the lemma follows from the classification in [11] of projective representations of $PSL(2, p)$.

Lemma 4. *Let* L *be a subgroup of a finite group* G *and* $i_p(L) = i_p(G) = p$ *for a prime* $p \geqq 5$. *Let* X *be a faithful, irreducible representation over* C *of* G *of degree* $n < p-1$. *Let* Y *be the unique constituent of* $X|L$ *which is irreducible and satisfies* $i_p(Y(L)) = p$. *Let* $m = \deg Y$. *Then* $m = n$ *or* $m = n-1 = (p-1)/2$. (*Actually, by* [12], $m = n$.)

Proof. Let $X|L = W \oplus Y$ for some constituent W of $Y|L$. Then $i_p(W(L)) = 1$, by Lemma 1. By [1], $O^{p'}(W(L)) = O_p(W(L))$ is abelian and $(O^{p'}(W(L)))' = \langle 1 \rangle$. For any p-element M in $Y(L)$ we may find $x \in L$ with $Y(x) = M$. Then $Y([x]_p) = [Y(x)]_p$ and we may take x to be a p-element. Therefore, $Y(O^{p'}(L)) = O^{p'}(Y(L))$. Then by Lemma 2, we may find $y \in (O^{p'}(L))'$ with $Y(y)$ having order p and m distinct eigenvalues, one of which is 1 if and only if $m \geqq (p-1)/2$. Also, $W(y) \in (O^{p'}(W(L)))' = \langle 1 \rangle$. Applying Lemma 2 to $y \in G$ and the representation X, we see that $X(y)$ has distinct eigenvalues. This implies the conclusion of Lemma 4.

Lemma 5. *Let* X *be a faithful, reducible representation of the finite group* G. *Let* $X = X_1 \oplus X_2$ *where* $\deg X_1 \leqq (p+1)/2$, $\deg X_2 < p-1$, p *is a prime greater than* 4, X_i *is irreducible and* $i_p(X_i(G)) = p$ *for* $i = 1, 2$; *and* $i_p(G) > p$. *Then there exists* $x \in (O^{p'}(G))' \cap \ker X_2$ *of order* p *with* $X_1(x)$ *having exactly* $(p-1)/2$ *eigenvalues unequal to* 1. *Furthermore, if* $\deg X_1 = \deg X_2 = (p-1)/2$, *then* $G = Z(G)(G_1 \times G_2)$ *where for* $i = 1, 2$, $G_i \subset \ker X_i \cap O^{p'}(G)'$ *and* $G_i \simeq PSL(2, p)$ *if* $(p-1)/2$ *is odd,* $G_i \simeq SL(2, p)$ *if* $(p-1)/2$ *is even.*

Proof. Let α be the natural homomorphism $G \to Y_1(G) \times Y_2(G)$ where $Y_i(G) = X_i(G)/Z(X_i(G))$ for $i = 1, 2$. Then $\ker \alpha = Z(G)$ and by Lemma 1, $i_p(\alpha(G)) = i_p(G) > p$. By Lemma 2, $p \parallel |Y_i(G)|$ for $i = 1, 2$. Then $p^2 \mid |\alpha(G)|_p = |Y_2(G)|_p |(\ker Y_2)/Z(G)|_p$ and $p \mid |\ker Y_2/Z(G)|$. Let $K = \ker Y_2$. Then $K \lhd G$, $Y_1(K) \lhd Y_1(G)$, and $X_2(K) \subset Z(X_2(K))$. Since $p \mid |Y_1(K)|$, by Lemma 2, $O^{p'}(Y_1(G)) \subset Y_1(K)$ and $O^{p'}(X_1(G)) \subset X_1(K)Z(X_1(G))$. Then $K' \subset \ker X_2$ and $X_1(K') \supset (O^{p'}(X_1(G)))'$ which by Lemma 2 contains an element x of order p with exactly $(p-1)/2$ eigenvalues unequal to 1. If $\deg X_1 = (p-1)/2$, then by Lemma 3, $(O^{p'}(X_1(G)))' \simeq (P)SL(2, p)$ and $X_1(G) = (O^{p'}(X_1(G)))'Z(X_1(G))$. Defining $G_2 = K'$ and reversing the roles of X_1 and X_2 for $\deg X_1 = \deg X_2 = (p-1)/2$ finishes the proof of Lemma 5.

