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ON L^p ESTIMATES FOR INTEGRAL TRANSFORMS

BY
T. WALSH

Abstract. In a recent paper R. S. Strichartz has extended and simplified the proofs of a few well-known results about integral operators with positive kernels and singular integral operators. The present paper extends some of his results. An inequality of Kantorovič for integral operators with positive kernel is extended to kernels satisfying two mixed weak L^p estimates. The "method of rotation" of Calderón and Zygmund is applied to singular integral operators with Banach space valued kernels. Another short proof of the fractional integration theorem in weighted norms is given. It is proved that certain sufficient conditions on the exponents of the L^p spaces and weight functions involved are necessary. It is shown that the integrability conditions on the kernel required for boundedness of singular integral operators in weighted L^p spaces can be weakened. Some implications for integral operators in R^n of Young's inequality for convolutions on the multiplicative group of positive real numbers are considered. Throughout special attention is given to restricted weak type estimates at the endpoints of the permissible intervals for the exponents.

0. **Introduction.** The present paper attempts to answer a few questions raised by a paper of Strichartz [25] and the following example may be illustrative of its contents. As usual, for $x \in R^n$ let $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. It was shown by Stein [22] that if Ω is positively homogeneous of degree zero, bounded and of mean value zero on the unit sphere S^{n-1} in R^n and $K(x) = \Omega(x)|x|^{-n}$ then the singular integral operator T defined by $Tf(x) = \text{p.v. } K * f(x) = \lim_{\varepsilon \to 0} \int_{|x-t|>\varepsilon} K(x-t)f(t)\,dt$ satisfies

$$\int |Tf(x)|^p |x|^{\alpha p}\,dx \leqq A_p^p \int |f(x)|^p |x|^{\alpha p}\,dx$$

for $1 < p < \infty$, $-n/p < \alpha < n/p'$. Here and in what follows p' denotes the exponent conjugate to p, i.e., $1/p + 1/p' = 1$. Strichartz showed that it is sufficient to require $\Omega \in L^q(S^{n-1})$, $q = \max(p, p')$, $q > 2(n-1)$ and conjectured that the condition $q > 2(n-1)$ is not necessary (Theorem 6 of [25]). It is proved in the present paper that $1/q = 1 - |\alpha|/n$ is sufficient which in particular shows that $q > 2(n-1)$ is not needed (Proposition 7).

Notation. As in [25] let (X, Σ, μ), (Y, Σ', ν) be totally σ-finite measure spaces, $K(x, y)$ a measurable function on $X \times Y$ and define $Tf(x) = \int K(x, y) f(y)\,dy$ where

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for simplicity dy, dx are written for $d\nu(y)$, $d\mu(x)$. For f measurable let λ_f, f^* denote the distribution function and decreasing rearrangement of f on $(0, \infty)$ respectively. Further as in [13] set

$$\|f\|^*_{pq} = \|f\|^*[L^{pq}(X)] = \left(q/p \int_0^\infty [f^*(t)t^{1/p}]^q\, dt/t\right)^{1/q},$$
$$\|f\|_{pq} = \|f\|[L^{pq}(X)] = \|f^{**}\|^*_{pq}$$

where

$$f^{**}(t) = f^{**}(t, r) = \sup\left\{\left(|E|^{-1}\int |f(x)|^r\, dx\right)^{1/r} : |E| \geqq t\right\},$$
$$0 < r \leqq 1, \quad r \leqq q, \quad r < p$$

and for simplicity $|E| = \mu(E)$. If F is a Banach (an s-Banach) space $L^p(X, F)$ will denote the space of F-valued measurable functions on X such that $\|f\|_F \in L^p(X)$. Also let $L^{(p,q)}(X\times Y) = L^q(Y, L^p(X))$. The norm of $L^{(p,q)}$ will also be denoted $\|\cdot\|_{(p,q)}$ (cf. [2]). For f measurable on $X\times Y$ define

$$X^{p_1q_1}Y^{p_0q_0}f = \|\varphi\|_{p_1q_1} \quad \text{where } \varphi(x) = \|f(x, \cdot)\|_{p_0q_0}.$$

It will be convenient to define $X_*^{p_1q_1}Y_*^{p_0q_0}f$ analogously by replacing $\|\cdot\|_{p_iq_i}$ by the quasi-norm $\|\cdot\|^*_{p_iq}$ for $i=0, 1$.

1. **An extension of an inequality of Kantorovič.** It is well known (see [14]) that if

(1) $$X^{r_0}Y^{s_0}K = M_0 < \infty, \qquad Y^{s_1}X^{r_1}K = M_1 < \infty$$

then

(2) $$\|Tf\|_r \leqq M_0^{1-\theta}M_1^\theta\|f\|_s.$$

provided, for some θ, such that $0 \leqq \theta \leqq 1$

(3) $$1/r = (1-\theta)/r_0 + \theta/r_1, \qquad 1/s = (1-\theta)/s_0 + \theta/s_1,$$

(4) $$(1-\theta)/s_0 + \theta/r_1 \leqq 1, \qquad (s \geqq 1).$$

If the hypotheses are weakened by replacing the L^p (quasi-) norms in (1) by weak L^p (i.e., $L^{p\infty}$) (quasi-) norms there is a corresponding weaker conclusion given by

PROPOSITION 1. *Suppose* $X^{r_0\infty}Y^{s_0\infty}K = M_0$, $Y^{s_1\infty}X^{r_1\infty}K = M_1$ *and* (3) *is satisfied with* $0 \leqq \theta \leqq 1$. *Then*

$$\|Tf\|_{r\infty} \leqq BM_0^{1-\theta}M_1^\theta\|f\|_{s1}$$

provided $s > 1$ *and*

(5) $$(1-\theta)/s_0 + \theta/r_1 < 1.$$

Here B denotes a finite constant depending on r_0, r_1, s_0, s_1, and θ. If also $0<\theta<1$, $r_0 \neq r_1$, $0<u\leqq\infty$ (and M_0, $M_1 \geqq 1$) then

$$\|Tf\|_{ru} \leqq B(M_0+M_1)\|f\|_{s'u}.$$

This is Theorem 2 of [25] except that the assumptions on the exponents are less restrictive. Theorem 2 of [25] in fact follows from the Marcinkiewicz interpolation theorem for L^{pq} spaces (see [13]) and the following

LEMMA 1. *Suppose $M_0 = X^{qt} Y^{p's'} K < \infty$,*

$$1 < p,\ 1 \leqq s \quad or \quad p = s = 1 \quad and \quad q, t > 0 \tag{6}$$

then

$$\|Tf\|_{qt} \leqq BM_0\|f\|_{ps}. \tag{7}$$

If $M_1 = Y^{p's'} X^{qt} K < \infty$,

$$1 < p,\ 1 \leqq s \quad or \quad p = s = 1 \quad and \quad 1 < q,\ 1 \leqq t \quad or \quad q = t = 1 \tag{8}$$

then

$$\|Tf\|_{qt} \leqq BM_1\|f\|_{ps}. \tag{9}$$

Proof. Let $K, f \geqq 0$. (7) follows from $\int K(x,y)f(y)\,dy \leqq B\|K(x,\cdot)\|_{p's'}\|f\|_{ps}$. As for (9), by Minkowski's inequality for integrals, provided L^{qt} is a normed space

$$\left\|\int K(\cdot,y)f(y)\,dy\right\|_{qt} \leqq \int \|K(\cdot,y)\|_{qt} f(y)\,dy \leqq BM_1\|f\|_{ps}.$$

With the present notation which differs somewhat from that of [25], if $1<s_0<\infty$, $1<r_1<\infty$, $0<r_0\leqq\infty$, $1<s_1\leqq\infty$ then by Lemma 1

$$\|Tf\|_{r_0\infty} \leqq BM_0\|f\|_{s_0'1} \quad \text{and} \quad \|Tf\|_{r_1\infty} \leqq BM_1\|f\|_{s_1'1}$$

so that the Marcinkiewicz interpolation theorem implies Proposition 1 in this case.

Proof of Proposition 1. Consider $T(K_0, K_1, f)(x) = \int K_0(x,y)K_1(x,y)f(y)\,dy$. Suppose $X^{r_0u_0}Y^{s_0v_0}K_0 < \infty$, $Y^{s_1v_1}X^{r_1u_1}K_1 < \infty$ and furthermore that there exists t such that $0 \leqq t \leqq 1$ and $1-1/(s_0(1-t))$, $1-1/(v_0(1-t))$ in place of $1/p$, $1/s$ satisfy (6) and $1-1/(s_1t)$, $1-1/(v_1t)$, $1/(r_1t)$, $1/(u_1t)$ satisfy (8) in place of $1/p$, $1/s$, $1/q$, $1/t$. Then by Lemma 1

$$\begin{aligned}&\|T(K_0,K_1,f)\|_{r_0(1-t),u_0(1-t)}\\ &\qquad \leqq BX^{r_0(1-t),u_0(1-t)}Y^{s_0(1-t),v_0(1-t)}(K_0)\|K_1\|_\infty\|f\|_{[s_0(1-t)]',[v_0(1-t)]'}\end{aligned}$$

and

$$\|T(K_0,K_1,f)\|_{r_1t,u_1t} \leqq B\|K_0\|_\infty Y^{s_1t,v_1t}X^{r_1t,u_1t}(K_1)\|f\|_{(s_1t)',(v_1t)'},$$

hence by the complex method of interpolation (see 10.1 of [5] which extends to s-Banach spaces, see [9], [13], required since $L^{r_0u_0}$ is not necessarily a Banach space) with a possibly different B

$$\|T(K_0, K_1, f)\|_{ru} \leqq BX^{r_0u_0}Y^{s_0v_0}(K_0)Y^{s_1v_1}X^{r_1u_1}(K_1)\|f\|_{s'v'} \tag{10}$$

where $r^{-1}=r_0^{-1}+r_1^{-1}$, $u^{-1}=u_0^{-1}+u_1^{-1}$, $s^{-1}=s_0^{-1}+s_1^{-1}$, $v^{-1}=v_0^{-1}+v_1^{-1}$. By the above, t must satisfy $\max(r_1^{-1}, s_1^{-1})\leqq t\leqq 1-s_0^{-1}$ hence $s\geqq 1$ and

$$1/s_0+1/r_1 \leqq 1. \tag{11}$$

If on the other hand the latter two inequalities hold $0\leqq t\leqq 1$ and the inequalities for r_i, s_i can be satisfied. Thus if $r_i=u_i$, $s_i=v_i$, (11) (and $s\geqq 1$) is sufficient for (10). (In this cas• $B=1$.)

