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A NEW BASIS FOR UNIFORM ASYMPTOTIC SOLUTION OF DIFFERENTIAL EQUATIONS CONTAINING ONE OR SEVERAL PARAMETERS

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Introduction. The purpose of this paper is to put forward a ring-ideal theoretic basis for the theory of uniform asymptotic expansions, and to study a new elementary technique for the asymptotic solution of differential equations containing one or more paramcters. In Part 1 we formulate a new notion of asymptotic expansion and ancillary concepts of formal convergence and formal series. Because we consider this formulation to have significance beyond the uses exhibited here, we make it at a level of generality which seems to us to best reveal its essential nature. In Part 2, we use the notions of Part 1 to frame an investigation of some turning point problems for linear equations of arbitrary order containing a single parameter. Our analysis is based on a new classification of such problems depending on a refinement of the notion of a turning point. In Part 3, we study a second order equation containing two parameters. This analysis contains, as a special case, results for Bessel's equation as both the independent variable and the parameter tend to infinity. We obtain uniform asymptotic results under weak relative growth restrictions on the two parameters which differ qualitatively from any previously known to us.

This investigation is an outgrowth of a previous study [1] of turning point problems for second order linear differential equations. We are indebted to Professor Wolfgang Wasow for reading preliminary versions of Part 2 and to the late Professor R. E. Langer for calling the third order problem of §2.6 to our attention.

PART 1. UNIFORM ASYMPTOTIC EXPANSIONS IN RINGS OF FUNCTIONS SATISFYING ASYMPTOTIC A PRIORI ESTIMATES

1.1. **The role of ideal theory.** To fix our ideas we consider, for purposes of discussion, the system of ordinary differential equations

$$y' = P(x, y, \lambda), \qquad (\ )' = d/dx, \tag{1.1.1}$$

where y is an n-vector, λ is an m-vector parameter $(\lambda_1, \ldots, \lambda_m)$ and P is a polynomial in each of its arguments. We wish to obtain asymptotic information about solutions as one or more of the variables $x, \lambda_1, \ldots, \lambda_m$ tend to infinity. In a previous investigation [1], and in Parts 2 and 3 of this paper we show that the following formal

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scheme can lead to powerful asymptotic results. We momentarily consider the problem, $\sigma y' = P(x, y, \lambda)$, where σ is an additional parameter. This modified problem has formal σ-power series solutions of the form $y = \sum_{k=0}^{\infty} \sigma^k y_k(x, \lambda)$ where

$$\begin{aligned} P(x, y_0, \lambda) &= 0, \\ \frac{\partial P}{\partial y}(x, y_0, \lambda) y_1 - \frac{\partial}{\partial x} y_0 &= 0, \ldots \text{ etc.} \end{aligned} \tag{1.1.2}$$

Suppose that $y_0, y_1, \ldots$ satisfy this recursive set of equations. If we let $\sigma = 1$, we obtain as a "formal" solution of (1.1.1)

$$y = \sum_{k=0}^{\infty} y_k(x, \lambda). \tag{1.1.3}$$

Our object in Part 1 is to find a natural context for the study of such a series, to identify some of the suitable mathematical tools, and in particular to give a workable definition of what it means for such a series to have an asymptotic character.

The use of the parameter σ is merely an artifice and this scheme can be regarded algebraically as a formal solution of the problem $Dy = P$ $(D = d/dx)$ by a formal power series, $y = \sum D^k y_k$, in the operator D. For this reason it is perhaps not extravagant to think of this scheme as asymptotic solution by expansion in powers of d/dx. This makes it clear that we can hope to obtain results by this procedure precisely in situations where, in some sense, d/dx is asymptotically small.

If we take for granted that the series (1.1.3) has asymptotic significance and that $y_0(x, \lambda)$ is the dominant term, then a necessary prerequisite for an analysis of the differential equation is a study of the system of algebraic equations $P(x, u, \lambda) = 0$. Evidently this problem requires the resources of algebraic geometry and establishes a high lower bound for the complexity of our problem. For example, it is clear that an attack on (1.1.1) with techniques of local multiple power series expansion would implicitly contain a corresponding attack on the reduced algebraic equation. Since techniques of local power series expansion are very weak tools for this latter problem (except in the easy case of algebraic curves) we conclude that these techniques are not suitable for (1.1.1) either.

We now observe that the recursion scheme (1.1.2) fails, or at least becomes irregular, if the linear operator $\partial P(x, y_0, \lambda)/\partial y$ is not invertible. Moreover this operator also appears in the computation of the x-derivatives of y_0 by implicit differentiation of $P = 0$. Thus the series $\sum_{k=0}^{\infty} y_k$ is typically a series of functions with poles on the variety

$$\det |\partial P(x, y_0(x), \lambda)/\partial y| = 0. \tag{1.1.4}$$

For example

$$y_1 = -\{\partial P(x, y_0, \lambda)/\partial y\}^{-2} \, \partial P(x, y_0, \lambda)/\partial x.$$

Clearly when (x, λ) satisfies (1.1.4) we cannot expect the series $\sum_{k=0}^{\infty} y_k$ to have

any asymptotic significance. Thus equation (1.1.4) defines a variety which determines the termwise failure of the above formal scheme in the same way in which turning points (see Wasow [3, Chapter VII]) determine the breakdown of classical power series techniques for differential equations containing a single parameter. Accordingly we will tentatively call this subvariety of $P=0$ the turning variety of (1.1.1). However, we make no claims for this notion beyond the assertion that there are some significant cases where it is appropriate. We remark that at present even the simpler notion of turning point also has this tentative character.

We now consider what it means for the series $\sum_{k=0}^{\infty} y_k(x, \lambda)$ to be an asymptotic series. Evidently we must require that there exists a solution $y(x, \lambda)$ of the differential equation (1.1.1) such that the remainders

$$R_k(x, \lambda) = y(x, \lambda) - \sum_{j=0}^{k} y_j(x, \lambda)$$

are small in some sense. In the case when a single parameter tends to infinity the simplest useful condition is

$$\|R_k(x, \lambda)\| < M_k|\lambda|^{-k-1}, \qquad \| \quad \| = \text{vector norm}, \tag{1.1.5}$$

that is, $\|R_k\|$ has a zero at $\lambda=\infty$ which increases in order as k increases. In extending such a definition to the case of several parameters there is a new and essential difficulty to consider which has no counterpart in the case of a single parameter. The local theory of the zeros of functions of several variables is very deep and requires the heavy machinery of ideal theory in suitable rings. For this reason we consider that the use of similar resources in asymptotic theory is inevitable.