LEMMA 6. *Let p be a prime $\geqq 5$ and G be a finite group with a faithful representation X of degree $n=p-1$. Let $L\subset G$ with $L/Z(L)\simeq PSL(2,p)$, $(|Z(L)|,p)=1$, and $X|L=X_1\otimes I_2$ with* $\deg X_1=(p-1)/2$. *Let P be a p-Sylow subgroup of L and A be an abelian subgroup of G with $P\subset A$. If $AZ(G)$ (or A) is a trivial intersection set of $G/Z(G)$ (or G), then $X|\langle A, L\rangle$ is reducible.*

Proof. By [11], $X_1|N_L(P)$ is irreducible. By [4, Lemma 51.2], $C_{GL(n,C)}(X(N_L(P)))=I_{n/2}\otimes GL(2,C)=C_{GL(n,C)}(X(L))$. Therefore, $X|N_L(P)$ and $X|L$ have the same invariant subspaces. As $P\subset A$, $P\not\subset Z(G)$, and $AZ(G)$ (or A) is a T. I. S. of $G/Z(G)$ (or G) it follows that $N_L(P)\subset N(AZ(G))$. Furthermore,

$$|\langle N_L(P), AZ(G)\rangle/AZ(G)| \leqq |N_L(P)/P| = (p-1)/2.$$

By Clifford's Theorem, $X|\langle N_L(P), AZ(G)\rangle$ has an invariant space of dimension less than or equal to $(p-1)/2$. As this space is invariant for $N_L(P)$, it is invariant for L and for $\langle L, A\rangle$.

LEMMA 7. *Let Π be a set of primes, all of which are greater than n. Let G be a finite Π-group with a faithful representation X of degree n. Then G is abelian.*

Proof. Degrees of irreducible constituents of X divide $|G|=|G|_\Pi$ and are no larger than n. Such degrees are 1, and G is abelian.

LEMMA 8. *Let G have a faithful representation X. Let K be a normal subgroup of G with $X|K$ irreducible. Let $x\in G$ have order p, an odd prime not dividing $|K|$. Then there exists $\gamma\in C$ with $\gamma^p=1$ and the primitive pth roots of 1 appearing equally often as eigenvalues of $\gamma X(x)$.*

Proof. There are p extensions of $X|K$ to $\langle K, x\rangle$. Let Y be one of these. Then they are all of the form $Y\otimes A^i$ where A is a faithful linear representation of $\langle K, x\rangle/K$. Since the character of X does not vanish on x (as $\deg X\not\equiv 0 \pmod p$) and since the Galois group $\langle\sigma\rangle$ of the pth roots of 1 permutes these p representatives $Y\otimes A^i$ it follows that exactly one of them has a rational character, since $Y\otimes A^i=\sigma(Y\otimes A^i)=(Y\otimes A^k)\otimes A^{ti}=Y\otimes A^{k+ti}$ can be solved for i.

LEMMA 9. *Let X be a faithful, irreducible, quasiprimitive representation of a finite group G on an n-dimensional vector space V. Let H be a subgroup of G with $H/Z(H)\simeq PSL(2,p)$ for p a prime greater than four. Suppose that* $\dim C_V(H)=n-(p-1)/2$. *Then* $\dim C_V(H)\leqq 1$.

Proof. By Lemma 3, we may replace H by H' with $H\simeq(P)SL(2,p)$. Assume that $C_V(H)>1$. There are two overlapping cases:

Case 1. $(p-1)/2$ is even. In this case $H\simeq SL(2,p)$. Let $P=\langle x\rangle$ be a p-Sylow subgroup of H. By [13] we may write X in matrix form in the ring of local integers of some algebraic number field for a prime ideal dividing (2). As $(2, |P|)=1$, by

[13] and [10], we may further take $X|P$ to be diagonal. Let z be the involution in $Z(H)$. By Lemma 2, $X(x)$ has $n-(p-1)/2$ eigenvalues 1 and $(p-1)/2$ distinct eigenvalues $\varepsilon_1, \ldots, \varepsilon_{(p-1)/2}$ unequal to 1. We may write

$$X(x) = \operatorname{diag}(1, \ldots, 1, \varepsilon_1, \ldots, \varepsilon_{(p-1)/2}).$$