If $u_i=v_i=\infty$ for $i=0, 1$, Lemma 1 requires $\max(1/r_1, 1/s_1)<t<1-1/s_0$ hence $s>1$, $1/s_0+1/r_1<1$ is sufficient for

$$\|T(K_0, K_1, f)\|_{r\infty} \leqq BX^{r_0\infty}Y^{s_0\infty}(K_0)Y^{s_1\infty}X^{r_1\infty}(K_1)\|f\|_{s'1}.$$

Inequalities of Kantorovič's type can be deduced from this by letting

$$K_0 = K^{1-\theta}, \quad K_1 = K^{\theta} \quad \text{for some } 0 \leqq \theta \leqq 1. \tag{12}$$

In fact it follows from the definition of $\|\cdot\|_{pq}$ that $\|\,|f|^{\theta}\,\|_{pq}=\|f\|^{\theta}_{p\theta,q\theta}$ for $\theta>0$. (If $f^{**}(t, r)$ is used in the definition of $\|f\|_{pq}$ then r/θ has to take the place of r in the definition of $\|\,|f|^{\theta}\,\|_{pq}$.) From the hypotheses of Proposition 1

$$X^{r_0/(1-\theta),u_0/(1-\theta)}Y^{s_0/(1-\theta),v_0/(1-\theta)}(|K|^{1-\theta}) = M_0^{1-\theta},$$
$$Y^{s_1/\theta,v_1/\theta}X^{r_1/\theta,u_1/\theta}(|K|^{\theta}) = M_1^{\theta}.$$

Hence if (3) is satisfied and $r_i=u_i$, $s_i=v_i$ for $i=0, 1$, then $\|Tf\|_r\leqq M_0^{1-\theta}M_1^{\theta}\|f\|_{s'}$ provided $(1-\theta)/s_0+\theta/r_1\leqq 1$ and ($s\geqq 1$) which is Kantorovič's inequality (2). The proposition follows likewise from the above.

That in general the condition $(1-\theta)/s_0+\theta/r_1<1$ cannot be weakened beyond replacement of $<$ by $\leqq$ can be shown as follows.

PROPOSITION 2. *Let X, Y both have the property that for any two measurable subsets E_1, E_2, $E_1\subset E_2$ of finite measure there is another subset E_1' disjoint from E_2 such that the measures of E_1 and E_1' are the same. Then in order for*

$$T(K_0, K_1, f) = \int K_0(\cdot, y)K_1(\cdot, y)f(y)\,dy$$

to satisfy

$$\|T(K_0, K_1, f)\|_{q\infty} \leqq BX^{r_0u_0}Y^{s_0v_0}(K_0)Y^{s_1v_1}X^{r_1v_1}(K_1)\|f\|_{pw} \tag{13}$$

(*with B independent of K_0, K_1, f*), *it is necessary that*

$$1/r_0+1/s_1+1/p-1/q \geqq 0. \tag{14}$$

Proof. It seems slightly more convenient to work with the quasi-norms $\|\cdot\|^*$. Let then B be minimal in (13) with the norms starred and all of K_0, K_1, f vanishing outside sets of finite measure, of the special form $E\times F$ in the case of K_0, K_1. For $0<\eta<1$ there exist then simple functions K_0, K_1, f such that

$$(15)\qquad K_i = \sum_{k,l=1}^{k_1,l_1} a_i^{kl}\chi_k\varphi_l,\qquad f = \sum_{l=1}^{l_1} b_l\varphi_l,\qquad a_i^{kl},\ b_l\in R,\quad i = 0, 1$$

where χ_k, φ_l are the characteristic functions of two sequences of mutually disjoint sets $\{E_k\}_{1\leqq k\leqq k_1}$ and $\{F_l\}_{1\leqq l\leqq l_1}$ respectively, and for which

$$(16)\qquad \|T(K_0, K_1, f)\|^*_{q\infty} > B\eta\|K_0\|^*\|K_1\|^*\|f\|^*_{pw}\qquad (\|K_0\|^* = X_*^{r_0u_0}Y_*^{s_0v_0}(K_0)).$$

By hypothesis there exist sequences of mutually disjoint sets $\{E_k'\}_{1\leqq k\leqq k_1}$, $\{F_l'\}_{1\leqq l\leqq l_1}$ in X, Y contained in the complement of $\bigcup_{k=1}^{k_1} E_k$ and $\bigcup_{l=1}^{l_1} F_l$ respectively such that $|E_k|=|E_k'|$, $|F_l|=|F_l'|$ for $1\leqq k\leqq k_1$, $1\leqq l\leqq l_1$. Let χ_k', φ_l' be the characteristic functions of E_k', F_l'. Define

$$K_i' = \sum a_i^{kl}\chi_k'\varphi_l',\qquad f' = \sum_{l=1}^{l_1} b_l\varphi_l',$$
$$K_i^{\sim} = K_i+K_i',\qquad f^{\sim} = f+f',\qquad i = 0, 1,$$

then $\lambda_{f^{\sim}}(t)=2\lambda_f(t)$ hence $(f^{\sim})^*(t)=f^*(t/2)$. Thus

$$(17)\qquad \|f^{\sim}\|^*_{pw} = 2^{1/p}\|f\|^*_{pw}.$$

Similarly

$$\|K_0^{\sim}\|^* = 2^{1/r_0}\|K_0\|^*,\qquad \|K_1^{\sim}\|^* = 2^{1/s_1}\|K_1\|^*,$$
$$\|T(K_0^{\sim}, K_1^{\sim}, f^{\sim})\|^*_{q\infty} = \|T(K_0, K_1, f)+T(K_0', K_1', f')\|^*_{q\infty} = 2^{1/q}\|T(K_0, K_1, f)\|^*_{q\infty}.$$

So by the definition of B

$$2^{1/q}\|T(K_0, K_1, f)\|^*_{q\infty} \leqq B2^{1/r_0+1/s_1+1/p}\|K_0\|^*\|K_1\|^*\|f\|^*_{pw}.$$

This and (16) imply $2^{1/r_0+1/s_1+1/p-1/q}>\eta$. Since $\eta<1$ was arbitrary (14) follows.

Remark. Proposition 2 holds in particular if X, Y are noncompact locally compact (second countable) groups provided with left-invariant measure. The above argument was suggested by the proof of Theorem 1.1 of [12]. If $u<\infty$ the simple functions are dense in L^{pu}. Hence it follows from (17) that likewise if T is a continuous linear operator from $L^{pu}(G)$ to $L^{qv}(G)$ where G is a noncompact locally compact group, $T(f(a.))=(Tf)(a.)$ for any $a\in G$ and $q<p<\infty$, $v<\infty$, then $T=0$ (u may be replaced by any smaller positive exponent). In case $v=\infty$, $1<q<p<\infty$, $u>1$, it follows by consideration of the adjoint of T that $T=0$.

Lemma 2. *Let X, Y be, more particularly, continuous measure spaces of infinite measure such that* (13) *is valid for all K_0, K_1, f then*

$$(18)\qquad 1/q = 1/r_0+1/r_1,\qquad 1/p+1/s_0+1/s_1 = 1.$$

Proof. For any triple K_0, K_1, f of simple functions of the special form (15) and any $\varepsilon, \eta > 0$ there are mutually disjoint sets E_k^ε and F_l^η such that $|E_k^\varepsilon| = \varepsilon^{-1}|E_k|$, $|F_l^\eta| = \eta^{-1}|F_l|$. Let χ_k^ε, φ_l^η be the corresponding characteristic functions

$$f^\eta = \sum_{l=1}^{l_1} b_l \varphi_l^\eta, \qquad K_i^{\varepsilon,\eta} = \sum a_i^{kl} \chi_k^\varepsilon \varphi_l^\eta, \qquad i = 0, 1.$$

Then

$$\|f^\eta\|_{pw}^* = \eta^{-1/p}\|f\|_{pw}^*, \qquad \|K_i^{\varepsilon,\eta}\|^* = \varepsilon^{-1/r_i}\eta^{-1/s_i}\|K_i\|^*, \qquad i = 0, 1,$$

and

$$\|T(K_0^{\varepsilon,\eta}, K_1^{\varepsilon,\eta}, f^\eta)\|_{q\infty}^* = \eta^{-1}\varepsilon^{-1/q}\|T(K_0, K_1, f)\|_{q\infty}^*.$$

Hence $\varepsilon^{-1/q+1/r_0+1/r_1}\eta^{-1+1/s_0+1/s_1+1/p}$ is bounded for $\varepsilon, \eta > 0$ and (18) follows. Note that in the present situation (14) is equivalent to (11).

Corollary. *If for* $T(K,f)(x) = \int K(x, y)f(y)\, dy$,

$$\|T(K,f)\|_{q\infty} \leqq BX^{r_0u_0}Y^{s_0v_0}(K)^{1-\theta}Y^{s_1v_1}X^{r_1u_1}(K)^\theta\|f\|_{pw}$$

with B independent of K, f, then, in the situation of the preceding lemma, (3) *and* (4) *must be satisfied.*

This follows similarly by proceeding with K as with K_0, K_1 above.