We now sketch a crude preliminary formulation according to this point of view. Let $x=\lambda_0\xi$ and suppose that λ_0, λ, ξ range over a subset Ω of C^{m+2} on which the λ's are bounded away from 0 and unbounded. Suppose that on Ω all but a finite number of the functions y_k have norms in some ring B of bounded functions on Ω. Let us grant for the moment that the natural description of a zero of a function as one or more of the parameters tend to infinity is an ideal Z_0 in the ring B. We can then imitate the condition (1.1.5) by requiring that

$$\|R_k(\lambda_0\xi, \lambda)\| \in Z_0^{N(k)} \tag{1.1.6}$$

where $N(k)$ increases to infinity with k. In the case of a single parameter λ_1, there is essentially only one choice for Z_0, namely $Z_0=\lambda_1^{-1}B$. Even in this case we are led to a notion of asymptotic expansion which is more general than the usual one and which we will exploit in Part 2. However, in the multiple parameter case there are many inequivalent choices of Z_0, each of which gives rise to a natural kind of asymptotic character. For example, if $Z_0=\sum_{j=0}^{m} \lambda_j^{-1}B$, then (1.1.6) implies that R_k is small if *all* parameters (including λ_0 which describes the independent variable) are sufficiently large. If instead we use $Z_0'=\prod_{j=0}^{m} \lambda_j^{-1}B$, then R_k is small if any one parameter is sufficiently large. In the former case we have a precise formulation of

asymptotic character as all parameters tend to infinity, in the latter as at least one parameter tends to infinity. As another example, $Z_0'' = \lambda_0^{-1}B + \sum_{i \neq j} \lambda_i^{-1}\lambda_j^{-1}B$ defines asymptotic character as the independent variable and at least $m-1$ of the parameters $\lambda_1, \ldots, \lambda_m$ tend to infinity.

The above formulation is, however, too general for our purposes, and in the next section we turn to the problem of selecting the ring B in a more discerning way.

1.2. **The role of asymptotic a priori estimates.** We express (1.1.1) in the form

$$dy/d\xi = \lambda_0 P(\lambda_0\xi, y, \lambda), \qquad x = \lambda_0\xi, \tag{1.2.1}$$

and suppose that $(\xi, \lambda_0, \lambda)$ range over a set Ω in C^{m+2} on which ξ is bounded. We now observe that we must consider functions which respond to differentiation in an unbounded way. For example if there is a turning variety on which the functions y_k of (1.1.3) have poles, then differentiation will increase the order of these poles. The natural description of a pole of a function of several variables is again known to be an ideal Z_1 in a suitable ring. We are going to consider functions f which have a "pole-like" response to differentiation in the sense that there is an ideal Z_1 in the ring B of bounded functions on Ω such that

$$Z_1^k(d/dx)^k f \subset B, \qquad k = 0, 1, \ldots. \tag{1.2.2}$$

Estimates of the form (1.2.2) are also naturally related to *a priori estimates* satisfied by solutions of (1.2.1). For example, let the right-hand side of (1.2.1) be given in the form

$$\lambda_0 P(\lambda_0\xi, y, \lambda) = H_0 + H_1(y) + \cdots + H_k(y)$$

where H_j is a homogeneous j-form valued polynomial in ξ, λ_0, λ. Let $\pi(|\lambda_0|, \ldots, |\lambda_m|)$ be a nonzero polynomial satisfying the finite set of conditions

$$\|(d/d\xi)^i H_j\| < \pi(|\lambda_0|, \ldots, |\lambda_m|), \qquad i \geqq 0, \quad j = 1, \ldots, k,$$

(where $\| \quad \|$ indicates some vector norm and all j-form norms subordinate to it). Then (1.2.1) and the equations derived from it by repeated ξ-differentiation imply that any bounded solution y of (1.2.1) satisfies estimates of the form (1.2.2) with $Z_1 = \{\pi(|\lambda_0|, \ldots, |\lambda_m|)\}^{-1}B$. Such estimates, which can be deduced prior to any considerations of existence, are usually called a priori estimates and clearly in this case are asymptotic rather than numerical in nature. Because of the general occurrence of the above pattern in the theory of differential equations we are going to formulate a restricted theory of asymptotic expansions for functions satisfying such estimates which we will refer to as *asymptotic a priori estimates.*

Finally we observe that the recursion scheme (1.1.2) uses ring operations and differentiation, that is, the operations of differential algebra. For this reason we will find it natural in the end to define uniform asymptotic expansions for a class of functions which form a differential ring, that is, a ring closed under differentiation.

We now proceed to our precise formulation of an asymptotic theory.

1.3. **Formal and uniform asymptotic series.** We are going to establish notions of *formal convergence* and *uniform asymptotic expansion* for functions of the form $f(\xi, \lambda)$ which depend analytically on a complex vector variable $\xi \in \boldsymbol{C}^p$ and depend in addition in a quite arbitrary manner on a vector parameter $\lambda \in \boldsymbol{C}^m$. In particular we will not require analytic or even continuous dependence on λ. It will be most convenient to assume simply that these functions take on values in a complex Banach algebra β with identity I and norm $\| \quad \|$, of which a typical instance is the algebra of $n \times n$ matrices over $\boldsymbol{C}$. We remark that no essential changes are required in the following if we replace $\xi \in \boldsymbol{C}^p$ by a real variable $t \in \boldsymbol{R}^p$ and consider functions which are infinitely differentiable in t, and we will make use of this variant in Part 2. However, for purposes of exposition we restrict ourselves to the analytic case.

We first identify a preliminary enveloping ring $\beta(\Omega)$ which contains all the function rings in which we are interested.

DEFINITION 1.3.1. Let Ω be a subset of $\boldsymbol{C}^{p+m}$ on which the coordinate functions $\lambda_1, \ldots, \lambda_m$ are unbounded and are also bounded away from zero. Let $\beta(\Omega)$ be the ring of β-valued functions on Ω which restricted to each cross section $\Omega_\lambda = (\xi \mid (\xi, \lambda) \subset \Omega)$ are analytic in ξ. (We recall that a function is analytic on a set $X \subset \boldsymbol{C}^p$ if it is the restriction to X of a function analytic on some open set containing X.)

DEFINITION 1.3.2. Let $\beta_0(\Omega)$ be the subring of uniformly bounded elements of $\beta(\Omega)$.

We now introduce a priori asymptotic estimates depending on a subring β_1 of β_0 and an ideal ζ_1 in β_1.

DEFINITION 1.3.3. Let β_1 be a subring of β_0. Let ζ_1 be an ideal in β_1 such that $\zeta_1 D \zeta_1 \not\subset \zeta_1$ for every first order ξ-differentiation D. Define $B_0(\beta_1, \zeta_1)$ to be the subset of elements f in β_1 satisfying $\zeta_1^k D_k f \subset \beta_1$ for every ξ-differentiation of order k. Let $Z(\beta_1, \zeta_1) = B_0(\beta_1, \zeta_1) \cap \zeta_1$.

It follows easily from this definition that B_0 is a ring and Z is an ideal in B_0. Moreover we have the following.

DERIVATIVE ESTIMATE. Functions in B_0 satisfy the derivative estimates $Z^k D_k f \subset B_0$ for every ξ-differentiation of order k.