As $z \in C(x)$ and $C_V(P) = C_V(H) \subset C_V(z)$, there exist γ_i with

$$X(z) = \operatorname{diag}(1, \ldots, 1, \gamma_1, \ldots, \gamma_{(p-1)/2}).$$

Let Y be the modular representation obtained by taking coefficients in X modulo the prime ideal dividing (2). Then $z \in K$, the kernel of Y. By [3], K is a two group. By quasiprimitivity, we may change coordinates to write $X(K) = U(K) \otimes I_m$ for some irreducible representation U of K and some integer m. By [8, Satz 3], there exist functions V and W from G to $GL(n/m, C)$ and $GL(m, C)$ respectively with $X(g) = V(g) \otimes W(g)$ for all $g \in G$. We may also take $V(x)$ to have order p. Then $V(x)$ normalizes $U(K)$. By Lemma 8, $V(x)$ is scalar or has as many as $(p-1)$ distinct eigenvalues. As $X(x)$ has only $(p+1)/2$ distinct eigenvalues, $V(x)$ is scalar. As ε_1 occurs only once as an eigenvalue of $X(x)$, $\dim V = 1$. Then $X(z)$ is scalar, a contradiction.

Case 2. Here $(p-1)$ is not a power of 2. In this case, there exists q, an odd prime dividing $p-1$. Let $P = \langle x \rangle$ be a p-Sylow subgroup of H. Then there exists y of order q in $N_H(P)$. The normal subgroup P^G generated by P contains H. If it is reducible, then some constituent of $X|P^G$ has H in the kernel, contrary to quasiprimitivity. Conjugates of $X(x)$ cannot permute spaces of imprimitivity nontrivially, for that would imply that $|\text{trace } X(x)| \leqq n-p$. Therefore, $X|P^G$ is primitive. We may replace G by P^G and assume $G = P^G$. Then $\langle y \rangle^G \supset H$, $\langle y \rangle^G \supset H^G \supset P^G = G$, and $G = \langle y \rangle^G$. Write $X|H = X_1 \oplus X_2$ where H is in the kernel of X_1 and $\deg X_2 = (p-1)/2$. The constituents of $X_2|P$ are distinct and nonprincipal. Also, y fixes only principal characters of P. Therefore, $X_2(y) = 0$ and the eigenvalue 1 occurs $(p-1)/2q$ times in $X_2(y)$ and $n-((p-1)/2-(p-1)/2q)$ times in $X(y)$. If u is any conjugate of y in G and $H_u = \langle H, u^{-1}Hu \rangle$, then

$$\begin{aligned} n - \dim C_V(H_u) &\leqq n - \dim C_V(\langle H, u \rangle) \\ &= n - \dim C_V(H) \cap C_V(u) \leqq n - \dim C_V(H) + n - \dim C_V(u) \\ &= (p-1)/2 + (p-1)/2 - (p-1)/2q < p-1. \end{aligned}$$

As $(p-1)/2 + (p-1)/2 + \dim C_V(H_u) > n$, by [6], $X|H_u$ has at most one constituent, say X_u acting on the subspace V_u, with $i_p(X_u(H_u)) \neq 1$. By [5], $i_p(X_u(H_u)) \leqq p$. Since $i_p(H) = p$, by Lemma 1, such an X_u exists, $i_p(X_u(H)) = i_p(X_u(u^{-1}Hu)) = p$, and X_u contains the nonprincipal constituent of $X|H$ and the nonprincipal constituent of $X|u^{-1}Hu$. Let Y_u be a complement to X_u for $X|H_u$. Then $X|H_u = Y_u \oplus X_u$. As H and $u^{-1}Hu$ are in the kernel of Y_u, H_u is in the kernel of Y_u and $V = C_V(H_u) \oplus V_u$. By Lemma 4 applied to X_u and $H \subset H_u$:

$$\deg X_u = (p \pm 1)/2 \quad \text{and} \quad \dim C_V(H_u) = n - (p \pm 1)/2.$$