2. **Variable kernel singular integral operators.** §4 of [25] is concerned with singular integrals of the form

$$Tf(x) = \text{p.v.} \int H(x-y)f(y)\, dy(x) \tag{19}$$

where $H(t, x) = H(x)(t) = \Omega(t, x')|x|^{-n}$ for $x' = |x|^{-1}x$ and Ω is a function on S^{n-1} with values in the space L_u^2 (L_u^q consists of the tempered distributions f on R^n such that $\hat{f}(1+|\cdot|^2)^{u/2} = \hat{g}$, where $g \in L^q$, in this case $\|f\|_{q,u} = \|g\|_q$, see [4]). By heavier use of the results of Calderón and Zygmund about singular integrals based on the "method of rotation" in [8] stronger results can be obtained for L_u^q valued kernels. In particular Ω need not satisfy any smoothness condition (as a function on S^{n-1}).

Proposition 3. *Let Z be some measure space,* $\Omega \in L^{(q,1)}(R^n \times S^{n-1})$, $1 < q < \infty$. *Suppose also the even part* Ω_0, *say, of* Ω *is in* $L \log^+ L(S^{n-1}, L^q(Z))$ *and has mean value zero* (*in* $L^q(Z)$) *on* S^{n-1}. *Let the completed tensor product of* $L^p(R^n)$ *and* $L^q(Z)$ *with the projective norm* (*see* [26]) *be denoted* $L^p \tilde{\otimes} L^q = L^p(R^n) \tilde{\otimes} L^q(Z)$. *Then, if also* $\Omega_1 = \Omega - \Omega_0$, $1 < p < \infty$,

$$\begin{aligned} &\|\text{p.v. } H * f\|[L^p \tilde{\otimes} L^q] \\ &\quad \leqq C_{p,q}(1 + \|\Omega_0\|[L \log^+ L(S^{n-1}, L^q(Z))] + \|\Omega_1\|[L(S^{n-1}, L^q(Z))])\|f\|_p. \end{aligned} \tag{20}$$

Proof. Suppose first of all Ω is odd, i.e., $\Omega(-x) = -\Omega(x)$. Then (see [8])

$$\int_{|x-y|>\varepsilon} H(\cdot, x-y)f(y)\, dy = \int_{S^{n-1}} \Omega(\cdot, y')\tilde{f}_\varepsilon(x, y')\, dy'$$

where $\tilde{f}_\varepsilon(x, y') = \int_{|t|>\varepsilon} f(x-ty')t^{-1}\,dt$ and dy' denotes the element of area on S^{n-1}. Hence

$$\left\| \int_{|\cdot-y|>\varepsilon} H(\cdot, \cdot-y)f(y)\,dy \right\| [L^q(Z) \tilde{\otimes} L^p(R^n)]$$
$$\leqq \int_{S^{n-1}} \|\Omega(\cdot, y')\tilde{f}_\varepsilon(\cdot, y')\| [L^q(Z) \otimes L^p(R^n)]\,dy'$$
$$= \int_{S^{n-1}} \|\Omega(\cdot, y')\|_q \|\tilde{f}_\varepsilon(\cdot, y')\|_p\,dy' \leqq C_p \|\Omega\|_{(q,1)} \|f\|_p.$$

It remains to consider the case of even Ω. It is sufficient to observe that the reduction to the case of odd kernels via the formula $f = \sum_{i=1}^n R_i(R_i f)$, where $Rf = (R_1 f, \ldots, R_n f)$ denotes the Riesz transform of f, given in [8], can be carried over to kernels with values in $L^q(Z)$ for $1<q<\infty$. This argument is based on the fact that the Riesz transform of a real valued function in $L(\log^+ L+1)(R^n)$ is locally integrable. The analogous result holds for L^q valued functions. For if K denotes the Riesz kernel $K(x) = c_n^{-1}|x|^{-n-1}x$ then for all $z \in Z$

$$\|K_\varepsilon * f(z, \cdot)\|_q^q \leqq C_q^q \|f(z, \cdot)\|_q^q$$

and integration over Z gives

$$\|K_\varepsilon * f\| [L^q(R^n, L^q(Z))] \leqq C_q \|f\| [L^q(R^n, L^q(Z))].$$

Also K satisfies the condition

$$\int_{|x| \geqq 2|y|} |K(x-y) - K(x)|\,dx \leqq C \quad \text{for all } y \in R^n. \tag{21}$$

Hence by Theorem 2 of [1]

$$\|Rf\| [L^p(R^n, L^q(Z))] \leqq \lim_{\varepsilon \to 0} \|K_\varepsilon * f\| \leqq C_{p,q} \|f\| [L^p(R^n, L^q(Z))]$$

where $C_{p,q} = O((p-1)^{-1})$ for $p \downarrow 1$. According to [27, p. 119, 4.41(ii)] the latter estimate implies that for $f \in L(\log^+ L+1)(R^n, L^q(Z))$, $Rf \in L_{\mathrm{loc}}(R^n, L^q(Z))$.

An extension of the argument of [8] to L^q valued kernels to prove (20) also requires an estimate analogous to (20) for an L^q valued maximal function defined by

$$T^*f(x) = \sup_{\varepsilon>0} \varepsilon^{-n} \left| \int_{|y| \leqq \varepsilon} \Omega^*(\cdot, y') f(x-y)\,dy \right| \quad \text{where } \Omega^* \in L^1(S^{n-1}, L^q(Z)).$$

That in fact

$$\|T^*f\| [L^p \tilde{\otimes} L^q] \leqq C_p \|\Omega\| [L(S^{n-1}, L^q(Z))] \|f\|_p$$

for $1<p\leqq\infty$, $1\leqq q\leqq\infty$ follows by analogous modification of the argument of [8].

By application of continuous linear operators corresponding to continuous bilinear operators on $L^p \times L^q$ to the left-hand side of (20) various corollaries can be obtained.

COROLLARY 1. *Suppose the measurable function* $\Omega \in L^1(S^{n-1}, L_u^b(R^n))$, *where* $u > q/n$, Ω *has mean value zero on* S^{n-1} *and the* L_u^q*-norm of the even part of* Ω *belongs to* $L \log^+ L(S^{n-1})$. *Then* T *defined by* (19) *satisfies* $\|Tf\|_p \leqq C_{p,q,u}(\Omega)\|f\|_p$ *for* $1 < p < \infty$.

Proof. Let J^u denote the isomorphism between L^q and L_u^q defined by $(J^u f)^\wedge = (1+|\cdot|^2)^{-u/2}\hat{f}$ and define Ω^{-u} by $\Omega^{-u}(\cdot, x) = J^{-u}\Omega(\cdot, x)$ and H^{-u} analogously. Then by Proposition 3

$$\|\text{p.v. } H^{-u} * f\|[L^p \tilde{\otimes} L^q] \leqq C_p(\Omega)\|f\|_p.$$

Hence

$$\|\text{p.v. } H * f\|[L^p \tilde{\otimes} L_u^q] \leqq C_p(\Omega)\|f\|_p.$$

Now by Sobolev's inequality for any $x \in R^n$,

$$\|\text{p.v. } H * f(\cdot, x)\|_\infty \leqq \|\text{p.v. } H * f(\cdot, x)\|_{p,u},$$

hence

$$\begin{aligned} \|Tf\|_p &\leqq C_{q,u}\|\text{p.v. } H * f\|[L^p(R^n, L_u^q(R^n))] \\ &\leqq C_{q,u}\|\text{p.v. } H * f\|[L^p \tilde{\otimes} L_u^q] \leqq C_{p,q,u}(\Omega)\|f\|_p. \end{aligned}$$

REMARK. Suppose $\Omega \in L^1(S^{n-1}, L^p(R^n))$ and now $f \in L_u^q$ while the remaining conditions are altered in the obvious manner. Let $Tf(x)$ be defined by evaluation of the (continuous) function $(\text{p.v. } H * f)(x, \cdot) = \text{p.v. } H * f(\cdot)(x) \in L_u^q$ at x. Then $\|Tf\|_p \leqq C_{p,q,u}(\Omega)\|f\|_{q,u}$. This follows similarly from $\|\tilde{f}_\varepsilon(\cdot, y')\|_{q,u} \leqq C_{q,u}\|f\|_{q,u}$ (a consequence of the fact that the operation $f \to \tilde{f}_\varepsilon(\cdot, y')$ commutes with J^u).

For simplicity the next corollary is stated only for odd kernels.

COROLLARY 2. *Suppose* $\Omega(x, y') = -\Omega(x, -y')$ *and* $\Omega \in L^{(q,1)}(R^n \times S^{n-1})$, $1 < p < \infty$, $1 \geqq 1/r = 1/p + 1/q$ *then* T *defined by* (19) *satisfies*

$$\|Tf\|_r \leqq C_p\|\Omega\|_{(q,1)}\|f\|_p. \tag{22}$$

Proof. By Proposition 3, (20) is satisfied ($Z = R^n$). By Hölder's inequality the bilinear map $(f, g) \to fg$ is continuous from $L^p(R^n) \times L^q(R^n)$ to L^r. This implies (22).

Corollary 2 might be compared with Theorem 4.1 of [7] which deals with kernels in $L^{(s,q)}(S^{n-1} \times R^n)$ where $1 < s$.

3. **Fractional integration in weighted norms.** First a result needed repeatedly later will be proved.

LEMMA 3. *Let* f *be measurable on* $X \times Y$ *then*

$$\|f\|_{p\infty}^* \leqq X^p Y_*^{p\infty} f. \tag{23}$$

(*For the case* $f(x, y) = f_1(x)f_2(y)$ *see O'Neil* [19, *Theorem* 11.4].)

