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ON EMBEDDINGS WITH LOCALLY NICE CROSS-SECTIONS(1)

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Abstract. A k-dimensional compactum X^k in euclidean space E^n $(n-k \geqq 3)$ is said to be locally nice in E^n if $E^n - X^k$ is 1-ULC. In this paper we prove a general theorem which implies, in particular, that X^k is locally nice in E^n if the intersection of X^k with each horizontal hyperplane of E^n is locally nice in the hyperplane. From known results we obtain immediately that a k-dimensional polyhedron P in E^n $(n-k \geqq 3$ and $n \geqq 5)$ is tame in E^n if each $(E^{n-1} \times \{w\}) - P$ $(w \in E^1)$ is 1-ULC. However, by strengthening our general theorem in the case $n=4$, we are able to prove this result for $n=4$ as well. For example, an arc A in E^4 is tame if each horizontal cross-section of A is tame in the cross-sectional hyperplane (that is, lies in an arc that is tame in the hyperplane).

In this note we give a sufficient condition that a k-dimensional compactum X in euclidean n-space E^n $(n-k \geqq 3)$ have a 1-ULC complement in E^n. A principal application of our result is that $E^n - X$ is 1-ULC provided each $(E^{n-1} \times \{w\}) - X$ $(w \in E^1)$ is 1-ULC, where we consider E^n as $E^{n-1} \times E^1$.

THEOREM 1. *Suppose that X is a k-dimensional compactum in $E^{n+1} = E^n \times E^1$ $(n \geqq 3$ and $n-k \geqq 2)$ such that for each $w \in E^1$ and each $\varepsilon > 0$ there exists an ε-push h of (E^{n+1}, X) (see* [4]) *such that $(E^n \times \{w\}) - h(X)$ is 1-ULC. Then E^{n+1} is 1-ULC.*

THEOREM 2. *Suppose that X is a 1-dimensional compactum in E^4 satisfying the hypothesis of Theorem* 1 *and having the additional property that, for some $\delta > 0$, no component of $(E^3 \times \{w\}) \cap X$ contains a nontrivial (Čech) 1-cycle of diameter less then δ for any $w \in E^1$.*

Then for each 2-complex K in E^4 and for each $\varepsilon > 0$, there exists an ε-push g of $(E^4, X \cap K)$ such that $g(K) \cap X = \varnothing$.

COROLLARY 1. *Suppose that $h_t: M \to Q$ $(t \in [0, 1])$ is an isotopy of locally flat embeddings of the topological m-manifold M in the interior of the topological q-manifold Q $(q-m \geqq 3)$. Then the embedding $H: M \times I \to Q \times I$ defined by $H(x, t) = (h_t(x), t)$ is a locally flat embedding.*

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COROLLARY 2. *Suppose that A is an arc in E^{n+1} ($n \geqq 3$) such that $A \cap (E^n \times \{w\})$ has a 1-ULC complement in $E^n \times \{w\}$. Then A is tame.*

More generally, if for each $w \in E^1$ there exists a small push h of (E^{n+1}, A) such that $(E^n \times \{w\}) - h(A)$ is a tame subset of $E^n \times \{w\}$ of dimension $\leqq 0$, then A is tame.

COROLLARY 3. *If P is a k-dimensional polyhedron in E^{n+1} ($n \geqq 3$ and $n-k \geqq 2$) such that $(E^n \times \{w\}) - P$ is 1-ULC for each $w \in E^1$, then P is tame.*

COROLLARY 4. *Suppose that X is a k-dimensional compactum in E^{n+1} ($n \geqq 3$, $2k+1 \leqq n$) and that $f\colon X \to E^{n+1}$ is an embedding such that both X and $f(X)$ satisfy the hypotheses of Theorem 1 if $n \geqq 4$ and Theorem 2 if $n=3$. Then there exist a compact set C and an isotopy h_t ($t \in [0, 1]$) of E^{n+1} such that $h_0 = \text{identity}$, $h_1 | X = f$, and each h_t is the identity outside C.*

The proofs of the corollaries in the case $n \geqq 4$ are each obtained as a direct application of Theorem 1 and the results of [2] and [3]. The cases in which $n=3$ follow from Theorem 2 and the methods of [1].