If deg $X_u=(p-1)/2$, then $u^{-1}C_V(H)=C_V(u^{-1}Hu)=C_V(H_u)=C_V(H)$. As $C_V(H)$ is not invariant for $G=\langle y\rangle^G$ we may find u_0 conjugate to y with deg $X_{u_0}=(p+1)/2$ (actually, deg $X_{u_0}=(p+1)/2$ is impossible by [12], but we go on, anyway) and dim $C_V(H_{u_0})=n-(p+1)/2>0$. Since $G=H^G$ and $G=\langle y\rangle^G$, $G=\langle v^{-1}Hv \mid v=u_1\cdots u_r$ where u_i is a conjugate of y in G for $i=1,\ldots,r\rangle$. As $C_V(H_{u_0})$ is not invariant under $X(G)$, we may find $v=u_1\cdots u_r$, u_i conjugate to y for $i=1,\ldots,r$, with $C_V(H_{u_0})$ not invariant under $v^{-1}Hv$. Then $C_V(H_{u_0})\not\subset C_V(v^{-1}Hv)$ and $C_V(\langle v^{-1}Hv, H_{u_0}\rangle)\neq C_V(H_{u_0})$. Take v so that $C_V(\langle v^{-1}Hv, H_{u_0}\rangle)\neq C_V(H_{u_0})$ and r is minimal. Then $r\geqq 1$. Define $w=vu_r^{-1}=u_1\cdots u_{r-1}$. Then $C_V(\langle w^{-1}Hw, H_{u_0}\rangle)=C_V(H_{u_0})$. Letting $w^{-1}Hw$ play the role of H and u_r play the role of u, we have

$$C_V(\langle w^{-1}Hw, v^{-1}Hv\rangle) = C_V(\langle w^{-1}Hw, u_r^{-1}(w^{-1}Hw)u_r\rangle) = n-(p\pm 1)/2.$$

As $C_V(\langle w^{-1}Hw, v^{-1}Hv\rangle)\subset C_V(w^{-1}Hw)$ and $C_V(H_{u_0})\subset C_V(w^{-1}Hw)$,

$$\begin{aligned}\dim C_V(\langle v^{-1}Hv, H_{u_0}\rangle) &\geqq \dim C_V(\langle w^{-1}Hw, v^{-1}Hv\rangle)\cap C_V(H_{u_0})\\ &\geqq \dim C_V(\langle w^{-1}Hw, v^{-1}Hv\rangle)+\dim C_V(H_{u_0})-\dim C_V(w^{-1}Hw)\\ &\geqq n-(p+1)/2+n-(p+1)/2-(n-(p-1)/2) > n-(p-1).\end{aligned}$$

As with X_u, by [6], $X|\langle v^{-1}Hv, H_{u_0}\rangle$ has exactly one irreducible constituent W acting on U_w with $i_p(W(\langle v^{-1}Hv, H_{u_0}\rangle))\neq 1$. By [5], $i_p(W(\langle v^{-1}Hv, H_{u_0}\rangle))=p$. This constituent W contains the nonprincipal constituents of $X|v^{-1}Hv$ and $X|H_{u_0}$, so $V=C_V(\langle v^{-1}Hv, H_{u_0}\rangle)\oplus U_w$. By Lemma 4 applied to W and $H_{u_0}\subset\langle v^{-1}Hv, H_{u_0}\rangle$: deg $W=(p+1)/2$, dim $C_V(\langle v^{-1}Hv, H_{u_0}\rangle)=n-(p+1)/2=\dim C_V(H_{u_0})$, and

$$C_V(\langle v^{-1}Hv, H_{u_0}\rangle) = C_V(H_{u_0}),$$

a contradiction.

Proof of the theorem. We use induction on $n=\deg X$ and assume that the finite group G with representation X is a counterexample to the theorem with n minimal for a fixed set of primes, Π. By Lemma 7 we may let H be an abelian Π-Sylow subgroup of G.

(A). If $L\subset G$ and $X|L$ is reducible, then L satisfies the conclusion of the theorem. In particular $X|G$ is irreducible.

Proof. Let L contradict (A) and $X|L=Y_1\oplus Y_2$. Then deg $Y_i<n\leqq p-1$ for all $p\in\Pi$ and $i=1, 2$. By the minimality of n, $i_\Pi(Y_i(L))$ is not composite for $i=1, 2$. If for $i=1$ or 2, deg $Y_i<(p-1)/2$ for all $p\in\Pi$, then by [6], $i_\Pi(Y_i(L))=1$, and by Lemma 1, $i_\Pi(L)$ is not composite. Therefore, deg $Y_1=n/2=p-1$ for some $p\in\Pi$ and $i_\Pi(Y_i(L))=p$ for $i=1, 2$. Then $i_\Pi(L)=p$, or Lemma 5 applied to L gives the conclusion.