Proof.

$$\|f\|_{p\infty}^{*p} = \sup_{t>0} t^p\lambda_f(t) = \sup_{t>0} t^p \int_X \lambda_{f(x,\cdot)}(t)\,dx \leqq \int_X \sup t^p\lambda_{f(x,\cdot)}(t)\,dx$$
$$= \int_X \|f(x,\cdot)\|_{p\infty}^{*p}\,dx.$$

REMARKS. Obviously, if for a.e. x and all $t>0$, $t^p\lambda_{f(x,\cdot)}(t)=\|f(x,\cdot)\|_p^{*p}$ then equality holds in (23). Thus if $f=f_1f_2$, $f_1\in L^p(X)$, f_2 measurable on Y and such that $\lambda_{f_2}(t)=Ct^{-p}$ then $t^p\lambda_{f(x,\cdot)}(t)=t^p\lambda_{f_2}(t/|f_1(x)|)=C|f_1(x)|^p$ and hence $\|f\|_{p\infty}^*=C\|f\|_p$ (cf. [19]). Since $\|\cdot\|_{p\infty}^*$ and $\|\cdot\|_{p\infty}$ are equivalent it is also true that $\|f\|_{p\infty}\leqq C_pX^pY^{p\infty}f$. In fact $C_p=1$. For if $r\leqq 1$, $r<p$ and $E_x=\{y:(x,y)\in E\}$; then

$$\|f\|_{p\infty} = \sup_{t>0} t^{1/p}f^{**}(t,r) = \sup_{|E|\geqq t} t^{1/p}\left(|E|^{-1}\int_E |f(x,y)|^r\,dx\,dy\right)^{1/r}$$
$$\leqq \sup_{|E|\geqq t} t^{1/p}\left(|E|^{-1}\int_X |E_x|^{-r/p+1}\|f(x,\cdot)\|_{p\infty}^r\,dx\right)^{1/r}$$
$$\leqq \sup_{|E|\geqq t} t^{1/p}|E|^{-1/r}\left(\int_X |E_x|\,dx\right)^{1/r-1/p} X^pY^{p\infty}f.$$

For $p<\infty$ there exist functions f such that $\|f\|_{p\infty}=\infty$ while $X^{p\infty}Y^pf<\infty$. It is sufficient to show this for $p=1$. Let $X=Y=R_+$ and f the characteristic function of $\{(x,y):xy<1\}$ then $\lambda_f=\infty$ while $X_*^{1\infty}Y^1f=\|\,|\cdot|^{-1}\,\|_{1\infty}^*=1$.

COROLLARY. *For f measurable on $X\times Y$, $1\leqq p<\infty$, $X^pY_*^{p1}f\leqq\|f\|_{p1}^*$.*

Proof.

$$X^pY_*^{p1}f = \sup_{\|g\|_{p'}\leqq 1}\int \|f(x,\cdot)\|_{p1}^*g(x)\,dx$$
$$= p^{-1}\sup_{X^{p'}Y_*^{p'\infty}g\leqq 1}\int f(x,y)g(x,y)\,dx\,dy$$
$$\leqq p^{-1}\sup_{\|g\|_{p'\infty}^*\leqq 1}\int f(x,y)g(x,y)\,dx\,dy$$
$$= \|f\|_{p1}^*.$$

By interpolation between these inequalities and $\|f\|_p=X^pY^pf$ it follows that there is a $B_p>0$ such that $\|f\|_{pu}\leqq B_pX^pY^{pu}f$ for $0<p\leqq u\leqq\infty$ and $X^pY^{pu}f\leqq B_p\|f\|_{pu}$ for $1\leqq u\leqq p<\infty$.

EXAMPLE. Consider $g(x)=\Omega(x)|x|^{-\lambda}$ where Ω is positively homogeneous of degree zero and $\|\Omega\|[L^{n/\lambda}(S^{n-1})]<\infty$. Since $dx=dx'r^{n-1}\,dr$ for $x=rx'$, $|x'|=1$, it follows that

$$\|r^{-\lambda}\|[L^{n/\lambda\,\infty}(R_+,r^{n-1}\,dr)] = \sup t^{\lambda/n}\int_{r^{-\lambda}>t} r^{n-1}\,dr$$
$$= \sup t^{\lambda/n}\int_0^{t^{-1/\lambda}} r^{n-1}\,dr = n^{-1},$$

hence $\|g\|^*_{n/\lambda,\infty} \leqq n^{-1}\|\Omega\|_{n/\lambda}$. (By the above remark equality holds.) This implies Muckenhoupt's theorem [17]:

(24) $\quad \|g*f\|_q \leqq C\|f\|_p, \qquad 1/q = 1/p-(n-\lambda)/n, \qquad 0 < 1/q < 1/p < 1,$

(cf. [19, 11.17]).

Another short proof of the fractional integration theorem in weighted norms can be given by means of the multiplication and convolution theorems for L^{pq} spaces with a somewhat enlarged range of values for p, q (see also [10]). As usual, let $\alpha^+ = \max(\alpha, 0)$, $\alpha^- = \max(-\alpha, 0)$.

PROPOSITION 4. *Suppose* $1/p'+1/q=(\alpha+\beta+\gamma)/n$,

(25) $\quad 0 \leqq \gamma < n, \quad \alpha+\beta \geqq 0, \quad 1/p' \geqq \alpha^+/n, \quad 1/q \geqq \beta^+/n, \quad 0 < u \leqq v$

except if $1/p'=\alpha^+/n$ *or* $1/q=\beta^+/n$ (*or both, hence* $\alpha, \beta \geqq 0$, $\gamma=0$) *in which case* $1=u=v'$ *and* $I_\gamma f(x)=\int |x-t|^{-\gamma}f(t)\,dt$, $\|f\|_{pu,\alpha}=\|f|\cdot|^\alpha\|_{pu}$. *Then*

$$\|I_\gamma f\|_{qv,-\beta} \leqq C\|f\|_{pu,\alpha}.$$

Proof. Let $M_\alpha f(t)=|t|^{-\alpha}f(t)$, $I_\alpha f(t)=|\cdot|^{-\alpha}*f(t)$, $Tf(x)=\int K(x,y)f(y)\,dy$ where $K(x,y)=K_{\alpha,\gamma,\beta}(x,y)=|x|^{-\beta}|x-y|^{-\gamma}|y|^{-\alpha}$ and distinguish three cases.

(I) Suppose $\alpha, \beta \geqq 0$. Then by the multiplication and convolution theorems (see [13, Theorems 4.5, 4.10]),

$$\|Tf\|_{qv} = \|M_\beta I_\gamma M_\alpha f\|_{qv} \leqq C\|I_\gamma M_\alpha f\|_{p_2 v} \leqq C\|M_\alpha f\|_{p_1 u} \leqq C\|f\|_{pu}$$

where $1/q=1/p_2+\beta/n$, $1/p_2=1/p_1+\gamma/n-1$, $1/p_1=1/p+\alpha/n$, since by assumption $1 \geqq 1/p_1 > 1/p_2 = 1/p+(\alpha+\gamma)/n-1=1/q-\beta/n \geqq 0$ with $1/p_1=1$ or $1/p_2=0$ only if $u=v'=1$.

If one of $\alpha, \beta<0$ estimate $K(x,y)$ separately according as (a) $|x|/|y|<1/2$, (b) $1/2 \leqq |x|/|y| \leqq 2$, (c) $2<|x|/|y|$.

(II) $\alpha<0$, now $1/p' \geqq 0$, $1/q \geqq \beta/n > (\alpha+\beta)/n$. If (a) or (b) holds then $K(x,y) \leqq C|x-y|^{-\gamma}|y|^{-\alpha-\beta}$, if (c) $K(x,y) \leqq C|x-y|^{-\alpha-\gamma}|y|^{-\beta}$, $(\alpha+\gamma)/n=1/p'+1/q-\beta/n \geqq 0$. Hence by case (I) T maps L^{pu} boundedly into L^{qv}.

(III) $\beta<0$, now

(26) $$1/p' \geqq \alpha/n > (\alpha+\beta)/n, \qquad 1/q \geqq 0.$$

If (a) then $K(x,y) \leqq C|x|^{-\alpha}|x-y|^{-\gamma-\beta}$, $(\gamma+\beta)/n=1/p'+1/q-\alpha/n \geqq 0$, if (b) or (c) then $K(x,y) \leqq C|x|^{-\alpha-\beta}|x-y|^{-\gamma}$. By (26) and by case (I) the assertion follows.

REMARKS. In those cases when L^{qv} is a normed space (III) can be obtained from (II) by passing to the adjoint. Suppose $\Omega, \Omega_0, \Omega_1 \geqq 0$ are homogeneous of degree 0. Since $\Omega \in L^{n/\alpha}(S^{n-1})$ implies $\Omega|\cdot|^{-\alpha} \in L^{n/\alpha,\infty}$ (by Lemma 3) it is easy to see that if instead of $|\cdot|^\alpha$, $|\cdot|^{-\beta}$ the more general weight functions $|\cdot|^\alpha\Omega_0$, $|\cdot|^{-\beta}\Omega_1$ are used, then

$$\|[(\Omega|\cdot|^{-\gamma}]*f)\Omega_1|\cdot|^{-\beta}\|_{qv} \leqq C\|f\Omega_0|\cdot|^\alpha\|_{pu}$$

provided in addition $\Omega_0^{-1} \in L^{n/(\alpha-\alpha^--\beta^-)}(S^{n-1})$, $\Omega_1 \in L^{n/(\beta-\alpha^--\beta^-)}(S^{n-1})$,

$$\Omega \in L^{n/(\gamma-\alpha^--\beta^-)}(S^{n-1}).$$

Proposition 4 raises the question whether the conditions $p \leqq q$ (if $p=u$, $q=v$), $\alpha+\beta \geqq 0$ are necessary.

