ON EMBEDDINGS WITH LOCALLY NICE CROSS-SECTIONS(1)

BY

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Abstract. A k-dimensional compactum \( X^k \) in euclidean space \( E^n \) \((n-k \geq 3)\) is said to be locally nice in \( E^n \) if \( E^n - X^k \) is 1-ULC. In this paper we prove a general theorem which implies, in particular, that \( X^k \) is locally nice in \( E^n \) if the intersection of \( X^k \) with each horizontal hyperplane of \( E^n \) is locally nice in the hyperplane. From known results we obtain immediately that a k-dimensional polyhedron \( P \) in \( E^n \) \((n-k \geq 3 \text{ and } n \geq 5)\) is tame in \( E^n \) if each \( (E^{n-1} \times \{w\}) - P \) \((w \in E^1)\) is 1-ULC. However, by strengthening our general theorem in the case \( n=4 \), we are able to prove this result for \( n=4 \) as well. For example, an arc \( A \) in \( E^4 \) is tame if each horizontal cross-section of \( A \) is tame in the cross-sectional hyperplane (that is, lies in an arc that is tame in the hyperplane).

In this note we give a sufficient condition that a k-dimensional compactum \( X \) in euclidean n-space \( E^n \) \((n-k \geq 3)\) have a 1-ULC complement in \( E^n \). A principal application of our result is that \( E^n - X \) is 1-ULC provided each \( (E^{n-1} \times \{w\}) - X \) \((w \in E^1)\) is 1-ULC, where we consider \( E^n \) as \( E^{n-1} \times E^1 \).

Theorem 1. Suppose that \( X \) is a k-dimensional compactum in \( E^{n+1} = E^n \times E^1 \) \((n \geq 3 \text{ and } n-k \geq 2)\) such that for each \( w \in E^1 \) and each \( \epsilon > 0 \) there exists an \( \epsilon \)-push \( h \) of \( (E^{n+1}, X) \) (see [4]) such that \( (E^n \times \{w\}) - h(X) \) is 1-ULC. Then \( E^{n+1} \) is 1-ULC.

Theorem 2. Suppose that \( X \) is a 1-dimensional compactum in \( E^4 \) satisfying the hypothesis of Theorem 1 and having the additional property that, for some \( \delta > 0, \) no component of \( (E^3 \times \{w\}) \cap X \) contains a nontrivial (Čech) 1-cycle of diameter less than \( \delta \) for any \( w \in E^1 \).

Then for each 2-complex \( K \) in \( E^4 \) and for each \( \epsilon > 0 \), there exists an \( \epsilon \)-push \( g \) of \( (E^4, X \cap K) \) such that \( g(K) \cap X = \emptyset \).

Corollary 1. Suppose that \( h_t : M \to Q \) \((t \in [0, 1])\) is an isotopy of locally flat embeddings of the topological m-manifold \( M \) in the interior of the topological q-manifold \( Q \) \((q-m \geq 3)\). Then the embedding \( H : M \times I \to Q \times I \) defined by \( H(x, t) = (h_t(x), t) \) is a locally flat embedding.

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Corollary 2. Suppose that $A$ is an arc in $E^{n+1} (n \geq 3)$ such that $A \cap (E^n \times \{w\})$ has a 1-ULC complement in $E^n \times \{w\}$. Then $A$ is tame.

More generally, if for each $w \in E^1$ there exists a small push $h$ of $(E^{n+1}, A)$ such that $(E^n \times \{w\}) - h(A)$ is a tame subset of $E^n \times \{w\}$ of dimension $\leq 0$, then $A$ is tame.

Corollary 3. If $P$ is a $k$-dimensional polyhedron in $E^{n+1} (n \geq 3$ and $n - k \geq 2)$ such that $(E^n \times \{w\}) - P$ is 1-ULC for each $w \in E^1$, then $P$ is tame.

Corollary 4. Suppose that $X$ is a $k$-dimensional compactum in $E^{n+1} (n \geq 3, 2k + 1 \leq n)$ and that $f: X \to E^{n+1}$ is an embedding such that both $X$ and $f(X)$ satisfy the hypotheses of Theorem 1 if $n \geq 4$ and Theorem 2 if $n = 3$. Then there exist a compact set $C$ and an isotopy $h_t (t \in [0, 1])$ of $E^{n+1}$ such that $h_0 = \text{identity}$, $h_1 | X = f$, and each $h_t$ is the identity outside $C$.