Proof. To establish the given inclusion it is required to show that if f is in B_0 then

$$\zeta_1^j D_j(Z^k D_k f) \subset \beta_1$$

for each j and each ξ-differentiation D_j of order j. The identities

$$\begin{aligned} z_{j+1} z_j \cdots z_1 D_{j+1} f &= z_{j+1} \cdots z_1 D_1 D_j f \\ &= z_{j+1} D_1 z_j \cdots z_1 D_j f - \left(\sum_{i=1}^{j} (z_{j+1} D_1 z_i) z_1 \cdots \hat{z}_i \cdots z_j \right) D_j f \end{aligned}$$

show (using $\zeta_1 D_1 \zeta_1 \subset \zeta_1$) that

$$\zeta_1^{j+1} D_{j+1} f \subset \zeta_1 D_1(\zeta_1^j D_j f) + \zeta_1^j D_j f.$$

From this it follows that if $f \subset \zeta_1$, then if $\zeta_1^j D_j f$ is in ζ_1 so is $\zeta_1^{j+1} D_{j+1} f$. Hence by induction it follows that $\zeta_1^j D_j \zeta_1 \subset \zeta_1$. Now the product rule for differentiation implies that $\zeta_1^j D_j fg \subset \sum_{r+s=j} (\zeta_1^r D_r f)(\zeta_1^s D_s g)$ where the sum extends over all differentiations of the indicated orders. Combining this with the previous estimate we easily find that $\zeta_1^j D_j \zeta_1^k \subset \zeta_1^k$. Finally for f in B_0 we have

$$\zeta_1^j D_j Z^k D_k f \subset \sum_{r+s=j} (\zeta_1^r D_r Z^k) \zeta_1^s D_s D_k f \subset \sum_{r+s=j} \zeta_1^k \zeta_1^s D_s D_k f \subset \beta_1.$$

DEFINITION 1.3.4. Let B be the set of elements of $\beta(\Omega)$ satisfying $Z^k f \subset B_0$ for some integer k (depending on f).

The set B is easily seen to be a ring and is, moreover, closed under differentiation, that is, B is a differential ring. The ring B now has sufficient structure to serve as the starting point for an asymptotic theory.

We next introduce the topology of formal convergence in B induced by the set Z in order to obtain a formal theory of series of elements of B. We need this theory because asymptotic series invariably possess characteristic formal properties, and because, as in classical asymptotic theory, formal solutions of differential equations frequently turn out to be asymptotic solutions.

DEFINITION 1.3.5. A sequence f_k, $k=0, 1, \ldots$, of elements of B is *formally convergent to zero* if given any positive integer N, there is a $k(N)$ such that $f_k \in Z^N$ for $k \geqq k(N)$.

DEFINITION 1.3.6. A series of functions in B, $\sum_{k=0}^{\infty} f_k$, is *formally convergent* if the sequence f_k is formally convergent to zero. A series is *formally convergent to zero* if its sequence of partial sums is formally convergent to zero. Let $\mathscr{B}$ be the collection of formally convergent series. Let $\mathscr{Z}$ be the collection of series formally convergent to zero.

PROPOSITION 1.3.1. *Define addition and multiplication in $\mathscr{B}$ by $\sum_{k=0}^{\infty} f_k + \sum_{k=0}^{\infty} g_k = \sum_{k=0}^{\infty} (f_k + g_k)$, $(\sum_{k=0}^{\infty} f_k)(\sum_{k=0}^{\infty} g_k) = \sum_{k=0}^{\infty} (\sum_{i+j=k} f_i g_j)$. Define ξ-differentiation by termwise differentiation. Then $\mathscr{B}$ is a ring closed under differentiation and $\mathscr{Z}$ is an ideal in $\mathscr{B}$ closed under differentiation.*

The proof is a direct consequence of the previous definitions. The language of differential algebra (see [2]) is appropriate here according to which $\mathscr{B}$ is a differential ring with derivations $\partial/\partial\xi_i$, $i=1, \ldots, p$, and $\mathscr{Z}$ is a differential ideal in $\mathscr{B}$.

We now observe that the ring $\mathscr{B}$ is too large for the purposes of asymptotic analysis since, for example, $f+0+0+\cdots$ and $0+f+0+\cdots$ are different elements of $\mathscr{B}$. We eliminate such redundancies by forming the quotient ring $\mathscr{B}/\mathscr{Z}$. Here it is essential that $\mathscr{Z}$ is closed under differentiation in order that the $\partial/\partial\xi_k$ act naturally on the quotient. However, since we will invariably be working with specific representatives in $\mathscr{B}$ rather than cosets in $\mathscr{B}/\mathscr{Z}$ it is most convenient to introduce the following terminology.

DEFINITION 1.3.7. The symbol "$\doteq$" or the term "formal equality" denotes equality in $\mathscr{B}$ modulo $\mathscr{Z}$.

The preceding definitions utilize a standard method of using an ideal to topologize a ring. This method appears in its purest form in the analytic theory of local rings [9]. This theory in turn is abstracted from algebraic geometry (see [10] for example) where one uses the ideal of nonunits in a valuation ring to induce convergence in an algebraic function field in order to retain the advantages of function theoretic methods without extraneous analytic hypotheses. However, the rings which appear naturally in our problem do not resemble local rings or even the rings of infinitely differentiable functions of [6].

We observe that we are now in a position to consider formal solutions in $\mathscr{B}$ of partial differential equations, $\pi(y) \doteq 0$, where π is a polynomial in y and its ξ-derivatives with coefficients in B or $\mathscr{B}$. Moreover this problem corresponds in a well defined way to a problem $\tilde{\pi}(w)=0$ in $\mathscr{B}/\mathscr{Z}$ with coefficients in this quotient ring. This means that formal solutions of $\pi(w) \doteq 0$ are well defined in the sense that if $\pi(w_1) \doteq 0$ and $w_1 \doteq w_2$, then we know that $\pi(w_2) \doteq 0$ also.

For example, we are now able to raise the question: in what sense is the series given by (1-3), $\sum_{k=0}^{\infty} y_k(\xi\lambda_0, \lambda)$, a formal solution of (2-1), $dy/d\xi = \lambda_0 P(\xi\lambda_0, y, \lambda)$? However, there appears to be great arbitrariness in this question because the above scheme depends on the domain Ω, the subring β_1 of $\beta_0(\Omega)$, and the ideal ζ_1 in β_1, which have yet to be specified. The following remarks will show why this generality is required, and why, when we attack a specific problem, it is quickly used up. To begin with, the discussion in §1.1 above makes it plain that the ideal ζ_1 determines the kind of asymptotic character required of asymptotic solutions and is therefore actually part of the data of the problem. Next, we call attention to the fact that the above scheme is based on the use of the same ideal to define both formal convergence and derivative estimates. However, while the formal convergence again merely reflects the mode of asymptotic character, the derivative estimates are determined by the differential equation and the domain Ω, and there is no reason in general for these to agree. For example, if there is a turning variety on which y_1 has poles and Ω intersects the turning variety, then *no* derivative estimates are possible. Since we want Ω to be as large as possible, roughly speaking, Ω should simply be the largest domain on which the necessary asymptotic a priori estimates can be obtained. Finally, the choice of β_1 is somewhat less important. The choice $\beta_1 = \beta_0(\Omega)$ gives the simplest and most general formal scheme depending only on the required mode of asymptotic character and the domain natural to the problem. However, we will see in Part 2 that there can be smaller subrings which take into account very special features of the problem in a way which is useful as an intermediate technical device.

