SEMIGROUPS ON FINITELY FLOORED SPACES

BY

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Abstract. This paper is concerned with certain aspects of acyclicity in a compact connected topological semigroup, and applications to the admissibility of certain multiplications on continua. The principal result asserts that if $S$ is a semigroup on a continuum, finitely floored in dimension 2, then $S = ESE$ implies $S = K$.

Introduction. We are concerned here with some aspects of acyclicity in a compact connected semigroup, and applications to the admissibility of a semigroup structure on "finitely floored" spaces (Definition 2). The main result is that if $S$ is a semigroup on a continuum, finitely floored in dimension 2, then $S = ESE$ implies $S = K$. This is a generalization of the result of Cohen and Koch [1], where the same conclusion is obtained assuming that $S$ is a floor for each nonzero $h \in H^2(S)$, $\text{cd} S \leq 2$, and $S$ is locally Euclidean except possibly at one point.

A preliminary result which may be of interest in itself is that if $S$ is a semigroup with a zero satisfying $S = ESE$ on a continuum, then for each nonzero $h \in H^2(S)$ there exists a pair of idempotents $e$ and $f$ such that $h|Se \cup Sf \neq 0$.

The notation is that of [4]. In particular, $S$ denotes a topological semigroup, $K$ the minimal ideal, and $E$ the set of idempotents. For a closed set $A$ of $S$, $S/A$ denotes the space obtained by shrinking $A$ to a point. The cohomology used is that of Alexander-Wallace-Spanier with coefficient group arbitrary. Throughout the paper we shall use reduced groups in dimension 0. We denote by $A \setminus B$ the complement of $B$ in $A$, the closure of $A$ by $A^*$ and the empty set by $\emptyset$. If $\leq$ is a relation from a space $X$ to a space $Y$, and $y \in Y$, then $L(y) = \{x \in X : x \leq y\}$; if $B \subseteq Y$, then $L(B) = \{x \in X : x \leq b$ for some $b \in B\}$.

The author would like to express his gratitude to Professor R. J. Koch for his many useful suggestions during the preparation of this paper.

Topological preliminaries. If $A$ is a closed subset of a space $X$ and $h$ is an element of $H^n(X)$, then $h|A$ denotes the image of $h$ under the natural homomorphism $H^n(X) \to H^n(A)$. The following theorem is due to Wallace [5]:

**Theorem 1.** Let $X$ and $Y$ be compact Hausdorff spaces and $\leq$ a closed relation from $X$ to $Y$ such that $L(A) \cap L(B)$ is connected for each pair of closed subsets $A$ and $B$ in $X$.

Received by the editors June 29, 1970.

AMS 1970 subject classifications. Primary 22A15; Secondary 22A15.

Key words and phrases. Semigroups on continua, acyclicity in compact semigroups, admissibility of multiplication on two-manifolds.

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B of Y. If \( h \in H^1(X) \) has the property that \( h|L(y) = 0 \) for each \( y \in Y \), then \( h|L(A) = 0 \) for each closed subset of \( Y \).

**Definition 2.** Let \( X \) be a space and \( L \neq \emptyset \) a subset of \( H^n(X) \). A closed subset \( F \) of \( X \) is called a floor for \( L \) if (i) \( h|F \neq 0 \) for each \( h \in L \) and (ii) for each proper closed subset \( A \) of \( F \), there exists an element \( h \) in \( L \) such that \( h|A = 0 \). If \( L \) is finite and \( F = X \), \( X \) is said to be finitely floored in dimension \( n \).

It should be observed that if \( X \) is compact and \( L \subset H^n(X) \) does not contain \( 0 \), then there exists a floor \( F \) for \( L \). In particular, if \( A \) is a closed subset of \( X \) such that \( h|A \neq 0 \) for each \( h \in L \), then \( F \) may be chosen as a subset of \( A \).

The following topological lemma is a modified version of that due to Cohen and Koch [1].

**Lemma 3.** Let \( X \) be a continuum such that \( H^n(X) \neq 0 \), and suppose \( X \) is a floor for some subset \( L \) of \( H^n(X) \). If \( A \) is a proper retract of \( X \), then \( H^n(X/A) \neq 0 \) and \( X/A \) is a floor for some subset \( L_0 \) of \( H^n(X/A) \). Moreover, \( L_0 \) may be chosen to be finite if \( L \) is finite.