(B). We may choose X and G so that X is unimodular. Then $H\cap Z(G)=\langle 1\rangle$.

Proof. By [1], we may find a finite group L with a faithful, unimodular representation Y of dimension n with $X(G)ZGL(n, C)=Y(L)ZGL(n, C)$. Then Y is irreducible.

Now G has a Π-Sylow subgroup and $L/Z(L) \simeq G/Z(G)$ has a Π-Sylow subgroup. Let $U \supset Z(L)$ and $UZ(L)$ be a Π-Sylow subgroup of $L/Z(L)$. As $[Z(L)]_{\Pi'}$ is a normal Π'-Sylow subgroup of $UZ(L)$, by Schur-Zassenhaus, $UZ(L)$ has a Π-Sylow subgroup, and this is a Π-Sylow subgroup for L. Now $i_\Pi(G) = i_\Pi(G/Z(G)) = i_\Pi(L/Z(L)) = i_\Pi(L)$ is composite. If $V = V_1 \oplus V_2$ gives spaces of imprimitivity for $Y(L)$, then $X(G)$ has the same spaces of imprimitivity and a normal subgroup K of index 1 or 2 leaves V_1 and V_2 invariant. As $O_\Pi(K)$ is characteristic in K and $K \lhd G$, $O_\Pi(K) \subset O_\Pi(G)$. Then G satisfies whichever alternative of the conclusion of the theorem that K satisfies by (A). Therefore, Y and L are a counterexample to the theorem and may be used to replace X and G. Then X may be taken to be unimodular. Then if $x \in H \cap Z(G)$, $X(x) = \gamma I_n$ where $\gamma^n = 1$, and γ must be 1.

(C). We may further choose G with $G = O^{\Pi'}(G) = H^G$, and with X being primitive.

Proof. Both $O^{p'}(L)$ and H^G are the subgroup of G generated by all Π-elements. Also, $H \subset H^G$. If $i_\Pi(H^G)$ is not composite, then as $O_\Pi(H^G)$ is characteristic in $H^G \lhd G$, $O_\Pi(H^G) \subset O_\Pi(G)$, and $i_\Pi(G)$ is not composite, a contradiction. Suppose that $V = V_1 \oplus V_2$ and $O_\Pi(H^G)$ is imprimitive on the V_i, $i = 1, 2$. As a subgroup of H^G of index 2 contains all Π-elements of H^G and, hence of G and must equal H^G, it follows that V_1 and V_2 are invariant for H^G. As $i_\Pi(H^G)$ is composite, by (A), $X(H^G)$ satisfied II of the theorem. As V_1 and V_2 are the unique invariant subspaces of dimension $n/2$ for $H^G \lhd G$, G is imprimitive on the V_i, $i = 1, 2$; and satisfies II of the theorem, a contradiction. As $O^{\Pi'}(O^{\Pi'}(G))$ contains all Π-elements of $O^{\Pi'}(G)$ and of G, $O^{\Pi'}(O^{\Pi'}(G)) = O^{\Pi'}(G)$. As $O^{\Pi'}(G)$ is a contradiction to the theorem, we may replace G by $O^{\Pi'}(G)$. Then we have $G = O^{\Pi'}(G) = H^G$. If $V_1, \ldots, V_m$ form spaces of imprimitivity for G, then $m \leqq n < p$ for all $p \in \Pi$ and Π-elements must fix the V_i. Then $G = H^G$ fixes the V_i. Then, by (A), $m = 1$.

(D). If x is a Π-element with an eigenvalue occurring more than $n/2$ times in $X(x)$, then $x = 1$.