Let D^2 denote the unit disk in E^2 and let $S^1 = \text{Bd } D^2$. A subset U of E^n is said to be 1-ULC if to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that every map $f\colon S^1 \to U$ with $\operatorname{diam} f(S^1) < \delta$ extends to a map $F\colon D^2 \to U$ with $\operatorname{diam} F(D^2) < \varepsilon$. We use I to denote the interval $[-1, 1]$ considered as a subset of the first factor of E^n. Let

$$D^* = \{(x, t) \in E^n \times E^1 \mid x \in I \quad \text{and} \quad 0 \leqq t \leqq 1\}.$$

LEMMA 1. *Suppose that $E^n - X^k$ ($n-k \geqq 3$) is 1-ULC and that $F\colon D^2 \to E^n$ is a map with $F(S^1) \cap X = \varnothing$. Then for each $\varepsilon > 0$ there exists a map $G\colon D^2 \to E^n - X$ such that $d(F(x), G(x)) < \varepsilon$ and $G|S^1 = F|S^1$.*

For a proof of this lemma, see the proof of Lemma 1 of [1].

LEMMA 2. *Suppose that $X^k \subset E^{n+1} = E^n \times E^1$ ($n \geqq 3$ and $n-k \geqq 2$) and that $f\colon D^* \to E^n \times [0, 1]$ is a level-preserving map (that is, $f(x, t) \in E^n \times \{t\}$ for each $(x, t) \in D^*$) such that $f|\text{Bd } I \times [0, 1] = \text{inclusion}$ and $f(\text{Bd } D^*) \cap X = \varnothing$. Then for each neighborhood U of $f(D^*)$ in E^{n+1}, there exist numbers*

$$t_0 = 0 < t_1 < t_2 < \cdots < t_m = 1$$

and embeddings $\alpha_i\colon I \to E^n$ ($i = 1, 2, \ldots, m$) such that

(a) $\alpha_i | \text{Bd } I = \text{inclusion}$,

(b) $(\alpha_1(x), 0) = f(x, 0)$ *and* $(\alpha_m(x), 1) = f(x, 1)$, *and*

(c) $\alpha_i(I) \times [t_{i-1}, t_i] \subset U - X$ *for each* $i = 1, 2, \ldots, m$.

Proof. Let f_t be $f|I \times \{t\}$ followed by the projection into E^n. We may assume that f_0 and f_1 are embeddings ($n \geqq 3$). Since $f(I \times \{0\}) \cap X = f(I \times \{1\}) \cap X = \varnothing$, there exist numbers $0 < s_0 \leqq s_1 < 1$ such that $(f_0(I) \times [0, s_0]), (f_1(I) \times [s_1, 1]) \subset U - X$. For each $t \in [s_0, s_1]$ there exists an embedding $\alpha_t\colon I \to E^n$ such that $\alpha_t | \text{Bd } I = \text{inclusion}$ and $\alpha_t(I) \times \{t\} \subset U - X$ (since $n-k \geqq 2$). Hence, there exists $\delta_t > 0$ such that $\alpha_t(I) \times (t - \delta_t, t + \delta_t) \subset U - X$.

Let $\lambda>0$ be a Lebesgue number for the cover $\{(t-\delta_t, t+\delta_t) \mid s_0 \leqq t \leqq s_1\}$ of $[s_0, s_1]$. Choose $t_1=s_0<t_2<\cdots<t_{m-1}=s_1$ so that $t_i-t_{i-1}<\lambda$ $(i=2,\ldots,m-1)$. Taking $\alpha_1=f_0$ and $\alpha_m=f_1$, we see that for $i=1,\ldots,m$ there exist embeddings $\alpha_i: I\to E^n$ satisfying (a), (b), and (c).