It is appropriate to mention here that there exists a profound and extensive theory of classes of infinitely differentiable functions satisfying derivative estimates ([10], [11]) which has certain points in common with our scheme. The simplest representative of this theory is the class of real analytic function on $[a, b]$ which can be characterized as the class of functions satisfying the numerical derivative

estimates, $|f^{(n)}(x)| < k^n n!$ $(n \geqq 0,\ x \in [a, b])$, where k is a constant depending only on f. However, other classes are considered for which the growth of the maxima of successive derivatives is more rapid. An interesting part of the theory deals with the relationship between such a function and the asymptotic series given by its formal Taylor expansions.

We now define the related notion of uniform (more precisely, uniform in ξ) asymptotic expansion.

DEFINITION 1.3.8. The series of elements of B, $\sum_{k=0}^{\infty} f_k$, is a uniform asymptotic expansion of a function f in B if the sequence $f - \sum_{j=0}^{k} f_j$ is formally convergent to zero. We indicate this relationship by $f \sim \sum_{k=0}^{\infty} f_k$.

The following is again an immediate consequence of the definitions.

PROPOSITION 1.3.2. *A uniform asymptotic expansion is a formally convergent series. If $f \sim \sum_{k=0}^{\infty} f_k$, then $f \sim \sum_{k=0}^{\infty} g_k$ if and only if $\sum_{k=0}^{\infty} f_k \doteq \sum_{k=0}^{\infty} g_k$. Uniform asymptotic expansions are unique as elements of $\mathscr{B}/\mathscr{Z}$.*

This proposition shows another reason why the theory of formal series is a necessary part of the theory of asymptotic series. There is no natural way of assigning any termwise uniqueness properties to such a complicated series as $\sum_{k=0}^{\infty} y_k(x, \lambda)$, the terms of which have no simple distinguishing algebraic properties as in the case of power series expansions. However, Proposition 1.3.2 shows that the formal structure accounts precisely for the nonuniqueness in asymptotic expansions.

PROPOSITION 1.3.3. *If $f \sim \sum_{k=0}^{\infty} f_k$, then*

$$\frac{\partial f}{\partial \xi_i} \sim \sum_{k=0}^{\infty} \frac{\partial f_k}{\partial \xi_i}.$$

Proof. We must show that the sequence $\partial f/\partial \xi_i - \sum_{j=0}^{k} (\partial f_j/\partial \xi_i)$ is formally convergent to zero. But $f - \sum_{j=0}^{k} f_j$ is formally convergent to zero and differentiation preserves this convergence.

As a consequence, if we establish uniform asymptotic expansions for the solutions of a differential equation, we also establish expansions for all its derivatives.

We finally establish that in certain cases, every formally convergent series is an asymptotic expansion.

PROPOSITION 1.3.4. *Let B be determined by the domain Ω, the ring $\beta_0(\Omega)$, and an ideal in $\beta_0(\Omega)$ which has a finite basis $z_1(\lambda), \ldots, z_q(\lambda)$ consisting of functions of λ alone. Then every formally convergent series of elements of B is a uniform asymptotic expansion for some element of B.*

Proof. Suppose $\sum_{j=0}^{\infty} f_j(\xi, \lambda)$ is formally convergent. Then for some j_0, $\zeta^{j_0+j} D_j f_k \subset \zeta^{\psi(k)}$, where $\zeta = z_1(\lambda)\beta_0 + \cdots + z_q(\lambda)\beta_0$, D_j is any ξ-differentiation of order j, and

$\psi(k)$ can be supposed to be a nondecreasing integer valued function of k tending to infinity with k. This implies that

$$\|D_j f_k\| < M(D_j, k)(|z_1| + \cdots + |z_q|)^{\psi(k) - j_0 - j} \tag{1.3.1}$$

where $M(D_j, k)$ is a constant, which, without loss of generality, we can suppose is greater than 1.

Let $\theta(u)$ be any function of a real variable satisfying $0 \leqq \theta \leqq 1$ which is identically 1 near zero and vanishes for $u \geqq 1$. Let $F_k(\xi, \lambda)$ be defined by

$$F_k(\xi, \lambda) = f_k(\xi, \lambda)\theta\left(k!\,(|z_1| + \cdots + |z_q|) \sum_{D_l, l \leqq k} M(D_l, k)\right)$$

where the indicated sum extends over all ξ differentiations of order $\leqq k$.

The sum $f = \sum_{k=0}^{\infty} F_k$ is finite for each λ, since either $|z_1(\lambda)| + \cdots + |z_q(\lambda)| = 0$, in which case all but at most a finite number of the f_k's are zero, or else $|z_1(\lambda)| + \cdots + |z_q(\lambda)| > 0$, in which case for k sufficiently large the θ factor in F_k is zero. In either case F_j is identically 0 unless

$$k!\,(|z_1(\lambda)| + \cdots + |z_q(\lambda)|) \sum_{D_l, l \leqq k} M(D_l, k) < 1,$$

which implies that

$$k!\,(|z_1(\lambda)| + \cdots + |z_q(\lambda)|) M(D_l, k) < 1$$

for $k \geqq l$. Combining this with (1.3.1) we find that either $D_j F_k$ is zero for $k \geqq j$ or

$$\|D_j f_k\| < (1/k!)(|z_1| + \cdots + |z_q|)^{\psi(k) - j_0 - j - 1} \quad \text{for } k \geqq j.$$

In either case

$$\|D_j F_k\| \leqq (1/k!)(|z_1| + \cdots + |z_q|)^{\psi(k) - j_0 - j - 1}$$

for $k \geqq j$.

This implies that for large j

$$D_j\left(f - \sum_{i=0}^{k-1} F_i\right) \in \zeta^{\psi(k) - j_0 - j - 1}$$

which in turn implies that $f \sim \sum_{k=0}^{\infty} F_k$. But $f_k - F_k$ has the form

$$[1 - \theta(C_k[|z_1| + \cdots + |z_q|])]f_k$$

which is divisible by any power of $|z_1| + \cdots + |z_q|$ since $\theta(u)$ is identically 1 near zero. Hence $\sum f_i \doteq \sum F_i$ and we conclude that $f \sim \sum_{k=0}^{\infty} f_k$.