Proof. Otherwise, we may take $x \in H$ of order p, a prime, with $X(x)$ having eigenvalues $\alpha, \alpha, \ldots, \alpha, \alpha_1, \ldots, \alpha_m$, $m < n/2$. If $\langle x \rangle^G$ is abelian, then by quasi-primitivity, (C), $X|\langle x \rangle^G$ has identical linear constituents and $\langle x \rangle^G \subset Z(G)$, a contradiction. Therefore, we may find y, a conjugate of x not in $C(x)$. Let $K = \langle x, y \rangle$. By Lemma 7, K is not a p-group, and $i_p(K) \neq 1$. Therefore, by Lemma 1, there exists an irreducible constituent Y of $X|K$ with $i_p(Y(K)) > 1$. Now, $C_V(\alpha^{-1}x) \cap C_V(\alpha^{-1}y)$ is a sum of linear constituents for $X|K$. Also,

$$\begin{aligned} &n - \dim C_V(\alpha^{-1}x) \cap C_V(\alpha^{-1}y) \\ &\qquad \leqq n - \dim C_V(\alpha^{-1}x) + n - \dim C_V(\alpha^{-1}y) \leqq 2n - 2m < n. \end{aligned}$$

Therefore, $\deg Y < n$ and by minimality of n, $i_p(Y(K)) = p$. As $Y(x) \notin Z(Y(K))$, by Lemma 2, $Y(x)$ has distinct eigenvalues. Let d be the number of $\alpha_1, \ldots, \alpha_m$ occurring as eigenvalues in $Y(x)$. Then

$$n/2 \leqq (p-1)/2 \leqq \deg Y = \operatorname{var} Y(x) \leqq 1 + d \leqq 1 + m < 1 + (n/2).$$

Then $(p-1)/2=\deg Y=\operatorname{var} Y(x)=1+d=1+m$. Replacing x by y above, we see that a complement U to Y for $X|K$ has $U(x)=U(y)=\alpha I_{n-(p-1)/2}$. By Lemma 2 applied to $Y(K)$, there exists u of order p in K' with $Y(x)Y(u^{-1})\in Z(Y(K))$ and $Y(u)$ having $(p-1)/2$ distinct eigenvalues, all unequal to 1. As $u\in K'$, $U(u)=I_{n-(p-1)/2}$. Then $\operatorname{var} X(u)=1+(p-1)/2$. As $Y(xu^{-1})$ and $U(xu^{-1})$ are both scalar of order dividing p, $\operatorname{var} Y(xu^{-1})\leqq 2$. As X is primitive and $p\geqq 7$, by [1], $X(xu^{-1})$ is scalar. Then

$$\operatorname{var} X(x) = \operatorname{var} X(u) = 1+(p-1)/2 \geqq 1+n/2 > 1+m,$$

a contradiction.

(E). If x is a nonidentity Π-element, then $i_\Pi(C(x))=1$.

Proof. Otherwise, by Lemma 1, $X|C(x)$ has an irreducible constituent Y with $i_\Pi(Y(C(x)))\neq 1$. By [6], $\deg Y\geqq(p-1)/2$ for some $p\in\Pi$. Then some eigenvalue occurs in $X(x)$ with multiplicity $m\geqq(p-1)/2\geqq n/2$. By (D), $m=(p-1)/2=n/2$ and $(n+1)\in\Pi$. Let U be a complementary constituent to Y for $X|C(x)$. If $U(C(x))$ does not have a normal abelian p-Sylow subgroup, then by [6], U is irreducible, $\operatorname{var} X(x)\leqq 2$, $\operatorname{var} X(x)=1$ by [1] and primitivity, $x\in Z(G)$, and by (B) $x=1$, a contradiction. Therefore, $(O^{p'}(C(x)))'$ is in the kernel of U. By Lemma 3,

$$Y((O^{p'}(C(x)))') \simeq (P)SL(2,p).$$

Then $(O^{p'}(C(x)))'$ contradicts Lemma 9.

(F). H is a trivial intersection set (T. I. S.) in G.

Proof. Let $x\in H^{\#}\cap g^{-1}Hg$. Then H, $g^{-1}Hg\in C(x)$. By (E), $i_\Pi(C(x))=1$. Then $H=O_\Pi(C(x))=g^{-1}Hg$.

(G). If $K\subset G$ and $O_\Pi(K)\neq\langle 1\rangle$, then $i_\Pi(K)=1$.

Proof. Let K contradict (G). If x is a Π-element of K, then $\langle x, O_\Pi(K)\rangle$ is a Π-group, and by Lemma 7, $x\in C(O_\Pi(K))$. Therefore, $O^{\Pi'}(K)\subset C(O_\Pi(K))\subset C(y)$ for some nonidentity Π-element y in $O_\Pi(K)$. By (E) and Lemma 1, $i_\Pi(O^{\Pi'}(K))=1$. Then $O_\Pi(O^{\Pi'}(K))$ is a normal Π-Sylow subgroup of K.