PROPOSITION 5. *Suppose*

$$Tf(x) = \int \Omega(x-y)|x-y|^{-\gamma} f(y)\, dy, \qquad \omega_0(x) = \Omega_0(x)|x|^{\alpha},$$

$$\omega_1(x) = \Omega_1(x)|x|^{-\beta}$$

where Ω, $\Omega_i \geqq 0$, Ω, Ω_0, Ω_1 *are positively homogeneous of degree* 0 *and none is* 0 *a.e. If* $\|f\|_{p,\omega_0} = \|f\omega_0\|_p$ *then in order for*

$$\|Tf\|_{q,\omega_1} \leqq C\|f\|_{p,\omega_0} \tag{27}$$

to hold it is necessary that $p \leqq q$.

Proof. Define the continuous linear maps $A\colon L^p(R_+^\times, L^p(S^{n-1})) \to L^p_{\omega_0}$ where the multiplicative group of positive real numbers $R_+^\times$ is provided with Haar measure dt/t, by $(Ag)(x) = g(|x|, x')|x|^{-\alpha-n/p}\Omega_0^{-1}(x)$, $(x' = |x|^{-1}x)$ and

$$B\colon L^q_{\omega_1} \to L^q(R_+^\times, L^q(S^{n-1}))$$

by $(Bg)(t)(\sigma) = g(t\sigma)t^{-\beta+n/q}\Omega_1(\sigma)$ for $\sigma \in S^{n-1}$. Then BTA is continuous, linear and different from 0. Also, if for $\mu > 0$, $g^{\mu}(t) = g(\mu t)$, then

$$BTAg^{\mu} = \mu^{\alpha+n/p}BT(Ag)^{\mu} = \mu^{\alpha+n/p-n+\gamma}B(TAg)^{\mu} = \mu^{\alpha+n/p-n+\gamma+\beta-n/q}(BTAg)^{\mu}$$

hence necessarily $1/q = 1/p + (\alpha+\beta+\gamma)/n - 1$ and BTA is invariant under multiplication. Theorem 1.1 of [12] is valid for any noncompact locally compact group instead of R^n, hence if $p < \infty$ then $p \leqq q$.

The same argument works for relatively dilation invariant operators, i.e., such that for some real number ρ $T(f^{\mu}) = \mu^{\rho}(Tf)^{\mu}$, in particular, for singular integrals. In the present case of integrals, if $p = \infty$, $q < \infty$, BTA must vanish on those g which satisfy $\lim_{t\to\infty} g(t) = \lim_{t\to 0} g(t) = 0$, hence by Fatou's lemma $BTA = 0$. It follows that $T = 0$ whenever $p > q$.

For simplicity let now $\omega_0(x) = |x|^{\alpha}$, $\omega_1(x) = |x|^{-\beta}$ and

$$L^{pq}_{\alpha} = \{f : \|f\|_{pq,\alpha} = \|f|\cdot|^{\alpha}\|_{pq} < \infty\}.$$

PROPOSITION 6. *Let T be continuous, linear and translation invariant from L^p_{α} to L^q_{β}. If T is not* 0 *and* $p < \infty$ *then* $\alpha+\beta \geqq 0$.

Proof. Let $A_{\varepsilon} = \{x : \varepsilon < |x| < \varepsilon^{-1}\}$ then there are $\varphi \in L^p_{\alpha}$, $\varepsilon > 0$ such that $\operatorname{supp}\varphi \subset A_{\varepsilon}$ and $\|(T\varphi)\chi_{A_{\varepsilon}}\|_{q,-\beta} \geqq \varepsilon$. If e is any unit vector

$$\|\varphi(\cdot - \mu e)\|^p_{p,\alpha} \leqq \int |\varphi(x)|^p|x|^{\alpha p}\, dx \max_{\varepsilon \leqq |x| \leqq \varepsilon^{-1}} (|x+\mu e|^{\alpha p}|x|^{-\alpha p}) \leqq C^p_{\varepsilon,\alpha}\mu^{\alpha p}$$

for $\mu \geqq 2\varepsilon^{-1}$, say. Likewise

$$\|T\varphi(\cdot - \mu e)\|_{q,-\beta} \geqq \|T\varphi\|_{q,-\beta} \min_{\varepsilon \leqq |x| \leqq \varepsilon^{-1}} (|x+\mu e|^{-\beta}|x|^{\beta}) \geqq D_{\varepsilon,\beta}\mu^{-\beta} \quad \text{for } \mu \geqq 2\varepsilon^{-1}$$

where $D_{\varepsilon\beta} > 0$, hence $\mu^{-\alpha-\beta}$ must remain bounded as $\mu \to \infty$, thus $\alpha+\beta \geqq 0$.

For singular integral operators $1/q=1/p+(\alpha+\beta)/n$, $\alpha+\beta\geqq 0$, $p\leqq q$ imply $p=q$, $\alpha=-\beta$.

REMARKS. Proposition 6 holds for $L^{p,q}$ spaces. Specifically $T\colon L^{pu}_{\alpha}\to L^{qv}_{-\beta}$ under the remaining hypotheses of Proposition 6, if $u<\infty$, implies $T=0$ or $\alpha+\beta\geqq 0$.

Propositions 4, 5, 6 are easily extended to the case of metrics and kernels with generalized homogeneity in the sense of [20] and integration over subspaces of R^n as follows. Let P be an $n\times n$ real matrix and for $\lambda>0$ denote the linear transformation of R^n $e^{P\log\lambda}$ by $\pi(\lambda)$. It is assumed that $|\pi(\lambda)x|\leqq\lambda|x|$ for $0<\lambda\leqq 1$ and any $x\in R^n$. Consequently the relation $|\pi(r(x)^{-1})x|=1$ for $x\neq 0$ defines a metric with the property $r(\pi(\lambda)x)=\lambda r(x)$. Assume further that $X=\{x: x_j=0,\ n_1+1\leqq j\leqq n\}$ and $Y=\{x: x_j=0,\ 1\leqq j\leqq n-n_2\}$ are invariant under P. Let $P_1=P|X$, $P_2=P|Y$ and let a_j be the trace of P_j for $j=1, 2$. If

$$\begin{aligned} a_1/p'+a_2/q &= \alpha+\beta+\gamma, \qquad 0\leqq\gamma<\min(a_1,a_2), \qquad \alpha+\beta\geqq 0,\\ \alpha^+/a_1 &\leqq 1/p', \qquad \beta^+/a_2\leqq 1/q, \qquad 0<u\leqq v \end{aligned} \tag{28}$$

except if $1/p'=\alpha^+/a_1$ or $1/q=\beta^+/a_2$ when $1=u=v'$ and if $If(x)=\int_Y r(x-y)^{-\gamma}f(y)\,dy$ then $\|(If)r^{-\beta}\|[L^{qv}(X)]\leqq C\|fr^{\alpha}\|[L^{pu}(Y)]$.

The proof is as that of Proposition 4 except that in place of the convolution theorem Krée's generalization of Kantorovič's inequality (see [16]) contained in Proposition 1 is used. Note that if $K(x,y)=r(x)^{-\beta}r(x-y)^{-\gamma}r(y)^{-\alpha}$ then

$$Y^{\infty}X^{a_1/\gamma,\infty}K<\infty, \qquad X^{\infty}Y^{a_2/\gamma,\infty}K<\infty.$$

By means of the preceding remarks it is easy to construct an example, somewhat similar to that of [3] showing that the Marcinkiewicz interpolation theorem does not hold above the main diagonal. Let $K_{\alpha,\gamma,\beta}(x,y)=|x|^{-\beta/a_1}(|x|^{1/a_1}+|y|^{1/a_2})^{-\gamma}|y|^{-\alpha/a_2}$ for $X=Y=R$. Then it is easy to see that for $T_{\alpha\gamma\beta}f(x)=\int K_{\alpha\gamma\beta}(x,y)f(y)\,dy$, $\|T_{\alpha\gamma\beta}f\|_{qv}\leqq C\|f\|_{pu}$ provided (28) holds (if $a_1, a_2\geqq 1$, $|x|^{1/a_1}+|y|^{1/a_2}$ is a metric equivalent to r constructed above for $P=\operatorname{diag}(a_1,a_2)$). Let $(1/p_i, 1/q_i)$ $(i=0,1)$ be such that $1/p_0<1/p_1$, $1/q_0<1/q_1$, $0\leqq 1/p_i\leqq 1$, $0\leqq 1/q_i<\infty$ and

$$1/p_1-1/p_0,\ 1/q_1-1/q_0<1. \tag{29}$$

Let the line through $(1/p_i, 1/q_i)$ $(i=0,1)$ have equation $a_1/p'+a_2/q=c\geqq 0$. By (29) $a_1(1/p_0'-1/p_1')=a_2(1/q_1-1/q_0)<\min(a_1,a_2)$. Let γ, α satisfy $a_1(1/p_0'-1/p_1')\leqq\gamma<\min(a_1,a_2)$, $\gamma\leqq c$, $a_1/p_0'-\gamma\leqq\alpha\leqq a_1/p_1'$, thus $\beta=a_1/p_0'+a_2/q_0-\alpha-\gamma\leqq a_2/q_0$; also $\alpha+\beta=\gamma-c\geqq 0$. Hence $T_{\alpha\gamma\beta}$ is of restricted weak types (p_i,q_i) for $i=0,1$, but it is not of type (p_t,q_t) for $p_t^{-1}=(1-t)p_0^{-1}+tp_1^{-1}$, $q_t=(1-t)q_0^{-1}+q_1^{-1}$, $0\leqq t\leqq 1$, unless $p_t\leqq q_t$ by the argument of Proposition 5.