**Proof.** Let \( \varphi : X \to X/A \) denote the natural map and \( p = \varphi(A) \in X/A \). Consider the commutative diagram in which the top row is exact and \( \varphi^* \) is induced by the natural map \( \varphi_0 : (X, A) \to (X/A, p) \).

\[
\begin{array}{ccc}
H^n(X, A) & \longrightarrow & H^n(X) \longrightarrow H^n(A) \\
\varphi^* \downarrow & & \varphi^* \downarrow \\
H^n(X/A, p) & \longrightarrow & H^n(X/A)
\end{array}
\]

Note that \( \varphi^* \) is an isomorphism by the map excision theorem [6]. Let \( L_1 \subset L \) be those elements \( h \) in \( L \) satisfying \( h|A = 0 \). For each \( h \in L_1 \) there exists an element \( h_0 \) of \( H^n(X/A) \) such that \( \varphi^*(h_0) = h \). Let \( L_0 \subset H^n(X/A) \) be a "section" of \( L_1 \); i.e. for each \( h \in L_1 \) there exists a unique element \( h_0 \) in \( L_0 \) such that \( \varphi^*(h_0) = h \). Clearly such a set exists and is finite if \( L \) is finite. We shall show that \( X/A \) is a floor for \( L_0 \).

By the manner in which \( h_0 \) was chosen, it is clear that \( 0 \notin L_0 \). Suppose that \( B \) is a proper closed subset of \( X/A \) and consider the commutative diagram

\[
\begin{array}{ccc}
H^n(X) & \longrightarrow & H^n(\varphi^{-1}(B)) \\
\varphi^* \uparrow & & \varphi^* \uparrow \\
H^n(X/A) & \longrightarrow & H^n(B)
\end{array}
\]

\( \varphi^* \) is induced by the restriction of \( \varphi \) to \( \varphi^{-1}(B) \). Since \( X \) is connected, \( \varphi^{-1}(B) \cup A \) is a proper closed subset of \( X \). Hence there is an element \( h \) in \( L \) such that \( h|\varphi^{-1}(B) \cup A = 0 \). In particular, \( h|\varphi^{-1}(B) = 0 \) and \( h|A = 0 \), \( h \in L_1 \), and so there exists an element \( h_0 \) in \( L_0 \) such that \( h_0|B = 0 \). Our intention is to show that \( h_0|B = 0 \). Since \( h|\varphi^{-1}(B) = 0 \) and \( h|\varphi^{-1}(B) = \varphi^*(h_0)|\varphi^{-1}(B) = \varphi^*_0(h_0|B) \), it suffices to show that \( \varphi^*_0 \) is injective.

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With this end in mind, two cases are considered:

(i) $p = \varphi(A) \in B$ and

(ii) $p = \varphi(A) \notin B$.

In case (i) $\varphi_1$ is a homeomorphism of $\varphi^{-1}(B)$ onto $B$ and thus $\varphi_1^*$ is an isomorphism. In case (ii) we consider the diagram

\[
\begin{array}{ccc}
H^n(B) & \xrightarrow{\varphi_1^*} & H^n(\varphi^{-1}(B)) \\
k^* & & i_0^* \\
H^n(B, p) & \xrightarrow{\varphi_2^*} & H^n(\varphi^{-1}(B), A),
\end{array}
\]

where $\varphi_2^*$ is induced by the restriction of $\varphi_0$ to $(\varphi^{-1}(B), A)$. Now $k^*$ and $\varphi_2^*$ are isomorphisms by the map excision theorem. It remains to show that $i_0^*$ is injective. Consider the exact sequence

\[
H^{n-1}(\varphi^{-1}(B)) \xrightarrow{m^*} H^{n-1}(A) \xrightarrow{i_0^*} H^n(\varphi^{-1}(B), A) \rightarrow H^n(\varphi^{-1}(B)).
\]

Since $A$ is a retract of $X$, $m^*$ is onto. Therefore $i_0^*$ is injective. This completes the proof.

**Principal results.** The following theorem is of use in the sequel and is of independent interest in itself.

Theorem 4. Let $S$ be a continuum with a zero satisfying $S = ESE$. If $h$ is a nonzero element of $H^2(S)$, then there exists a pair of idempotents $e$ and $f$ of $S$ such that $h|S_e \cup S_f \neq 0$.

**Proof.** Let $F$ be a subset of $E$, minimal relative to (i) being closed and (ii) satisfying $h|SF \neq 0$. (Here we are using the hypothesis that $S = ESE$.) Since $S$ has a zero, $H^2(Se) = 0$, so $F$ does not consist of a single element. Thus $F$ may be expressed as the union of two proper closed subsets $A$ and $B$ of $F$. Consider the Mayer-Vietoris sequence

\[
\begin{array}{ccc}
\Delta & \rightarrow & H^1(SA \cap SB) \\
& \rightarrow & H^2(Se) \\
& \rightarrow & H^2(SA) \times H^2(SB) \rightarrow .
\end{array}
\]