The proofs of the corollaries in the case $n \geq 4$ are each obtained as a direct application of Theorem 1 and the results of [2] and [3]. The cases in which $n = 3$ follow from Theorem 2 and the methods of [1].

Let $D^2$ denote the unit disk in $E^2$ and let $S^1 = \text{Bd } D^2$. A subset $U$ of $E^n$ is said to be 1-ULC if to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that every map $f: S^1 \to U$ with $\text{diam } f(S^1) < \delta$ extends to a map $F: D^2 \to U$ with $\text{diam } F(D^2) < \varepsilon$. We use $I$ to denote the interval $[-1, 1]$ considered as a subset of the first factor of $E^n$. Let

$$D^* = \{(x, t) \in E^n \times E^1 \mid x \in I \text{ and } 0 \leq t \leq 1\}.$$

Lemma 1. Suppose that $E^n - X^k (n - k \geq 3)$ is 1-ULC and that $F: D^2 \to E^n$ is a map with $F(S^1) \cap X = \emptyset$. Then for each $\varepsilon > 0$ there exists a map $G: D^2 \to E^n - X$ such that $d(F(x), G(x)) < \varepsilon$ and $G| S^1 = F| S^1$.

For a proof of this lemma, see the proof of Lemma 1 of [1].

Lemma 2. Suppose that $X^k \subset E^{n+1} = E^n \times E^1 (n \geq 3$ and $n - k \geq 2)$ and that $f: D^* \to E^n \times [0, 1]$ is a level-preserving map (that is, $f(x, t) \in E^n \times \{t\}$ for each $(x, t) \in D^*$) such that $f| \text{Bd } I \times [0, 1] = \text{inclusion}$ and $f(\text{Bd } D^*) \cap X = \emptyset$. Then for each neighborhood $U$ of $f(D^*)$ in $E^{n+1}$, there exist numbers

$$t_0 = 0 < t_1 < t_2 < \cdots < t_m = 1$$

and embeddings $\alpha_i: I \to E^n (i = 1, 2, \ldots, m)$ such that

(a) $\alpha_i| \text{Bd } I = \text{inclusion},$
(b) $(\alpha_1(x), 0) = f(x, 0)$ and $(\alpha_m(x), 1) = f(x, 1)$, and
(c) $\alpha_i(I) \times [t_{i-1}, t_i] \subset U - X$ for each $i = 1, 2, \ldots, m$.

Proof. Let $f_t$ be $f| I \times \{t\}$ followed by the projection into $E^n$. We may assume that $f_0$ and $f_1$ are embeddings ($n \geq 3$). Since $f(I \times \{0\}) \cap X = f(I \times \{1\}) \cap X = \emptyset$, there exist numbers $0 < s_0 \leq s_1 < 1$ such that $(f_0(I) \times [0, s_0]), (f_1(I) \times [s_1, 1]) \subset U - X$. For each $t \in [s_0, s_1]$ there exists an embedding $\alpha_t: I \to E^n$ such that $\alpha_t| \text{Bd } I = \text{inclusion}$ and $\alpha_t(I) \times \{t\} \subset U - X$ (since $n - k \geq 2$). Hence, there exists $\delta_t > 0$ such that $\alpha_t(I) \times (t - \delta_t, t + \delta_t) \subset U - X$. 

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Let $\lambda > 0$ be a Lebesgue number for the cover $\{((t-\delta_i, t + \delta_i) \mid s_0 \leq t \leq s_1}\}$ of $[s_0, s_1]$. Choose $t_1 = s_0 < t_2 < \cdots < t_{m-1} = s_1$ so that $t_i - t_{i-1} < \lambda$ $(i = 2, \ldots, m-1)$. Taking $\alpha_1 = f_0$ and $\alpha_m = f_1$, we see that for $i = 1, \ldots, m$ there exist embeddings $\alpha_i: I \to E^n$ satisfying (a), (b), and (c).