1.3.4. EXAMPLES.

1. Let $\Omega = \{\xi \mid |\xi_i| \leqq 1, i = 1, \ldots, p\} \times \{\lambda \mid \lambda_i \geqq 1, i = 1, \ldots, m\}$. Let β_1 be the ring of functions which are jointly analytic in ξ and infinitely differentiable in λ for $1 \leqq \lambda_i \leqq +\infty$. Let ζ_1 be the ideal $\lambda_1^{-1}\beta_1 + \cdots + \lambda_m^{-1}\beta_1$. Let $\hat{f}$ be the formal power series of f with respect to λ,

$$\hat{f} = \sum f_{i_1 \cdots i_m}(\xi)\lambda_1^{-i_1} \cdots \lambda_m^{-i_m}.$$

Then with respect to the asymptotic character induced by ζ_1, $f \sim \hat{f}$. It is obvious in this example that termwise differentiation of this formula with respect to ξ is legitimate. Thus our notions generalize the asymptotic multiple power series expansion of an infinitely differentiable function. However, in our context this asymptotic power series is merely the simplest representative of infinitely many equivalent expansions. We note that our definition is actually more special than the usual Poincaré definition since expansions in the latter sense cannot be so differentiated.

2. The following examples illustrate the scope of our definition of formal convergence. Let Ω be as in the previous example, let $\beta_1 = \beta_0(\Omega)$ and let $\zeta_1 = \lambda_1^{-1}\beta_0 + \lambda_2^{-1} \cdots \lambda_m^{-1}\beta_0$. Let $a_{ij}(\xi)$ be analytic for $|\xi_i| \leqq 1$, let $b_{ij}(\lambda)$ be bounded functions of λ, and let $\psi(k)$ be any real valued function on the integers which tends to infinity with k. Then

$$\sum_{i,j \geqq 0} a_{ij}(\xi) b_{ij}(\lambda) \lambda_1^{-\psi(i)} (\lambda_2, \ldots, \lambda_m)^{-\psi(k)}$$

is formally convergent. By Proposition 4 of §1.3 this series is the uniform asymptotic expansion of some function on Ω which evidently has a "power like" asymptotic character as λ_1 and at least one of $\lambda_2, \ldots, \lambda_m$ tend to infinity. In our view it is the statement that convergence is induced by $\lambda_1^{-1}\beta_0 + (\lambda_2 \cdots \lambda_n)^{-1}\beta_0$ which gives precise mathematical significance to this last description of the asymptotic character.

3. In the previous examples, derivative estimates played no role. We now consider a simple example in which derivative estimates appear. Suppose $\varphi(\xi)$ has an ordinary asymptotic power series expansion at $\xi = 0$ in the sector $|\arg \xi| \leqq \theta$, $\varphi(\xi) \sim \sum a_k \xi^k$. Let λ_1 and λ_2 be real positive parameters. We consider asymptotic expansions of the function $\varphi(\lambda_1^{-1}\lambda_2^{-2}[\xi - \lambda_1^{-1} - \lambda_2^{-1}]^{-1})$. We establish the following:

Let Ω be the domain

$$\{(\xi, \lambda_1, \lambda_2) \mid \lambda_1 \geqq a,\ \lambda_2 \geqq a,\ |\xi - \lambda_1^{-1} - \lambda_2^{-1}| \geqq (\lambda_1^{-1} + \lambda_2^{-2})^{3-\delta} |\arg(\xi - \lambda_1^{-1} - \lambda_2^{-1})| < \theta' < \theta\},$$

where $\delta > 0$ and a is sufficiently large. Let $\beta_1 = \beta_0(\Omega)$ and let $\zeta_1 = (\lambda_1^{-1}\beta_0 + \lambda_2^{-1}\beta_0)^3$. Then on Ω, with the convergence induced by ζ_1,

$$\varphi(\lambda_1^{-1}\lambda_2^{-2}[\xi - \lambda_1^{-1} - \lambda_2^{-1}]^{-1}) \sim \sum_{k=0}^{\infty} a_k \lambda_1^{-k} \lambda_2^{-2k} [\xi - \lambda_1^{-1} - \lambda_2^{-1}]^{-k}. \tag{1.4.1}$$

To prove this formula we observe that on Ω,

$$|\lambda_1^{-1}\lambda_2^{-2}[\xi - \lambda_1^{-1} - \lambda_2^{-1}]^{-1}| < (\lambda_1^{-1} + \lambda_2^{-1})^3 (\lambda_1^{-1} + \lambda_2^{-1})^{\delta - 3} < (\lambda_1^{-1} + \lambda_2^{-1})^{\delta}.$$

Similarly

$$|(\lambda_1^{-1} + \lambda_2^{-1})^{3k} (d/d\xi)^k \lambda_1^{-1}\lambda_2^{-2}[\xi - \lambda_1^{-1} - \lambda_2^{-1}]^{-1}| < M_k (\lambda_1^{-1} + \lambda_2^{-1})^{\delta}$$

which shows that

$$\lambda_1^{-1}\lambda_2^{-2}[\xi - \lambda_1^{-1} - \lambda_2^{-1}] \in (\lambda_1^{-1} + \lambda_2^{-1})^{\delta} B_0(\Omega)$$

where $B_0(\Omega)$ is the ring of functions in β_0 which satisfy the derivative estimates induced by the ideal ζ_1 in β_0. This implies that

$$\lambda_1^{-k}\lambda_2^{-2k}[\xi-\lambda_1^{-1}-\lambda_2^{-1}]^{-k}\in Z^{[(k/3)\delta]}$$

where $[u]$ is the greatest integer in u, and $Z=(\lambda_1^{-1}B_0+\lambda_2^{-1}B_0)^3$. This shows that the series in (1.4.1) is formally convergent.

To establish the asymptotic character of the series we observe that on any sector $|\arg\xi|\leqq\theta'<\theta$, $|\xi|\leqq M$, $f(\xi)$ has bounded derivatives. It then follows that

$$f^{(k)}(\lambda_1^{-1}\lambda_2^{-2}[\xi-\lambda_1^{-1}-\lambda_2^{-1}]^{-1})\in B_0.$$

Finally Taylor's formula with remainder shows

$$\begin{aligned}f(\lambda_1^{-1}\lambda_2^{-2}[\xi-\lambda_1^{-1}-\lambda_2^{-1}])-\sum_{j=0}^{k}c_j\lambda_1^{-j}\lambda_2^{-2j}[\xi-\lambda_1^{-1}-\lambda_2^{-1}]^{-j}\\ =\frac{1}{k!}(\lambda_1^{-1}\lambda_2^{-2}[\xi-\lambda_1^{-1}-\lambda_2^{-2}]^{-1})^{k+1}\\ \cdot\int_0^1(1-s)^{k+1}f(s\lambda_1^{-1}\lambda_2^{-2}[\xi-\lambda_1^{-1}-\lambda_2^{-1}]^{-1})\,ds\in Z^{[((k+1)/3)\delta]},\end{aligned}$$

which shows that the remainders converge formally to zero.

The asymptotic formula (1.4.1) is a very simple example of an asymptotic formula which can be formally reduced to various multiple power series expansions by expanding $(\xi-\lambda_1^{-1}-\lambda_2^{-1})^{-k}$ by the binomial theorem in different ways depending on the relative sizes of ξ, λ_1^{-1}, and λ_2^{-1}. However, the resulting formulas will be valid asymptotic formulas only on proper subdomains of Ω. Therefore in this simple example we get a glimpse of how multiple power series techniques fragment the asymptotic analysis in a way which can be disastrous in harder problems.