LEMMA 3. *Suppose that $X^k\subset E^{n+1}=E^n\times E^1$ $(n\geqq 3$ and $n-k\geqq 2)$ satisfies the hypothesis of Theorem* 1 *and that f: $D^*\to E^n\times[0,1]$ is a level-preserving map with $f|\mathrm{Bd}\, I\times[0,1]=$ inclusion and $f(\mathrm{Bd}\, D^*)\cap X=\varnothing$. Then for each neighborhood U of $f(D^*)$ there exists a map g: $D^*\to U-X$ such that $g|\mathrm{Bd}\, D^*=f|\mathrm{Bd}\, D^*$.*

Proof. Let U be a neighborhood of D^* and assume that $f|I\times\{0,1\}$ is an embedding. From Lemma 2 we obtain numbers $t_0=0<t_1<t_2<\cdots<t_m=1$ and embeddings $\alpha_i: I\to E^n$ $(i=1,2,\ldots,m)$ satisfying

$$\alpha_i|\mathrm{Bd}\, I = \text{inclusion},$$
$$\alpha_1=f_0 \quad\text{and}\quad \alpha_m=f_1$$

(again, f_t is $f|I\times\{t\}$ followed by the projection into E^n), and

$$\alpha_i(I)\times[t_{i-1}, t_i]\subset U-X$$

for each $i=1,2,\ldots,m$. Moreover, we may assume that each

$$[\alpha_i(I)\cup\alpha_{i+1}(I)]\times\{t_i\}$$

is a simple closed curve that bounds a singular disk D_i' in $U\cap(E^n\times\{t_i\})$ $(i=1,\ldots,m-1)$.

By hypothesis, there exists a small push h_1 of (E^{n+1}, X) such that

$$(E^n\times\{t_1\})-h_1(X)$$

is 1-ULC. If h_1 is a sufficiently small push, then we will have

$$h_1\Big|\bigcup_{i=1}^{m}(\alpha_i(I)\times[t_{i-1}, t_i]) = \text{identity}.$$

By Lemma 1, we may replace D_1' by a singular disk D_1'' in $[U\cap(E^n\times\{t_1\})]-h_1(X)$. Then $h_1^{-1}(D_1'')\cap X=\varnothing$ and $\mathrm{Bd}\,(h_1^{-1}(D_1''))=(\alpha_1(I)\cup\alpha_2(I))\times\{t_1\}$. Thus

$$[\alpha_1(I)\cup\alpha_2(I)]\times\{t_1\}$$

bounds a disk D_1 $(=h_1^{-1}(D_1''))$ in $U-X$.

Continuing in this manner, we obtain singular disks $D_1, D_2,\ldots,D_{m-1}$ in $U-X$ such that $\mathrm{Bd}\,(D_i)=[\alpha_i(I)\cup\alpha_{i+1}(I)]\times\{t_i\}$ for $i=1,\ldots,m-1$. Taking the union

$$(\alpha_1(I)\times[t_0, t_1])\cup D_1\cup(\alpha_2(I)\times[t_1, t_2])\cup\cdots\cup D_{m-1}\cup(\alpha_m(I)\times[t_{m-1}, t_m]),$$

we obtain a singular disk D in $U-X$ such that $\mathrm{Bd}\, D=f(\mathrm{Bd}\, D^*)$.

Proof of Theorem 1. Suppose that $X\subset E^{n+1}=E^n\times E^1$ satisfies the hypothesis of Theorem 1. Let Γ' be a small simple closed curve in $E^{n+1}-X$. We may change

Γ' by a small homotopy to a simple closed curve Γ in $E^{n+1}-X$ such that Γ is the union of a finite collection of small line segments, each being parallel to either $E^n\times\{0\}$ or to $\{0\}\times E^1$.

Let v_0 be a vertex of Γ. We are going to construct a small homotopy of Γ in $E^{n+1}-X$ that takes Γ into $E^n\times\{v_0\}$.

Let J be a "horizontal" segment in Γ (that is, J is parallel to $E^n\times\{0\}$), and let u and v be the endpoints of J. Let $f_t\colon J\to E^{n+1}$ ($t\in[0,1]$) be the natural homotopy that slides J into the plane $E^n\times\{v_0\}$, oriented so that $f_0=$inclusion and $f_1(J)\subset E^n\times\{v_0\}$.