**Lemma 3.** Suppose that $X^k \subset E^{n+1} = E^n \times E^1$ $(n \geq 3$ and $n-k \geq 2)$ satisfies the hypothesis of Theorem 1 and that $f: D^* \to E^n \times [0, 1]$ is a level-preserving map with $f|\Bd I \times [0, 1] = \text{inclusion}$ and $f(\Bd D^*) \cap X = \emptyset$. Then for each neighborhood $U$ of $f(D^*)$ there exists a map $g: D^* \to U - X$ such that $g|\Bd D^* = f|\Bd D^*$.

**Proof.** Let $U$ be a neighborhood of $D^*$ and assume that $f|I \times \{0, 1\}$ is an embedding. From Lemma 2 we obtain numbers $t_0 = 0 < t_1 < t_2 < \cdots < t_m = 1$ and embeddings $\alpha_i: I \to E^n$ $(i = 1, 2, \ldots, m)$ satisfying

- $\alpha_i|\Bd I = \text{inclusion}$,
- $\alpha_1 = f_0$ and $\alpha_m = f_1$ (again, $f_i$ is $f|I \times \{t\}$ followed by the projection into $E^n$), and
- $\alpha_i(I) \times [t_{i-1}, t_i] \subset U - X$

for each $i = 1, 2, \ldots, m$. Moreover, we may assume that each $[\alpha_i(I) \cup \alpha_{i+1}(I)] \times \{t_i\}$ is a simple closed curve that bounds a singular disk $D'_i$ in $U \cap (E^n \times \{t_i\})$ $(i = 1, \ldots, m-1)$.

By hypothesis, there exists a small push $h_1$ of $(E^n \times \{t_1\}, X)$ such that $(E^n \times \{t_1\}) - h_1(X)$ is 1-ULC. If $h_1$ is a sufficiently small push, then we will have

$$h_1\bigcup_{i=1}^{m} (\alpha_i(I) \times [t_{i-1}, t_i]) = \text{identity}.$$

By Lemma 1, we may replace $D'_i$ by a singular disk $D''_i$ in $[U \cap (E^n \times \{t_i\})] - h_1(X)$. Then $h_1^{-1}(D''_i) \cap X = \emptyset$ and $\Bd (h_1^{-1}(D''_i)) = (\alpha_i(I) \cup \alpha_0(I)) \times \{t_i\}$. Thus

$$[\alpha_0(I) \cup \alpha_0(I)] \times \{t_i\}$$

bounds a disk $D_i (= h_1^{-1}(D''_i))$ in $U - X$.

Continuing in this manner, we obtain singular disks $D_1, D_2, \ldots, D_{m-1}$ in $U - X$ such that $\Bd (D_i) = [\alpha_i(I) \cup \alpha_{i+1}(I)] \times \{t_i\}$ for $i = 1, \ldots, m-1$. Taking the union

$$(\alpha_1(I) \times [t_0, t_1]) \cup D_1 \cup (\alpha_2(I) \times [t_1, t_2]) \cup \cdots \cup D_{m-1} \cup (\alpha_m(I) \times [t_{m-1}, t_m]),$$

we obtain a singular disk $D$ in $U - X$ such that $\Bd D = f|\Bd D^*$.

**Proof of Theorem 1.** Suppose that $X \subset E^{n+1} = E^n \times E^1$ satisfies the hypothesis of Theorem 1. Let $\Gamma'$ be a small simple closed curve in $E^{n+1} - X$. We may change
Γ" by a small homotopy to a simple closed curve Γ in $E^{n+1} - X$ such that Γ is the union of a finite collection of small line segments, each being parallel to either $E^n \times \{0\}$ or to $\{0\} \times E^1$.

Let $v_0$ be a vertex of Γ. We are going to construct a small homotopy of Γ in $E^{n+1} - X$ that takes Γ into $E^n \times \{v_0\}$.

Let J be a “horizontal” segment in Γ (that is, J is parallel to $E^n \times \{0\}$), and let u and v be the endpoints of J. Let $f_t: J \to E^{n+1}$ ($t \in [0, 1]$) be the natural homotopy that slides J into the plane $E^n \times \{v_0\}$, oriented so that $f_0 =$ inclusion and $f_1(J) \subset E^n \times \{v_0\}$.