(H). If $x\notin Z(G)$, then $i_\Pi(C(x))=1$.

Proof. As $C([x]_\Pi)\supset C(x)$, by (E) and Lemma 1, we may assume that x is a Π'-element contradicting (H). By (G), $O_\Pi(C(x))=\langle 1\rangle$. By (A) applied to $C(x)$, $i_\Pi(C(x))=p$ for some $p\in\Pi$; otherwise, the II of the theorem gives a subgroup contradicting Lemma 9. Therefore, $|C(x)|_\Pi=p$. Replacing x by a conjugate of x there exists a p-Sylow subgroup $P=\langle y\rangle$ of $C(x)$ contained in H.

If $X|C(x)$ has two constituents X_1 and X_2 with $i_p(X_i(C(x)))=p$ for $i=1,2$, then, by [6], $X|C(x)=X_1\oplus X_2$ with $p=(n+1)\in\Pi$, $\deg X_i=(p-1)/2$, and $X_i(C(x))/Z(X_i(C(x)))\simeq PSL(2,p)$ for $i=1,2$. By Lemma 3, there is a subgroup K of $C(x)'$ with $K\simeq(P)SL(2,p)$, and X_i are either the two distinct $(p-1)/2$ dimensional representations of $(P)SL(2,p)$ or are identical. In the first case, $X(y)$ has mutually distinct eigenvalues and $C(y)$ is abelian. Then $\langle H, x\rangle\subset C(y)$ and $H\subset C(x)$ contrary to $|C(x)|_\Pi=p$. In the second case, we may change coordinates to write

$X|K=X_1 \otimes I_2$ and apply Lemma 6 with $A=H$, $L=K$ to conclude that $X|\langle H, K\rangle$ is reducible. We may apply (A) to $\langle H, K\rangle$. As II of the theorem gives a subgroup contradicting Lemma 9, $i_\Pi(\langle H, K\rangle)$ is not composite, $O_\Pi(\langle H, K\rangle)\neq\langle 1\rangle$. By (G), $i_\Pi(\langle H, K\rangle)=1$. Then $p\leqq i_\Pi(K)\leqq i_\Pi(\langle H, K\rangle)=1$, a contradiction.

Therefore, $X|C(x)$ has exactly one irreducible constituent, say Y acting on the subspace S, with $i_p(Y(C(x)))\neq 1$. Let U, acting on the subspace T, be a complement to Y for $X|C(x)$. Let $K=(O^{p'}(C(x)))'$. Then $K\subset \ker U$. By Lemma 2, there exists u of order p in K with $Y(u)$ having $m\geqq(p-1)/2$ eigenvalues unequal to 1. As $|C(x)|_p=p$, we may choose u to be y. Let $W=\sum_{\beta\neq 1} C_V(\beta^{-1}y)$. Then $W\subset S$. Also, $m=\dim W$, and $m\geqq(p-1)/2$. Furthermore, $X(x)$ acts as a scalar on S, and, therefore, also on W. As $\langle H, x\rangle\subset C(y)$, W is invariant under $\langle H, x\rangle$. For any $h\in H$, $X((h, x))$ acts as a scalar on W and $X((h, x))$ has at least m eigenvalues equal to 1. As H is a T. I. S. and $y\in H\cap C(x)$, $x\in N(H)$ and $(h, x)\in H$. By (D), $(p-1)/2\leqq m\leqq n/2$ or $(h, x)=1$. Therefore, $2m+1=p=(n+1)\in\Pi$ or $H\subset C(x)$. As $|C(x)|_\Pi=p$, $2m+1=p=(n+1)\in\Pi$. Then $\deg Y=(p+1)/2$, otherwise, K contradicts Lemma 9. Now $T\oplus S=V=C_V(y)\oplus W$ with $C_V(y)$ and W invariant under $\langle H, x\rangle\subset C(y)$. By Lemma 2, $Y|K$ is irreducible. Let h be any element of H. Then $C_V(h^{-1}Kh)=h^{-1}C_V(K)\subset h^{-1}(C_V(y))=C_V(y)$. Then

$$\begin{aligned}\dim C_V(\langle K, h^{-1}Kh\rangle)\\ &= \dim C_V(K)\cap C_V(h^{-1}Kh) \geqq \dim C_V(K)+\dim C_V(h^{-1}Kh)-\dim C_V(y)\\ &= (p-3)/2+(p-3)/2-(p-1)/2 = (p-5)/2 > 0.\end{aligned}$$