Since $n+1\geqq 4$ and $n-k\geqq 2$, we may assume that the vertical segments $f(\{u\}\times[0,1])$ and $f(\{v\}\times[0,1])$ (where $f(x,t)=f_t(x)$) miss X. (A small argument is required to see that this can be accomplished with a homeomorphism that preserves $(n+1)$th coordinates.) Since $n-k\geqq 2$, we may also assume that $f_1(J)\cap X=\varnothing$ by slightly altering the homotopy f_t on the interior of J as t approaches 1.

Lemma 3 now allows us to replace f_t ($t\in[0,1]$) by a homotopy $g_t\colon J\to E^n-X$ ($t\in[0,1]$) having the properties that $g_0=f_0$, $g_1=f_1$, $g|\mathrm{Bd}\,J\times[0,1]=f|\mathrm{Bd}\,J\times[0,1]$, and $g(J\times[0,1])$ lies in any preassigned neighborhood U of $f(J\times[0,1])$.

Thus the natural homotopy of Γ into the plane $E^n\times\{v_0\}$ can be replaced by a homotopy $h_t\colon\Gamma\to E^{n+1}-X$ ($t\in[0,1]$) that takes Γ into $E^n\times\{v_0\}$. Applying the hypothesis of Theorem 1 once again to the plane $E^n\times\{v_0\}$ as in the proof of Lemma 3, we see that $h_1(\Gamma)$ can be contracted to a point in $E^{n+1}-X$ by a small homotopy. The combination of these two homotopies gives a small homotopy of Γ to a point in $E^{n+1}-X$.

Before we prove Theorem 2, we must prove a strengthened version of Lemma 3. We also need a little more notation. Let

$$B^4=\{(x_1,x_2,x_3,x_4)\in E^4\mid -1\leqq x_1,x_2,x_3\leqq 1\quad\text{and}\quad 0\leqq x_4\leqq 1\}$$

(notice that D^* is embedded properly in B^4), and let

$$B_t^3=\{x\in B^4\mid x_4=t\}\qquad(t\in[0,1]).$$

LEMMA 4. *Suppose that X is a 1-dimensional compactum in E^4 satisfying the hypothesis of Theorem 1 such that $X\cap\mathrm{Bd}\,D^*=\varnothing$ and every component of $X\cap B_t^3$ is acyclic for each $t\in[0,1]$. Then there exists a homeomorphism $h\colon B^4\to B^4$ such that $h|\mathrm{Bd}\,B^4=$ identity and $h(D^*)\cap X=\varnothing$.*

Proof. Let $B^3=\{(x_1,x_2,x_3)\in E^3\mid -1\leqq x_i\leqq 1\ (i=1,2,3)\}$. Let

$$C_1=\{(x_1,x_2,x_3)\in\mathrm{Bd}\,B^3\mid x_1=0\},$$

and for each $t\in(0,1]$, let $C_t=\{tx\mid x\in C_1\}$ and $S_t=\{(-1,0,0),(1,0,0)\}*C_t$ (where $A*B$ denotes the join of A and B). Each S_t is a 2-sphere in B^3 obtained by suspending the simple closed curve C_t from the points $(-1,0,0)$ and $(1,0,0)$.

For each $t\in(0,1)$ the set $X\cap(S_t\times\{t\})$ does not separate $S_t\times\{t\}$. Thus we can find an embedding $\alpha_t\colon I\to S_t$ such that $\alpha_t|\mathrm{Bd}\,I=$inclusion and $\alpha_t(I)\times\{t\}\cap X=\varnothing$.

Applying the methods in the proof of Lemma 3, we can find numbers $t_0=0<t_1<\cdots t_m=1$ and embeddings $\alpha_i: I\to B^3$ $(i=1,\ldots,m)$ such that

$$\begin{aligned}
&\alpha_i|\mathrm{Bd}\, I = \text{inclusion},\\
&\alpha_1 = \alpha_m = \text{inclusion},\\
&\alpha_i(I)\times[t_{i-1}, t_i]\cap X = \varnothing,\\
&\alpha_i(I)\cap\alpha_j(I) = \mathrm{Bd}\, I \quad \text{if } i\neq j,
\end{aligned}$$

and for $i=2,\ldots,m-1$, $\alpha_i(I)\subset S_t$ for some $t\in(0,1)$.