4. **Singular integrals in weighted norms.** The following proposition weakens the integrability conditions on Ω given in Theorem 6 of [25]. (The conditions have been weakened still further by recent independent joint results of Muckenhoupt and Wheeden.)

PROPOSITION 7. *Suppose* Ω *is homogeneous of degree* 0 *and has mean value* 0 *on* S^{n-1}, $K(x)=\Omega(x)|x|^{-n}$. *Let*

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x-y)f(y)\,dy, \qquad Tf = \lim_{\varepsilon\to 0} T_\varepsilon f = \text{p.v. } K*f.$$

Suppose T_ε *is bounded in* L^p *independently of* $\varepsilon>0$ *for some* p, $1<p<\infty$. *If* $-n/p<\alpha<n/p'$ *and* $\Omega\in L^q(S^{n-1})$ *where*

$$1/q \leqq 1-|\alpha|/n \tag{30}$$

then T *is bounded in* L^p_α. *If* $\alpha=-n/p$ *or* n/p' *then* T *is bounded from* L^{p1}_α *to* $L^{p\infty}_\alpha$.

Finally if T *is of weak type* (1, 1), *i.e.*, $\|T_\varepsilon f\|_{1\infty}\leqq C\|f\|_1$, C *independent of* $\varepsilon>0$, $\Omega\in L^{q,1}(S^{n-1})$ *and* (30) *holds, then* T *is bounded from* L^1_α *to* $L^{1\infty}_\alpha$.

The proof is based on

LEMMA 4. *Let* $\{T_j\}$, $j=0,\pm 1,\pm 2,\dots$, *be a family of linear operators on measurable functions on* R^n *satisfying*

$$\|T_j f\|_{qs} \leqq A\|f\|_{pr}, \qquad 1\leqq r\leqq p\leqq q\leqq s\leqq\infty, \tag{31}$$

and such that $T_jf(x)=0$ *unless the support of* f *has a nonempty intersection with* $\{y: 2^j\leqq|y|<2^{j+1}\}$ *and* $2^{j-1}\leqq|x|<2^{j+2}$. *If* $w\geqq 0$ *is a "weight" function on* R^n *satisfying* $w(x_1)/w(x_2)\leqq B$ *for* $\frac{1}{2}\leqq|x_1|/|x_2|\leqq 2$ *and* $T=\sum_{j=-\infty}^{\infty}T_j$, *then*

$$\|(Tf)w\|_{qs} \leqq \text{CAB}^2\,\|fw\|_{pr}$$

if $p=r$, $q=s$, *then* C *may be chosen to be* 3.

Proof. Let χ_j be the characteristic function of $\{x: 2^j\leqq|x|<2^{j+1}\}$, $f_j=f\chi_j$, then $\|f\|_p=(\sum_{-\infty}^{\infty}\|f_j\|_p^p)^{1/p}$, also

$$\|f\|^{*p}_{p\infty} = \sup t^p\lambda_f(t) = \sup t^p\sum_{-\infty}^{\infty}\lambda_{f_j}(t) \leqq \sum\|f_j\|^{*p}_{p\infty}$$

or $\|f\|^*_{p\infty}\leqq(\sum\|f_j\|^{*p}_{p\infty})^{1/p}$. Hence by the complex method of interpolation, $\|f\|_{ps}\leqq C(\sum\|f_j\|^p_{ps})^{1/p}$, $p\leqq s\leqq\infty$, and by duality $\|f\|_{pr}\geqq C(\sum\|f_j\|^p_{pr})^{1/p}$ for $1\leqq r\leqq p$. (This argument duplicates the proof of Lemma 3 and its corollary and the preceding assertion could have been deduced from them.)

$$\begin{aligned}
\|T(f)w\|^q_{qs} &\leqq C^q\sum_{j=-\infty}^{\infty}\sup_{2^j\leqq|x|<2^{j+1}} w(x)^q\Big\|\sum_{|k-j|\leqq 1}T_j(f_k)\Big\|^q_{qs}\\
&\leqq C^q\sum_{j=-\infty}^{\infty}3^{q-1}\sup_{2^j\leqq|x|<2^{j+1}} w(x)^q\sum_{|k-j|\leqq 1}\|T(f_j)\|^q_{pr}\\
&\leqq C^qA^qB^{2q}\sum_{j=-\infty}^{\infty}\|f_jw\|^q_{pr}\\
&\leqq (3AB^2C)^q\Big(\sum\|f_jw\|^p_{pr}\Big)^{q/p} \leqq (3AB^2C)^q\|fw\|^q_{pr}.
\end{aligned}$$

Proof of Proposition 7. Let $T^2(f)(x)=\text{p.v.}\int_{1/2<|x|/|y|<2} K(x-y)f(y)\,dy$, $T_j^2(f)=T^2(f\chi_j)$. Note that for $2^{j-1}\leqq|x|<2^{j+2}$, $2^j\leqq|y|<2^{j+1}$,

$$\{x:|x-y|\leqq 2^{j-2}\}\subset\{x:|x-y|\leqq|x|/2\}\subset\{x:\tfrac{1}{2}\leqq|y|/|x|\leqq 2\}$$
$$\subset\{x:|x-y|\leqq 3|x|\}\subset\{x:|x-y|<2^{j+4}\}.$$

Hence $T_j^2(f)(x)=0$ unless $2^{j-1}<|x|<2^{j+2}$ and

$$\begin{aligned}|T_j^2(f)(x)| &\leqq \left|\text{p.v.}\int_{|x-y|\leqq 2^{j-2}} K(x-y)f(y)\,dy\right| \\ &\quad+\int_{2^{j-2}<|x-y|<2^{j+4}}|\Omega(x-y)|\,|x-y|^{-n}|f(y)|\,dy \\ &= |(T-T_{2^{j-2}})(f)(x)|+2^{-n(j-2)}(|\Omega|\varphi_j)*f(x)\quad\text{where } \varphi_j=\sum_{|k-j|\leqq 3}\chi_k.\end{aligned}$$

Hence if $\sup_{\varepsilon>0}\|T_\varepsilon(f)\|_{ps}\leqq A\|f\|_{pr}$ then

$$\begin{aligned}\|T_j^2(f)\|_{ps} &\leqq 2A\|f\|_{pr}+2^{-n(j-2)}\|\Omega\varphi_j\|_1\|f\|_{pr} \\ &\leqq C(A+\|\Omega\|[L^1(S^{n-1})])\|f\|_{pr}.\end{aligned}$$

Thus Lemma 4 takes care of $T^2=\sum_{j=-\infty}^{\infty}T_j^2$.

Let χ denote the characteristic function of (0, 1). It remains to consider the kernels

$$K_1(x,y)=K(x,y)\chi(2|x|^{-1}|y|),\qquad K_3(x,y)=K(x,y)\chi(2|x|\,|y|^{-1}).$$

Let also $T^i f(x)=\int K_i(x,y)f(y)\,dy$ for $i=1,3$. If $0<\alpha<n/p'$ then $|K_1(x,y)|\leqq 2^\alpha|\Omega(x-y)|\,|x-y|^{-n+\alpha}|y|^{-\alpha}$ and so if $1<p<\infty$ then

$$\begin{aligned}\|T^1f\|_p &\leqq \|(|\Omega|\,|\cdot|^{-n+\alpha})*(|\cdot|^{-\alpha}f)\|_p\leqq C_{p,\alpha}\|\Omega\|_{n/(n-\alpha)}\|\,|\cdot|^{-\alpha}f\|_{(1/p+\alpha/n)^{-1}} \\ &\leqq C_{p,\alpha}\|\Omega\|_{n/(n-\alpha)}\|f\|_p.\end{aligned}$$

If $\alpha=n/p'>0$ then

$$\|(\Omega|\cdot|^{-n/p})*(|\cdot|^{-n/p'}f)\|_{p\infty}\leqq C_p\|\Omega\|_p\|\,|\cdot|^{-n/p'}f\|_1\leqq C_p\|\Omega\|_p\|f\|_{p1}.$$

Moreover for $0\leqq\alpha<n$,

$$\begin{aligned}\|T^1f\|_\infty &\leqq C_\alpha\sup_x\|\Omega(x-\cdot)|x-\cdot|^{-n}|x|^\alpha\chi(2|x|^{-1}|y|)\|_{n/(n-\alpha),1}\|\,|\cdot|^{-\alpha}f\|_{n/\alpha,\infty} \\ &\leqq C_\alpha\|\Omega\|_{n/(n-\alpha),1}\|f\|_\infty.\end{aligned}$$

(Hence for $-n<\alpha\leqq 0$, $\|T^3f\|_1\leqq C_\alpha\|\Omega\|_{n/(n-|\alpha|),1}\|f\|_1$.)

In case $\alpha\leqq 0$, $|K_1(x,y)|\leqq 2^\alpha|\Omega(x-y)|\,|x-y|^{-n}\chi(2|x|^{-1}|y|)$ hence $\|K(x,\cdot)\|_1\leqq C_\alpha\|\Omega\|_1$. Thus if $\alpha=0$ by (9) of Lemma 1, $\|T^1f\|_1\leqq C\|\Omega\|_1\|f\|_1$. Also

$$|K_1(x,y)|\leqq(3/2)^{|\alpha|}|\Omega(x-y)|\,|x-y|^{-n-|\alpha|}|y|^{|\alpha|}\chi(|x-y|^{-1}|y|)$$

hence for $\alpha<0$, $\|K_1(\cdot,y)\|_1\leqq C|\alpha|^{-1}\|\Omega\|_1$. Thus

$$X^\infty Y^1K_1+Y^\infty X^1K_1\leqq C|\alpha|^{-1}\|\Omega\|_1,$$

hence by Kantorovič's inequality (2), $\|T^1f\|_p \leqq C_\alpha\|\Omega\|_1\|f\|_p$ for $1\leqq p\leqq\infty$. By duality if $-n/p<\alpha<0$ then $\|T^3f\|_p\leqq C_{p,\alpha}\|\Omega\|_{n/(n+\alpha)}\|f\|_p$, and if $\alpha=-n/p$ then $\|T^3f\|_{p\infty}\leqq C_p\|\Omega\|_{p'}\|f\|_{p1}$. If $1\leqq p\leqq\infty$, $\alpha>0$, then $\|T^3f\|_p\leqq C_\alpha\|f\|_p$.