Part 2. Asymptotic Solutions Uniform in Many Scales for Some Higher Order Linear Differential Equations with Turning Points

2.1. **Introductory remarks.** The purpose of Part 2 is to obtain uniform asymptotic solutions for certain equations of the form

$$\varepsilon^{nq}y^{(n)}+\varepsilon^{(n-2)q}a_2(t,\varepsilon)y^{(n-2)}+\cdots+a_n(t,\varepsilon)y=0 \tag{2.1.1}$$

in cases in which the roots of the related algebraic equation

$$l^n+a_2(t,0)l^{n-2}+\cdots+a_n(t,0)=0 \tag{2.1.2$'$}$$

undergo an isolated n-fold degeneracy at $t=0$. We thus consider a problem which, according to common terminology, has a *turning point* at $t=0$. We shall present a purely real variable theory, assuming that t and ε range over $|t|\leqq t_0$, $0<\varepsilon\leqq\varepsilon_0$, q is a positive integer, and the coefficients a_k are complex valued functions infinitely differentiable on the closure of this domain.

Our analysis is based on the use of the equation

$$l^n + a_2(t, \varepsilon) l^{n-2} + \cdots + a_n(t, \varepsilon) = 0 \tag{2.1.2}$$

which, as we will show, can be decisive for the asymptotic behavior of (2.1.1) in certain cases where (2.1.2)′ is not. The main profit in our results is that they enable us to use the potentially intricate but frequently accessible behavior of the algebraic equation (2.1.2) to account for the more recondite behavior of the differential equation (2.1.1).

We must comment both on the general relation of our results to existing asymptotic theory (as represented, for example, by Chapters VII and VIII of Wasow [3]) and on the relation of our hypotheses (given in §2.3) to the requirement that (2.1.2)′ have distinct roots. In particular we must examine the relevance of the turning point dichotomy. Even though a really satisfactory definition of "turning point" has not yet been given, the problem outlined above seems to be regarded unequivocally as a turning point problem. Nonetheless we prefer to regard our results as fundamentally non-turning-point in nature for the following reasons.

We will obtain asymptotic solutions of (2.1.1) under the hypothesis that (for $\varepsilon \neq 0$) (2.1.2) has distinct roots which, moreover, do not coalesce too rapidly as $\varepsilon \to 0$. This is an evident generalization of the requirement that (2.1.2)′ have distinct roots, the usual condition precluding turning points. Also, certain $(t-\varepsilon)$ sets closely related to zeros of the discriminant of (2.1.2) and to the turning varieties considered in §1.1 will play the role of turning points in determining the way in which our asymptotic series can fail. Since these sets are typically curves in $(t-\varepsilon)$ space, we have proposed the term "turning variety" in a previous investigation (Stengle [1]) where we study second order equations according to this point of view. Because of the far greater complexity of the present problem we shall consider a subproblem which by no means exhausts the power of our methods, but which appears to be the simplest representative case. We shall assume that the roots of (2.1.2) coalesce in a symmetric manner in the following sense. Let $l_1, \ldots, l_n$ be the global continuous roots of (2.1.2). We shall assume that all quotients of the quantities $l_i - l_j$, $i \neq j$, are uniformly bounded. (This also generalizes the condition that (2.1.2)′ has distinct roots.) Within the confines of this symmetric case we can describe the results of Part 2 as: *uniform asymptotic solution of* (2.1.1) *in the case that* (2.1.1) *has a turning point but no turning varieties.*

In §2.2 we use a variant of the formal scheme of §1.1 to compute the formal solutions which constitute our main object of study. In §2.3 we classify the equations which fall within the scope of our method. In §2.4 we give a detailed termwise analysis of the series of §2.2. As is common in investigations of this kind, this formal analysis contains the main difficulties. In §2.5 we assign asymptotic significance to our formulas. Finally in §2.6 we apply our results to the illustrative problem (2.1.3).

2.2. **Preformal solutions.** We begin by computing the formulas ((2.2.8) and (2.2.9) below) that are our main object of study. We first transform (2.1.1) to an equivalent system by the transformation

$$w_1 = y, \quad w_2 = \varepsilon^q y', \ldots, w_n = \varepsilon^{(n-1)q} y^{(n-1)}. \tag{2.2.1}$$

The column vector w with components w_k then satisfies

$$\varepsilon^q w' = \underline{a} w \tag{2.2.2}$$

where

$$\underline{a} = \begin{bmatrix} 0 & 1 & \cdot & \cdots & \cdot \\ 0 & 0 & 1 & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & 1 \\ -a_n & -a_{n-1} & \cdot & \cdots & 0 \end{bmatrix}.$$

Since we assume that (2.1.2) has distinct roots for $\varepsilon \neq 0$, and the underlying domain $|t| \leqq t_0$, $0 < \varepsilon \leqq \varepsilon_0$, is contractible, the global continuous roots of (2.1.2), call them $l_1, \ldots, l_n$, are unique to within order. These roots are the eigenvalues of $\underline{a}$. Moreover $\underline{a}$ is reduced to diagonal form by the similarity

$$S^{-1} \underline{a} S = \Lambda, \tag{2.2.3}$$

where

$$S = (l_j^{k-1})$$

and Λ is the diagonal matrix with l_j in the jth diagonal entry. The linear change of variable

$$w = Sv \tag{2.2.4}$$

transforms (2.2.2) into

$$\varepsilon^q v' = (\Lambda - \varepsilon^q S^{-1} S') v. \tag{2.2.5}$$

We remark that this kind of reduction of (2.2.2) to (2.2.5) usually has the character of a preliminary normalization which can be taken for granted. This is not the case here since the matrix S is singular at $\varepsilon = 0$, and the coefficients of the transformed problem have a complicated behavior near $t = \varepsilon = 0$ which we will be obliged to examine in detail.

We now consider the problem of determining a linear change of dependent variable

$$v = Tu \tag{2.2.6}$$

which reduces (2.2.5) to the form $\varepsilon^q u' = Ru$ where R is a diagonal matrix. The matrices T and R must then satisfy the underdetermined system

$$\varepsilon^q T' - (\Lambda - \varepsilon^q S^{-1} S') T + TR = 0.$$

We make this system determinate by imposing the condition diag $T=I$ where diag is the operator projecting any matrix onto its diagonal part. This pair of equations implies that

$$R = \Lambda - \varepsilon^q \operatorname{diag}(S^{-1}S'T)$$

so that we obtain an equation for T alone:

$$\varepsilon^q T' + (\operatorname{ad}\Lambda)T + \varepsilon^q(S^{-1}S'T - T\operatorname{diag}[S^{-1}S'T]) = 0, \tag{2.2.7}$$

where ad Λ is the linear commutator operator

$$(\operatorname{ad}\Lambda)U = U\Lambda - \Lambda U.$$

We note that because the l_k's are distinct, ad $\Lambda(t, \varepsilon)$ is an invertible linear transformation on the linear space of matrices with vanishing diagonal elements. We denote its inverse by $(\operatorname{ad}\Lambda)^{-1}$.