Because of the minimal conditions of $F$, $h|SA = 0$ and $h|SB = 0$; thus $J^*(h|SF) = 0$. Hence there exists an element $h_0$ of $H^1(SA \cap SB)$ such that $\Delta(h_0) = h|SF$. Define a relation $\leq$ from $SA \cap SB$ to $A \times B$ as follows: For $x \in SA \cap SB$ and $(e, f) \in A \times B$, let $x \leq (e, f)$ if and only if $x \in Se \cap Sf$. It is easily verified that $\leq$ is a closed relation from $SA \cap SB$ to $A \times B$. For each subset $C$ of $A \times B$, $L(C) = \bigcup \{Se \cap Sf : (e, f) \in C\}$; therefore $L(C)$ is a left ideal of $S$. Because $S$ has a zero and $S = ESE$, it is easily verified that the left ideal $L(M) \cap L(N)$ is connected for each pair of closed subsets $M$ and $N$ of $A \times B$. Now $h_0|L(A \times B) = h_0|SA \cap SB = h_0 \neq 0$, so by Theorem 1 it
follows that there exists a pair \((e,f) \in A \times B\) such that \(h_0 | L((e,f)) \neq 0\). That is to say, \(h_0 | S e \cap S f \neq 0\). Consider the commutative diagram

\[
\begin{array}{ccc}
H^1(SA \cap SB) & \longrightarrow & H^2(SF) \\
\downarrow & & \downarrow \\
H^1(Se) \times H^1(Sf) & \longrightarrow & H^1(Se \cap Sf) \longrightarrow H^2(Se \cup Sf).
\end{array}
\]

Since \(S\) has a zero, \(\Delta_0\) is injective. Therefore \(h | Se \cup S f = \Delta_0 (h_0 | Se \cap S f) \neq 0\) and the proof is complete.

A point \(p\) in a space \(X\) is said to be *peripheral* if there exist small neighborhoods \(V\) containing \(p\) such that \(H^n(V^*, V^* \setminus V) = 0\) for all nonnegative \(n\). The following lemma follows from Hofmann and Mostert [3, p. 168] and the definition.

**Lemma 5.** Let \(S\) be a continuum and \(e\) an idempotent of \(S \setminus K\). If \(e \in (Se)^0\), then \(e\) is peripheral in \(S\).

**Lemma 6.** Suppose that \(X\) is a continuum and \(X\) is a floor for some subset \(L\) of \(H^n(X)\); then no point of \(X\) is peripheral.

**Proof.** For each neighborhood \(V\) of \(X\) the natural homomorphism of \(H^n(X) \rightarrow H^n(X \setminus V)\) is not injective. Indeed there exists a nonzero element \(h\) of \(L\) such that \(h | X \setminus V = 0\). The conclusion follows from the fact that \(H^n(X, X \setminus V)\) is isomorphic to \(H^n(V^*, V^* \setminus V)\) under the natural homomorphism.

**Theorem 7.** Let \(S\) be a continuum satisfying \(S = ESE\). If \(S\) is a floor for some finite subset \(L\) of \(H^2(S)\), then \(S = K\).

**Proof.** Since \(S\) is a floor for \(L\), \(H^2(S) \neq 0\). If \(S \neq K\), then \(K\) is a proper retract of \(S\) [7]. Thus the hypothesis of Lemma 3 is satisfied, and so \(S/K\), the Rees quotient modulo \(K\), satisfies the hypothesis of the theorem. We may now assume that \(S\) has a zero.

By Theorem 4 for each \(h \in L\) there exists a pair of idempotents \(e_h\) and \(f_h\) in \(S\) such that \(h | Se_h \cup S f_h \neq 0\). Let \(A = \bigcup \{Se_h \cup S f_h : h \in L\}\) and observe that \(A\) is closed because \(L\) is finite. Clearly \(h | A = 0\) for each \(h \in L\) and hence \(A = S\). We conclude that there exists a finite subset \(\{e_i\}_{i=1}^n\) of \(E\) such that \(S = \bigcup_{i=1}^n Se_i\), and also that \(e_i \notin Se_j\) for \(i \neq j\). Then \(e_1 \in (Se_1)^0\), so by Lemma 5, \(e_1\) is peripheral in \(S\). This establishes a contradiction to Lemma 6, and the proof is complete.

The class of compact connected 2-manifolds without boundary are covered by the preceding theorem, and other examples are readily constructed. Also, it should be noted that there is an example of a semigroup \(S\) with a zero satisfying \(S = ESE\) and \(H^2(S) \neq 0\) [2]. The underlying space of \(S\) is a 2-sphere with four closed intervals issuing from a common point \(z\) on the 2-sphere. The point \(z\) is a zero for \(S\) and the other idempotents for \(S\) are the free endpoints of the four arcs. In view of Theorems 4 and 7 this example is in a sense a prototype of any such example.
Bibliography


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