Since $n+1 \geq 4$ and $n-k \geq 2$, we may assume that the vertical segments $f({u} \times [0, 1])$ and $f({v} \times [0, 1])$ (where $f(x, t) = f_t(x)$) miss X. (A small argument is required to see that this can be accomplished with a homeomorphism that preserves $(n+1)$th coordinates.) Since $n-k \geq 2$, we may also assume that $f_t(J) \cap X = \emptyset$ by slightly altering the homotopy $f_t$ on the interior of J as t approaches 1.

Lemma 3 now allows us to replace $f_t (t \in [0, 1])$ by a homotopy $g_t: J \to E^{n} - X$ $(t \in [0, 1])$ having the properties that $g_0 = f_0$, $g_1 = f_1$, $g|\partial D \times [0, 1] = f|\partial D \times [0, 1]$, and $g(J \times [0, 1])$ lies in any preassigned neighborhood $U$ of $f(J \times [0, 1])$.

Thus the natural homotopy of Γ into the plane $E^n \times \{v_0\}$ can be replaced by a homotopy $h_t: \Gamma \to E^{n+1} - X$ $(t \in [0, 1])$ that takes Γ into $E^n \times \{v_0\}$. Applying the hypothesis of Theorem 1 once again to the plane $E^n \times \{v_0\}$ as in the proof of Lemma 3, we see that $h_t(\Gamma)$ can be contracted to a point in $E^{n+1} - X$ by a small homotopy. The combination of these two homotopies gives a small homotopy of Γ to a point in $E^{n+1} - X$.

Before we prove Theorem 2, we must prove a strengthened version of Lemma 3. We also need a little more notation. Let

$$B^4 = \{(x_1, x_2, x_3, x_4) \in E^4 \mid -1 \leq x_1, x_2, x_3 \leq 1 \text{ and } 0 \leq x_4 \leq 1\}$$

(notice that $D^*$ is embedded properly in $B^4$), and let

$$B^3_t = \{x \in B^4 \mid x_4 = t\} \quad (t \in [0, 1]).$$

**Lemma 4.** Suppose that X is a 1-dimensional compactum in $E^4$ satisfying the hypothesis of Theorem 1 such that $X \cap D^* = \emptyset$ and every component of $X \cap B^3_t$ is acyclic for each $t \in [0, 1]$. Then there exists a homeomorphism $h: B^4 \to B^4$ such that $h|\partial D^* \cap X = \emptyset$.

**Proof.** Let $B^3 = \{(x_1, x_2, x_3) \in E^3 \mid -1 \leq x_i \leq 1 \quad (i = 1, 2, 3)\}$. Let

$$C_1 = \{(x_1, x_2, x_3) \in \partial B^3 \mid x_1 = 0\},$$

and for each $t \in (0, 1)$, let $C_t = \{x \mid x \in C_1\}$ and $S_t = \{(-1, 0, 0), (1, 0, 0)\} \ast C_t$ (where $A \ast B$ denotes the join of A and B). Each $S_t$ is a 2-sphere in $B^3$ obtained by suspending the simple closed curve $C_t$ from the points $(-1, 0, 0)$ and $(1, 0, 0)$.

For each $t \in (0, 1)$ the set $X \cap (S_t \times \{t\})$ does not separate $S_t \times \{t\}$. Thus we can find an embedding $\alpha_t: I \to S_t$ such that $\alpha_t|\partial I =$ inclusion and $\alpha_t(I) \times \{t\} \cap X = \emptyset$. 


Applying the methods in the proof of Lemma 3, we can find numbers $t_0 = 0 < t_1 < \cdots < t_m = 1$ and embeddings $\alpha_i : I \to B^3$ ($i = 1, \ldots, m$) such that

- $\alpha_i|\text{Bd } I = \text{inclusion}$,
- $\alpha_1 = \alpha_m = \text{inclusion}$,
- $\alpha_i(I) \times [t_{i-1}, t_i] \cap X = \emptyset$,
- $\alpha_i(I) \cap \alpha_j(I) = \text{Bd } I$ if $i \neq j$,

and for $i=2, \ldots, m-1$, $\alpha_i(I) \subset S_t$ for some $t \in (0, 1)$.