We have now transformed the original equation to a form where we can apply a variant of the formal scheme of §1.1. We consider the problem, containing two parameters, ε and σ:

$$\sigma T' + (\operatorname{ad}\Lambda)T + \sigma(S^{-1}S'T - T\operatorname{diag}[S^{-1}S'T]) = 0.$$

This equation has formal σ-power series solutions of the form $T=\sum_{k=0}^{\infty}\sigma^k T_k$ where the T_k are defined by

$$\begin{aligned} T_0 &= I, \\ T_{k+1} &= -(\operatorname{ad}\Lambda)^{-1}\Big\{T_k' + S^{-1}S'T_k - \sum_{i+j=k} T_i \operatorname{diag}(S^{-1}S'T_j)\Big\}. \end{aligned} \tag{2.2.8}$$

It is important to note here that the expression in brackets is contained in the range of (ad Λ). Replacing σ by ε^q in the formal σ-power series solution we obtain

$$T(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{kq} T_k(t, \varepsilon) \tag{2.2.9}$$

as a potential formal solution of equation (2.2.7).

Our main task is to elucidate the formal properties of this series and to assign it precise asymptotic significance according to the ideas of Part 1. The terms of this series have no simple distinguishing algebraic features purely as functions of t and ε, and there is, in fact, a considerable arbitrariness in the above formal scheme. We could, for example, replace the roots of (2.1.2) by the roots of an equation obtained from (2.1.2) by replacing each $a_k(t, \varepsilon)$ by a sufficiently large finite section of its formal ε-power series. However, as we will see, our previous notions of formal equivalence will account for this nonuniqueness, at least in a general way.

2.3. **Basic hypotheses. Turning varieties.** We begin by describing some general restrictions on (2.1.1) which we shall take for granted without further mention. As we have already stated, a necessary hypothesis is that (2.1.2) must have distinct

roots for $\varepsilon \neq 0$. Accordingly the discriminant of (2.1.2), $\Delta(t, \varepsilon)$, is an infinitely differentiable function on $|t| \leqq t_0$, $0 \leqq \varepsilon \leqq \varepsilon_0$, satisfying $\Delta(0, 0) = 0$ but $\Delta(t, \varepsilon) \neq 0$ for $\varepsilon \neq 0$. In particular $\Delta(0, \varepsilon) \neq 0$ for $\varepsilon \neq 0$. In addition the hypothesis that $t = 0$ is the only turning point of (2.1.1) requires that $\Delta(t, 0) \neq 0$ for $t \neq 0$. Thus each of the functions $\Delta(t, 0)$, $\Delta(0, \varepsilon)$ has an isolated zero at $(0, 0)$. We shall require that each of these zeros is of finite order.

We next analyze $\Delta(t, \varepsilon)$ in terms of its formal power series at $t = \varepsilon = 0$. We will denote the formal power series of a given $\phi(t, \varepsilon)$ by $\hat{\phi}$. We will also use circumflexed quantities to denote formal power series not necessarily associated with a given function or to denote the formal product of an infinitely differentiable function and a formal power series.

Since $\Delta(t, 0)$ has a zero of finite order, say m, at $t = 0$, it follows from the Weierstrass preparation theorem for formal power series [4] that $\hat{\Delta}(t, \varepsilon)$ has the unique factorization

$$\hat{\Delta} = \hat{P}_0 \hat{U}_0, \qquad \hat{P}_0 = t^m + \sum_{j=0}^{m-1} t^j \hat{b}_j(\varepsilon), \tag{2.3.1}$$

where $\hat{U}_0$ is a unit in the ring of formal power series. By Puiseaux's Theorem [5], $\hat{P}_0$ can be resolved into linear factors of the form $t - \hat{t}(\varepsilon)$ where $\hat{t}$ is a formal power series in a fractional power of ε. We will assume that the reader is familiar with the constructive procedure for obtaining these factors by means of the Newton polygon, and we regard this as an essential prerequisite for following our rather intricate analysis. We will make extensive use of the set of roots $\{\hat{t}\}$ as an indexing or labeling set. Each root $\hat{t}$ has the form

$$\hat{t} = \sum_{k=1}^{\infty} c_k(\hat{t}) \varepsilon^{\delta_k(\hat{t})} \tag{2.3.2}$$

where $\delta_k(\hat{t})$ is an increasing sequence of positive rational numbers and $c_k(\hat{t})$ is a sequence of nonzero complex numbers. It is also convenient to define $\delta_0(\hat{t}) = c_0(\hat{t}) = c_{-1}(\hat{t}) = 0$. Then for each nonnegative integer k there is a change of variable from t to s given by

$$t = \sum_{j=1}^{k-1} c_j(\hat{t}) \varepsilon^{\delta_j(\hat{t})} + \varepsilon^{\delta_k(\hat{t})} s. \tag{2.3.3$_k$}$$

In terms of the variable s, $\hat{\Delta}$ can be expressed in the form

$$\hat{\Delta}(t, \varepsilon) = \varepsilon^{\gamma_k(\hat{t})} \alpha_k(s, \hat{t}) + \cdots \tag{2.3.4}$$

where $\gamma_k(\hat{t})$ is a positive rational number, $\alpha_k(s, \hat{t})$ is a polynomial in s if $k > 0$, $\alpha_0(s, \hat{t})$ is the discriminant of (2.1.2)′, and the dots indicate higher order terms in ε. To each root $\hat{t}$ we now assign a partial sum $\tau(\hat{t})$ in the following way.

DEFINITION 2.3.1. Let $N(\hat{t})$ be the largest integer such that

$$q - \delta_N(\hat{t}) - [n(n-1)]^{-1} \gamma_N(\hat{t}) > 0.$$

Let $\tau(\hat{t})$ be given by

$$\tau(\varepsilon, \hat{t}) = \sum_{k=1}^{N(\hat{t})} c_k(\hat{t})\varepsilon^{\delta_k(\hat{t})}.$$

The significance of the index $N(\hat{t})$ is that only the terms of $\hat{t}$ appearing in τ have essential bearing on the asymptotic theory of (2.1.1).

We can now give a definition of turning variety which is suited to the investigation of our symmetric case. However, we must again emphasize that we are definitely not attempting here to give a systematic general definition of turning variety.

DEFINITION 2.3.2. If for some root $\hat{t}$ of $\hat{P}_0$, $\tau(\hat{t})$ is real, then the algebraic variety in (t, ε) space given by $t=\tau(\varepsilon, \hat{t})$ is a turning variety of the differential equation (2.1.1).

We are now able to state our hypotheses.

I. Let l_i, $i=1, \ldots, n$, be the global continuous roots of (2.1.2). Then all quotients of the quantities $l_i - l_j$, $i \neq j$, are uniformly bounded.

II. The differential equation (2.1.1) has no turning varieties.

III. There are continuous determinations of $\arg(l_i - l_j)$, $i \neq j$, satisfying

$$|\arg(l_i - l_j)| \leqq \pi/2.$$

We have discussed condition I in §2.1. We will see that its main significance is that it permits us to make a detailed analysis of the roots of (2.1.2) near the point of degeneracy, $t=\varepsilon=0$, almost entirely in terms of the discriminant Δ.