Thus the proposition follows by collection of the preceding estimates. Note also that since $1<(|x|/|y|)^\varepsilon+(|x|/|y|)^{-\varepsilon}$ for $\varepsilon\in R$, $\|T^if\|_p\leqq C_{p,\alpha}(\Omega)\|f\|_p$ holds also for $1<p<\infty$, $\alpha=0$, $q>1$.

REMARKS. If instead of the weight $|x|^\alpha$ a weight function $\omega(|x|)$ is used satisfying the conditions: $\omega(\tau)\tau^{-\alpha_1}$ is almost decreasing for some $\alpha_1<n/p'$ (i.e., $\tau\leqq\tau'$ implies $\omega(\tau')\tau'^{-\alpha_1}\leqq C\omega(\tau)\tau^{-\alpha_1}$) while $\omega(\tau)\tau^{\alpha_3}$ is almost increasing for some $\alpha_3<n/p$, it follows similarly that if

$$1/q \leqq 1-n^{-1}\max(|\alpha_1|, |\alpha_3|).$$

Then T is a bounded operator on L^p_ω.

EXAMPLE. If $\omega(\tau)=(\tau+a)^\alpha$, $a\geqq 0$, $-n/p<\alpha<n/p'$, condition (30) is still sufficient.

It is, of course, well known that under the other assumptions T_ε is uniformly bounded in L^p, $1<p<\infty$. Proposition 7 as stated, however, immediately extends to kernels and metrics with generalized homogeneity in the sense of [20].

The proof of boundedness in L^p_α of singular integral operators with kernel $\Omega(x)|x|^{-n}$, $\Omega\in L^q(S^{n-1})$ in [25], is based on the boundedness of the integral operator associated with the kernel

$$H(x,y) = |\Omega(x-y)|\,|x-y|^{-n}(1-|x|^\alpha|y|^{-\alpha}). \tag{32}$$

This kernel may be of independent interest. Proposition 8 shows that the condition $q>2(n-1)$ can be replaced by $q>n$ or rather $\Omega\in L^{n,1}(S^{n-1})$.

LEMMA 5. *Let* $S_1=\{x : 1/2\leqq|x|<2\}$, $S_2=\{x : 1/4\leqq|x|<4\}$, $f^\lambda(x)=f(\lambda x)$, T *linear, such that* $\operatorname{supp}(f)\subset S_1$ *implies* $\operatorname{supp}(Tf)\subset S_2$, $T(f^\lambda)=\lambda^{n/r'}(Tf)^\lambda$ *and, for* f *supported in* S_1, $\|Tf\|_{qv}\leqq A\|f\|_{pu}$ *where* $1\leqq r\leqq\infty$, $1/q=1/p-1/r'$, $u\leqq p$, $v\geqq q$. *Then* $\|Tf\|_{qv}\leqq CA\|f\|_{pu}$ *for any* $f\in L^{pu}$ (*if* $u=p$, $v=q$ *then* C *can be taken to be* 3).

This follows from Lemma 4. Put $T_jf=T(f\chi_j)$.

Lemma 5 and Kantorovič's inequality imply

LEMMA 6. *Let* $X=Y=R^n$, $K(x,y)$ *be homogeneous of degree* $-n$ *and vanish unless* $\frac{1}{2}\leqq|y|/|x|\leqq 2$. *Suppose also*

$$\max\left(\sup_{1/4\leqq|x|<4}\int_{1/2\leqq|y|<2}|K(x,y)|\,dy,\ \sup_{1/4\leqq|y|<4}\int_{1/2\leqq|x|<2}|K(x,y)|\,dx\right) = M<\infty.$$

Then $\|Tf\|_p\leqq 3M\|f\|_p$ *for* $1\leqq p\leqq\infty$.

Now returning to the kernel H of (32),

$$\begin{aligned}\sup_{1/4\leqq|x|<4}\int_{1/2\leqq|y|<2} &|1-|x|^{\alpha}|y|^{-\alpha}|\,|\Omega(x-y)|\,|x-y|^{-n}\,dy\\ &\leqq 2^{|\alpha|}\sup_{1/4\leqq|x|\leqq4}\int_{1/2\leqq|y|\leqq2}\frac{|\,|y|^{\alpha}-|x|^{\alpha}\,|}{|\,|y|-|x|\,|}\cdot\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\,dy\\ &\leqq C_{\alpha}\sup_{|x|\leqq4}\int_{|y|\leqq2}|\Omega(x-y)|\,|x-y|^{-n+1}\,dy\\ &\leqq C_{\alpha}\int_{|y|\leqq6}|\Omega(y)|\,|y|^{-n+1}\,dy\\ &\leqq C_{n,\alpha}\|\Omega\chi(|\cdot|/6)\|_{n,1}\|\,|\cdot|^{-n+1}\chi(|\cdot|/6)\|_{n/(n-1),\infty}\\ &\leqq C_{n,\alpha}\|\Omega\|_{n,1}.\end{aligned}$$

Consequently there holds the following

PROPOSITION 8. *Suppose Ω is homogeneous of degree* 0, $\Omega\in L^{n,1}(S^{n-1})\cap L^{q}(S^{n-1})$, *and q satisfies* (30), *then for H defined by* (32),

$$\left\|\int H(\cdot,y)f(y)\,dy\right\|_{p}\leqq C_{p,q}(\Omega)\|f\|_{p}.$$

(*There are analogous weaker conclusions if* $p=1$, $\alpha=-n/p$ *or* $\alpha=n/p'$.)

In view of what was said about the kernels K_1, K_3 in the proof of Proposition 7 it may only be required to observe that, by the remark at the end of the proof of Proposition 7, $|\Omega(x-y)|\,|x|^{-n}\chi(2|y|\,|x|^{-1})$ and $|\Omega(x-y)|\,|y|^{-n}\chi(2|x|\,|y|^{-1})$, that is, K_1, K_3 with $\alpha=0$ give rise to integral operators of type (p,p) if $1<p<\infty$.

5. **On a generalization of Schur's inequality.** It may be worth pointing out that the proof of Strichartz's generalization of Schur's inequality (§1 of [25]) is implicitly based on Young's inequality for convolutions on the multiplicative group of positive real numbers $R_+^{\times}$ and two general simple lemmas.

Young's inequality for convolutions on $R_+^{\times}$ is equivalent to

LEMMA 7. *If the function K on $R_+\times R_+$ is homogeneous of degree $-1/r$, and satisfies*

$$\int_0^{\infty}|K(1,y)|^{r}y^{r/p'-1}\,dy=M^{r}<\infty$$

and the linear operator T is defined by $Tf(x)=\int K(x,y)f(y)\,dy$, then $\|Tf\|_q\leqq M\|f\|_p$ (norms are taken with respect to Lebesgue measure) provided p, q, r satisfy $1\leqq p$, q, $r\leqq\infty$, $1/q=1/p+1/r-1$.

In fact Young's inequality applied to $\varphi*\psi(x)=\int_0^{\infty}\varphi(xy^{-1})\psi(y)y^{-1}\,dy$ where $\varphi(x)=f(x)x^{1/p}$, $\psi(x)=g(x)x^{1/r}$ says

$$\left\|\int g(\cdot y^{-1})(\cdot)^{1/r-1/q}f(y)y^{1/p-1/r-1}\,dy\right\|_{q}\leqq\|f\|_{p}\|g\|_{r}.$$

Now set $g(y)=K(1, y^{-1})y^{-1/p'-1/r}$ then the left-hand side of the preceding inequality becomes $\|Tf\|_q$ by homogeneity of K while

$$\|g\|_r^r = \int_0^\infty |K(1, y^{-1})|^r y^{-r/p'-1}\, dy = \int_0^\infty |K(1, y)|^r y^{r/p'-1}\, dy.$$

LEMMA 8. *Let $\|K\|[L^p, L^q]$ denote the (quasi-) norm of the operator T, $(Tf(x)=\int K(x, y)f(y)\, dy)$ from $L^p(Y)$ to $L^q(X)$. Suppose $X=X_1\times X_2$, $Y=Y_1\times Y_2$, then for $q\geqq 1$*

$$\|K\|[L^p, L^q] \leqq \| \, \|K\|\, [L^p(Y_1), L^q(X_1)] \, \| \, [L^p(Y_2), L^q(X_2)],$$

where, of course, $\|K\|[L^p(Y_1), L^q(X_1)](x_2, y_2)=\|K((\cdot, x_2), (\cdot, y_2))\|\, [L^p(Y_1), L^q(X_1)]$.