Now fix $i (2 \leq i \leq m-1)$ and consider the simple closed curve

$$\Gamma_i = [\alpha_i(I) \cup \alpha_{i+1}(I)] \times \{t_i\}$$

in $B_3^3$. We may assume that $\Gamma_i$ is a polygonal curve. Our construction of $\alpha_i$ and $\alpha_{i+1}$ guarantees that $\Gamma_i$ bounds a nonsingular polyhedral disk $D_i$ in $B_3^3$ such that $D_i \cap \text{Bd } B_3^3 = \text{Bd } I \times \{t_i\}$. Let $h_i$ be a small push of $(E^3, X)$ such that $(E^3 \times \{t_i\}) - h_i(X)$ is 1-ULC. Let $B_i$ be a 3-cell in $B_3^3$ containing $D_i$ properly. From Lemma 1, we see that $\Gamma_i$ bounds a singular disk in $B_i - h_i(X)$, and so, by Dehn's lemma [5], $\Gamma_i$ bounds a nonsingular polyhedral disk $D_i$ in $B_i - h_i(X)$.

Since $h_i$ (and, hence, $h_i^{-1}$) is a stable homeomorphism of $E^4$, we may assume that $h_i^{-1}$ is the identity on $(\alpha_i(I) \times [t_{i-1}, t_i]) \cup (\alpha_{i+1}(I) \times [t_i, t_{i+1}])$ and outside $B^4 \cap E^3 \times [\frac{1}{2}(t_{i-1} + t_i), \frac{1}{2}(t_i + t_{i+1})]$. (This is the first time we have really needed stability.)

Observe that the 2-cell

$$D = \alpha_1(I) \times [t_0, t_1] \cup D_1 \cup \alpha_2(I) \times [t_1, t_2] \cup \cdots \cup D_{m-1} \cup \alpha_m(I) \times [t_{m-1}, t_m]$$

is properly embedded in $B^4$ and that the cell pair $(B^4, D)$ is unknotted. Let $h' : B^4 \to B^4$ be a homeomorphism that is fixed on $\text{Bd } B^4$ and takes $D^4$ to $D$, and let $h^* : B^4 \to B^4$ be defined by $h^*(x) = h_{t_i}^{-1}(x)$ if $x \in B^4 \cap E^3 \times [\frac{1}{2}(t_{i-1} + t_i), \frac{1}{2}(t_i + t_{i+1})]$ and $h^*(x) = x$ otherwise. Then the homeomorphism $h = h^* h' : B^4 \to B^4$ satisfies all of our requirements.

Proof of Theorem 2. Suppose that $X \subset E^4$ satisfies the hypotheses of Theorem 2, $\varepsilon > 0$, and $K$ is a 2-complex in $E^4$. Assume that $\varepsilon$ is small enough so that $(E^3 \times \{w\}) \cap X$ does not contain any nontrivial (Čech) 1-cycles of diameter less than $\varepsilon$ for any $w \in E^1$.

Subdivide the complex $K$ so that each of its simplexes has diameter less than $\varepsilon$.

First move the vertices of $K$ so that no two of them lie in a single horizontal hyperplane. Next move $X$ off the 1-skeleton of $K$ by an isotopy of $E^4$ that does not change the $x_k$-coordinate of any point of $E^4$.

Let $\sigma$ be a 2-simplex of $K$, let $u$ be the vertex of $\sigma$ with the smallest $x_k$-coordinate and let $v$ be the one with the largest. Let $r$ and $s$ denote the $x_k$-coordinates of $u$ and $v$, respectively. Choose numbers $t_0 > r$ and $t_1 < s$ such that $\sigma \cap X \subset E^3 \times [t_0, t_1]$.

Let $B$ be a 4-cell in $E^3 \times [t_0, t_1]$ of diameter less than $\varepsilon$ such that the pair
(B, σ ∩ (E^3 × [t_0, t_1])) is homeomorphic to (B^4, D^4) (as defined above) by a homeomorphism that takes each horizontal cross-section of B to a horizontal cross-section of B^4.

We may now apply Lemma 4 to get a homeomorphism k: B → B such that k|Bd B = identity and k(σ) ∩ X = ∅. Moreover, k is an ε-push of (E^4, σ ∩ X), since k is isotopic to the identity (keeping Bd B fixed) via the Alexander isotopy.

If we are careful to construct the 4-cells B corresponding to each 2-simplex σ of K so that any two intersect in a subset of the boundary of each, then the ε-pushes k that we obtain will piece together to give the desired ε-push h of (E^4, K ∩ X).

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