Now fix i $(2\leqq i\leqq m-1)$ and consider the simple closed curve

$$\Gamma_i = [\alpha_i(I)\cup\alpha_{i+1}(I)]\times\{t_i\}$$

in $B^3_{t_i}$. We may assume that Γ_i is a polygonal curve. Our construction of α_i and α_{i+1} guarantees that Γ_i bounds a nonsingular polyhedral disk D_i' in $B^3_{t_i}$ such that $D_i'\cap\mathrm{Bd}\, B^3_{t_i}=\mathrm{Bd}\, I\times\{t_i\}$. Let h_i be a small push of (E^4, X) such that $(E^3\times\{t_i\})-h_i(X)$ is 1-ULC. Let B_i be a 3-cell in $B^3_{t_i}$ containing D_i' properly. From Lemma 1, we see that Γ_i bounds a singular disk in $B_i-h_i(X)$, and so, by Dehn's lemma [5], Γ_i bounds a nonsingular polyhedral disk D_i in $B_i-h_i(X)$.

Since h_i (and, hence, h_i^{-1}) is a stable homeomorphism of E^4, we may assume that h_i^{-1} is the identity on $(\alpha_i(I)\times[t_{i-1}, t_i])\cup(\alpha_{i+1}(I)\times[t_i, t_{i+1}])$ and outside $B^4\cap E^3\times[\frac{1}{2}(t_{i-1}+t_i), \frac{1}{2}(t_i+t_{i+1})]$. (This is the first time we have really needed stability.)

Observe that the 2-cell

$$D = \alpha_1(I)\times[t_0, t_1]\cup D_1\cup\alpha_2(I)\times[t_1, t_2]\cup\cdots\cup D_{m-1}\cup\alpha_m(I)\times[t_{m-1}, t_m]$$

is properly embedded in B^4 and that the cell pair (B^4, D) is unknotted. Let $h': B^4\to B^4$ be a homeomorphism that is fixed on $\mathrm{Bd}\, B^4$ and takes D^* to D, and let $h'': B^4\to B^4$ be defined by $h''(x)=h_i^{-1}(x)$ if $x\in B^4\cap E^3\times[\frac{1}{2}(t_{i-1}+t_i), \frac{1}{2}(t_i+t_{i+1})]$ and $h''(x)=x$ otherwise. Then the homeomorphism $h=h''h': B^4\to B^4$ satisfies all of our requirements.

Proof of Theorem 2. Suppose that $X\subset E^4$ satisfies the hypotheses of Theorem 2, $\varepsilon>0$, and K is a 2-complex in E^4. Assume that ε is small enough so that $(E^3\times\{w\})\cap X$ does not contain any nontrivial (Čech) 1-cycles of diameter less than ε for any $w\in E^1$.

Subdivide the complex K so that each of its simplexes has diameter less than ε.

First move the vertices of K so that no two of them lie in a single horizontal hyperplane. Next move X off the 1-skeleton of K by an isotopy of E^4 that does not change the x_4-coordinate of any point of E^4.

Let σ be a 2-simplex of K, let u be the vertex of σ with the smallest x_4-coordinate and let v be the one with the largest. Let r and s denote the x_4-coordinates of u and v, respectively. Choose numbers $t_0>r$ and $t_1<s$ such that $\sigma\cap X\subset E^3\times[t_0, t_1]$. Let B be a 4-cell in $E^3\times[t_0, t_1]$ of diameter less than ε such that the pair

$(B, \sigma \cap (E^3 \times [t_0, t_1]))$ is homeomorphic to (B^4, D^*) (as defined above) by a homeomorphism that takes each horizontal cross-section of B to a horizontal cross-section of B^4.

We may now apply Lemma 4 to get a homeomorphism $k: B \to B$ such that $k|\mathrm{Bd}\, B = \text{identity}$ and $k(\sigma) \cap X = \varnothing$. Moreover, k is an ε-push of $(E^4, \sigma \cap X)$, since k is isotopic to the identity (keeping Bd B fixed) via the Alexander isotopy.

If we are careful to construct the 4-cells B corresponding to each 2-simplex σ of K so that any two intersect in a subset of the boundary of each, then the ε-pushes k that we obtain will piece together to give the desired ε-push h of $(E^4, K \cap X)$.

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