Then by [6] and Lemma 1, $X|\langle K, h^{-1}Kh\rangle$ has at most one constituent R with $i_p(R(\langle K, h^{-1}Kh\rangle))\neq 1$. The constituent R must contain the constituent Y for $R|K$. As $\deg R<n$, by minimality of n, $i_p(R(\langle K, h^{-1}Kh\rangle))=p$. By Lemma 4 applied to R and $K\subset\langle K, h^{-1}Kh\rangle$, $\deg R=\deg Y$. Then S is invariant under $X(h)$. As $X(x)$ is scalar on S, $S\subset C_V((h, x))$ and by (D), $(h, x)=1$. Then $H\subset C(x)$ a contradiction.

(I). Let $N_0=\{\bigcup_{1\neq y\in H} C(y)\}-Z(G)$. Then if $g\notin N(H)$, $N_0\cap g^{-1}N_0g$ is empty.

Proof. Let $x\in N_0\cap g^{-1}N_0g$. Then there exist h, $k\neq 1$, $h, k\in H$ with h, $g^{-1}kg\in C(x)$. By (H), $i_\Pi(C(x))=1$, so $h, g^{-1}kg\in O_\Pi(C(x))$. By Lemma 7, $O_\Pi(C(x))\subset C(h)$, $C(g^{-1}kg)$. As H is a T. I. S., $O_\Pi(C(x))\subset N(H)$, $N(g^{-1}Hg)$. Then $\langle O_\Pi(C(x)), H\rangle$, $\langle O_\Pi(C(x)), g^{-1}Hg\rangle$ are Π-groups, and $\langle h\rangle\subset O_\Pi(C(x))\subset H\cap g^{-1}Hg$. Then $H=g^{-1}Hg$.

(J). By (C) and (I), $H\subset G$ satisfies the hypothesis of Lemma 4.2 of [5], by which $n+1>|H|^{1/2}\geqq p$ for some p in Π, a contradiction.

Bibliography

1. H. F. Blichfeldt, *Finite collineation groups*, University of Chicago Press, Chicago, Ill., 1917.

2. R. Brauer, *On groups whose order contains a prime to the first power*. I, II, Amer. J. Math. **64** (1942), 401–440. MR **4**, 1; MR **4**, 2.

3. ———, *Über endlichen lineare Gruppen von Primzahlgrad*, Math. Ann. **169** (1967), 73–96 MR **34** #5913.

4. C. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. XI, Interscience, New York, 1962. MR **26** #2519.

5. W. Feit, *Groups which have a faithful representation of degree less than* $p-1$, Trans. Amer. Math. Soc. **112** (1964), 287–303. MR **28** #5110.

6. W. Feit and J. G. Thompson, *Groups which have a faithful representation of degree less than* $(p-1/2)$, Pacific J. Math. **11** (1961), 1257–1262. MR **24** #A3207.

7. G. Glauberman, *Correspondences of characters for relatively prime operation groups*, Canad. J. Math. **20** (1968), 1465–1488. MR **38** #1189.

8. B. Huppert, *Lineare auflösbare Gruppen*, Math. Z. **67** (1957), 479–518. MR **19**, 729.

9. N. Itô, *On a theorem of H. F. Blichfeldt*, Nagoya Math. J. **5** (1953), 75–77. MR **14**, 842.

10. I. Schur, *Über eine Klasse von endlichen Gruppe linearer Substitutionen*, S.-B. Preuss. Akad. Wiss. (1905), 77–91.

11. ———, *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **132** (1907), 85–137.

12. H. F. Tuan, *On groups whose orders contain a prime to the first power*, Ann. of Math. (2) **45** (1944), 110–140. MR **5**, 143.

13. H. Zassenhaus, *Neuer Beweis der Endlichkeit der Klassenzahl bei unimodularer Aquivalenz endlicher ganzzahliger Substitutionsgruppen*, Abh. Math. Sem. Univ. Hamburg **12** (1938), 276–288.

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