Proof. By Minkowski's inequality for integrals

$$\left\| \iint K(\cdot, \cdot, y_1, y_2) f(y_1, y_2)\, dy_1\, dy_2 \right\|_q$$

$$\leqq \left\| \iint \|K(\cdot, \cdot, \cdot, y_2)\| [L^p(Y_1), L^q(X_1)] \|f(\cdot, y_2)\|_p\, dy_2 \right\|_q$$

$$\leqq \| \, \|K\| [L^p(Y_1), L^q(X_1)] \| [L^p(Y_2), L^q(X_2)] \, \|f\|_p.$$

LEMMA 9. *Suppose furthermore $X_2=Y_2=G$, a compact or locally compact abelian group, $dx_2=dy_2=dg$ denotes Haar measure and*

$$K(x_1, x_2, y_1, y_2) = L(x_1, y_1, x_2 y_2^{-1}) \geqq 0, \qquad p = q \geqq 1.$$

Then

$$\|K\|\, [L^p(Y_1\times G), L^p(X_1\times G)] = \left\| \int_G L(\cdot, \cdot, g)\, dg \right\| [L^p, L^p].$$

Proof. For $\varphi_1 \in L^p(Y_1)$, $\psi_1 \in L^{p'}(X_1)$ let

$$l_{\varphi_1, \psi_1}(g) = \int_{X_1} \int_{Y_1} L(x_1, y_1, g)\varphi_1(y_1)\psi_1(x_1)\, dy_1\, dx_1.$$

Since for $h\geqq 0$ it is well known that $\sup\{\|h * f\|_p : \|f\|_p \leqq 1\}=\int_G h(g)\, dg$, it follows that

$$\left\| \int_G L(\cdot, \cdot, g)\, dg \right\| [L^p(Y_1), L^p(X_1)]$$

$$= \sup\left\{ \int_{X_1} \int_{Y_1} L(x_1, y_1, g)\varphi_1(x_1)\psi_1(y_1)\, dg\, dx_1\, dy_1 : \|\varphi_1\|_p, \|\psi_1\|_{p'} \leqq 1, \varphi_1, \psi_1 \geqq 0 \right\}$$

$$= \sup\left\{ \int_G \int_G l_{\varphi_1\psi_1}(x_2 y_2^{-1})\varphi_2(y_2)\psi_2(x_2)\, dy_2\, dx_2 : \|\varphi_i\|_p, \|\psi_i\|_{p'} \leqq 1, \varphi_i \geqq 0, \psi_i \geqq 0, i = 1, 2 \right\}$$

$$\leqq \|K\|[L^p(Y_1\times G), L^p(X_1\times G)].$$

The reverse inequality follows from Lemma 8 with X_1, Y_1, X_2, Y_2 replaced by G, G, X_1, Y_1 respectively.

As in [25] let now X, Y be cones in R^n, $X'=X\cap S^{n-1}$, $Y'=Y\cap S^{n-1}$. Suppose $K(\lambda x, \lambda y)=\lambda^{-n/r}K(x, y)$ for $\lambda>0$. Write

$$r^{n/q}\int_0^\infty \int_{Y'} K(rx', sy')f(sy')s^{n-1}\,dy'\,ds = \int_0^\infty \int_{Y'} r^{n/q}s^{n/p}K(rx', sy')[s^{n/p}f(sy')]\,dy'\,ds/s.$$

It is seen that $\|Tf\|_q\leqq M\|f\|_p$ is equivalent to

$$\|\tilde{T}g\|[L^q(X'\times R_+^\times)] \leqq M\|g\|[L^p(Y'\times R_+^\times)]$$

where $\tilde{T}$ is defined by $K(rx', sy')=r^{n/q}s^{n/p'}K(rx', sy')$, $g(sy')=s^{n/p}f(sy')$ and X', Y' $R_+^\times$ are provided with euclidean surface measure and multiplication invariant measure respectively. Hence by Lemmas 7 and 8

$$\|Tf\|_q \leqq \int_0^\infty \|K(\cdot, s\cdot)\|[L^p(Y'), L^q(X')]s^{n/p'-1}\,ds\|f\|_p \tag{33}$$

which is a slight simplification and generalization of Corollary 1 of Theorem 1 of [25]. If $K\geqq 0$ and $p=q$ then by Lemma 9

$$\left\|\int_0^\infty K(\cdot, s\cdot)s^{n/p'-1}\,ds\right\|[L^p(Y'), L^p(X')] = \|K\|[L^p, L^p].$$

The following restricted weak type results are related to (33) and could be used to prove results at the endpoints of the intervals for α, β, $1/p$, $1/q$ in the weighted norm estimates for fractional or singular integral operators.

LEMMA 10. *Let K be defined on $R_+\times R_+$, homogeneous of degree $-1/r$, $1\leqq p<\infty$, $\|K(1, \cdot)\|_{p'\infty}^*=M$ and T is defined as before then*

$$\|Tf\|_{q\infty}^* \leqq pM\|f\|_{p1}^* \tag{34}$$

provided $0\leqq 1/q=1/p+1/r^{-1}$. *For* $K\geqq 0$, pM *cannot be replaced by any smaller constant.*

This might be considered as a supplementary remark to Lemma 3 of [6] and can be proved as follows.

$$\int_0^\infty |K(s, t)|\,|f(t)|\,dt = \int_0^\infty K(1, t/s)s^{-1/r}f(t)\,dt \leqq s^{-1/r}M\int_0^\infty (t/s)^{-1/p'}f^*(t)\,dt$$
$$= s^{-1/r+1/p'}M\int_0^\infty f^*(t)t^{1/p}\,dt/t = s^{-1/q}pM\|f\|_{p1}^*.$$

Also for $f=f^*$, $K(1, t)=Mt^{-1/p'}$ equality holds throughout.

PROPOSITION 9. *Let $K(x, y)$ be positively homogeneous of degree $-n/r$, $p \geqq 1$, $q>1$, $1/q+1/p'=1/r$ and $\|K'\|[L^p(Y'), L^q(X')]=A$ where*

$$K'(x', y') = \|K(x', (\cdot)^{1/n}y')\|^*[L^{p'\infty}(R_+)].$$

Then T is of restricted weak type (p, q) and in fact

$$\|Tf\|^*_{q\infty} \leqq Apn^{-1/p'-1/q}\|f\|^*_{p1}. \tag{35}$$

Proof. Let $\varphi(sy')=f(s^{1/n}y')$, $(\tilde{T}\varphi)(rx')=(Tf)(r^{1/n}x')$. Then

$$\|\varphi\|^*[L^{p1}(Y'\times R_+)] = n^{1/p}\|f\|^*_{p1}, \qquad \|\tilde{T}\varphi\|^*[L^{q\infty}(X'\times R_+)] = n^{1/q}\|Tf\|^*_{q\infty} \tag{36}$$

where R_+ is provided with Lebesgue measure. For $f(s^{1/n}y')$ on $(R_+, n^{-1}\,ds)$ is equimeasurable with $f(sy')$ on $(R_+, s^{n-1}\,ds)$. Also

$$(\tilde{T}\varphi)(rx') = n^{-1}\int_{S^{n-1}\times R_+} K(r^{1/n}x', s^{1/n}y')\varphi(sy')\,dy'\,ds.$$

Hence by Lemma 3 and its corollary

$$\begin{aligned}\|\tilde{T}\varphi\|^*[L^{q\infty}(S^{n-1}\times R_+)] &\leqq \|\,\|\tilde{T}\varphi\|^*[L^{q\infty}(R_+)]\|[L^q(S^{n-1})]\\ &\leqq n^{-1}p\left\|\int\|K(\cdot,(\cdot)^{1/n}y')\|^*[L^{p'\infty}(R_+)]\|\varphi(\cdot, y')\|^*[L^{p1}(R_+)]\,dy'\right\|[L^q(S^{n-1})]\\ &\leqq n^{-1}pA\|\,\|\varphi\|^*[L^{p1}(R_+)]\|[L^p(S^{n-1})]\\ &\leqq n^{-1}pA\|\varphi\|^*[L^{p1}(S^{n-1}\times R_+)]\end{aligned}$$

where $q>1$ was needed to apply Minkowski's inequality for integrals (see the proof of Lemma 8). This together with (36) implies (35).

The special case $q^*=t^*=p^*=s^*$ of the following lemma is Lemma 1 of [25].

LEMMA 11. *Suppose for $T_rf(x)=\int |K(x, y)|^rf(y)\,dy$, $\|T_rf\|_{q^*t^*}\leqq M^r\|f\|_{p^*s^*}$ then $\|Tf\|_{qt}\leqq BM\|f\|_{ps}$ where B is a constant depending on r, p^*, q^*, s^*, t^* and*

$$\begin{aligned} &1 \leqq r < \infty, \qquad p = rp^*(1+(r-1)p^*)^{-1},\\ &s = rs^*(1+(r-1)s^*)^{-1}, \qquad q = rq^*, \qquad t = rt^*.\end{aligned} \tag{37}$$

Proof. Let $f\in L^{ps}$. It can be assumed that $K, f\geqq 0$ and f is simple. Then if

$$1/p = 1/p_0+1/p_1, \qquad 1/s = 1/s_0+1/s_1 \qquad (p_i, s_i > 0), \tag{38}$$

then by [13, §3] there exist (simple) H_0, $H_1\geqq 0$ such that $f=H_0H_1$ and

$$\|H_i\|_{p_is_i} \leqq B(\|f\|_{ps})^{s/s_i} \quad \text{for } i = 0, 1. \tag{39}$$

Hence

$$Tf(x) \leqq \left(\int K(x, y)^rH_0(y)^r\,dy\right)^{1/r}\left(\int H_1(y)^{r'}\,dy\right)^{1/r'},$$

$$\begin{aligned}\|Tf\|_{qt} &\leqq \|(T_r(H_0^r))^{1/r}\|_{qt}\|H_1\|_{r'} = \|T_r(H_0^r)\|^{1/r}_{q/r,t/r}\|H_1\|_{r'}\\ &\leqq M\|H_0^r\|_{p^*s^*}\|H_1\|_{r'} = M\|H_0\|_{p^*r,s^*r}\|H_1\|_{r'}.\end{aligned}$$

By the second and third relations of (37), $p_0=p^*r$, $p_1=r'$, $s_0=s^*r$, $s_1=r'$ satisfy (38) and the assertion follows from (39).

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PRINCETON UNIVERSITY,
PRINCETON, NEW JERSEY 08540