The meaning of condition II, beyond its evident resemblance to a condition ensuring that Δ has no zeros, is illuminated by considering the effect on (2.1.1) of the change of independent variable $(2.3.3)_k$. It can be verified (although we have no technical need to carry out the details of this change) that this leads to an equation of similar form in which the integer q is replaced by the rational number $q - \delta_k(\hat{t}) - [n(n-1)]^{-1}\gamma_k(\hat{t})$. Thus for $k \leqq N(\hat{t})$, the resulting transformed problem retains the singular kind of dependence on ε which naturally disposes towards asymptotic methods. Roughly speaking, the higher order terms with indices greater than $N(\hat{t})$ are too small in the natural asymptotic scales of the problem to have significance in our analysis.

Hypothesis III is a condition of a familiar kind which ensures that we will be able to construct solutions of (2.1.1) which do not undergo a violent transition from growth to decay as $\varepsilon \to 0$. In the absence of such a condition we would be forced to subdivide the (t, ε) domain of the problem into subdomains, on each of which the inequality of condition III would be satisfied. Here it is crucial that equality is permitted in condition III since this ensures that adjacent subdomains on which the inequality holds will have boundary points in common at which fundamental solutions of the system (2.2.2) on each subdomain can be compared. The arbitrariness in the selection of equation (2.1.2) discussed in §2.1 also appears in this condition.

Finally we illustrate the above classification with an example. It will appear in our subsequent treatment of examples that we have somewhat slighted certain constructive aspects of our analysis. For example it is not necessary to carry out the factorization $\hat{\Delta}=\hat{P}_0\hat{U}_0$ or to determine completely the roots of $\hat{P}_0$. However, we have considered it important to simplify an already complicated analysis at the expense of such details whenever possible.

We analyze the turning varieties of the problem

$$\varepsilon^{3r}w'''+\varepsilon^r(t^4-\varepsilon)^2w'+\varepsilon^3tw = 0 \tag{2.3.5}$$

where r is a positive rational number. The related algebraic equation is $l^3+(t^4-\varepsilon)^2l+\varepsilon^3t=0$ which has discriminant

$$\Delta = -4(t^4-\varepsilon)^6-27\varepsilon^6t^2.$$

In this case $P_0=-\frac{1}{4}\Delta$. The 24 roots of Δ are given by

$$\hat{t}_{jk} = e^{j\pi i/2}\varepsilon^{1/4}+3^{1/2}2^{-7/3}e^{(1/6+k/3)\pi i}\varepsilon^{1/3}+\cdots, \quad 0 \leqq j \leqq 3, \quad 0 \leqq k \leqq 5. \tag{2.3.6}$$

In terms of $s=t\varepsilon^{-1/4}$, Δ is given by

$$\Delta = -4\varepsilon^6[s^4-1]^6+\cdots.$$

Thus $\delta_1=\frac{1}{4}$, $\gamma_1=6$ and $r-\frac{1}{4}-(3(3-1))^{-1}6>0$ when $r>\frac{5}{4}$. Hence for $0<r\leqq\frac{5}{4}$, $N(\hat{t})=0$ and $\tau(\hat{t})=0$. Thus if $r\leqq\frac{5}{4}$, $t=0$ is a turning variety.

We now suppose that $r>\frac{5}{4}$. (In this case the change of variable $t=\varepsilon^{1/4}s$ transforms the original equation into

$$\varepsilon^{3(r-5/4)}\,d^3w/ds^3+\varepsilon^{(r-5/4)}(s^4-1)^2\,dw/ds+\varepsilon^{1/4}sw = 0$$

which has "secondary" turning points at $s=\pm 1$.) It is clear that in the search for turning varieties we need only consider the transformations arising from $\hat{t}_{0k}$ and $\hat{t}_{2k}$ since the remaining roots cannot give rise to (real) turning varieties. To these roots correspond the changes of variable $t=\pm\varepsilon^{1/4}+\varepsilon^{1/3}s$ under which Δ assumes the form $\Delta=\varepsilon^{13/2}(2^{14}s^6+27)+\cdots$. Here $\delta_2=\frac{1}{3}$, $\gamma_2=\frac{13}{2}$ and $r-\frac{1}{3}-\frac{13}{12}>0$ for $r>\frac{17}{12}$. Thus if $\frac{5}{4}<r\leqq\frac{17}{12}$, the differential equation has two turning varieties, $t=\pm\varepsilon^{1/4}$.

Finally if $r>\frac{17}{12}$, both of the terms of $\hat{t}_{jk}$ indicated explicitly in (2.3.6) appear in $\tau(\hat{t}_{jk})$. Since at least one of these is complex for each root, we conclude that for $r>\frac{17}{12}$ the problem has no turning varieties.

To summarize: problem (2.3.5) has the turning varieties $t=0$ if $0<r\leqq\frac{5}{4}$, $t=\pm\varepsilon^{1/4}$ if $\frac{5}{4}<r\leqq\frac{17}{12}$, and no turning varieties if $r>\frac{17}{12}$.

2.4. **Formal solutions.** The purpose of this section is to show that the preformal solutions of §2.2 are truly formal solutions in a precise technical sense of the kind formulated in §1.3. We briefly retrace this formulation in the special version which is suitable here. We consider functions on $\Omega(t_0,\varepsilon_0)=[-t_0,t_0]\times(0,\varepsilon_0]$ assuming values in the algebra β of complex $n\times n$ matrices endowed with a multiplicative norm $\|\ \ \|$. We take $\beta_1=\beta_0(\Omega)$ to be the ring of bounded matrix valued functions

$f(t, \varepsilon)$ which for each ε are infinitely differentiable functions of t, and we use the formal convergence and derivative estimates induced in β_0 by the ideal $\varepsilon^q\beta_0$. Thus $B_0(\Omega)$ is the subring of β_0 defined by the asymptotic a priori estimates

$$(\varepsilon^q D)^k f \in \beta_0, \qquad D = d/dt, \tag{2.4.1}$$

and $B(\Omega)$ is simply $\varepsilon^{-q}B_0 \cup \varepsilon^{-2q}B_0 \cup \cdots$.

Next a sequence f_k of elements of B is formally convergent to zero if the f_k simply become divisible by higher and higher powers of ε as $k \to \infty$, that is $f_k \in \varepsilon^{\psi(k)}B_0$ where $\psi(k) \to \infty$ as $k \to \infty$. From this specification of B_0 and formal convergence, the corresponding notions of formal and asymptotic series, and formal equivalence follow immediately and we will suppose that these underlie our use of these terms throughout Part 2.

The estimates (2.4.1) appear inevitable if we regard them as a priori estimates for bounded solutions T of (2.2.7) since it follows from this equation that if Λ and $\varepsilon^q S^{-1}S'$ satisfy (2.4.1) then so does T. We also remark that such derivative estimates for Λ and S are a technical necessity for analyzing the recursion scheme (2.2.8) which proceeds by repeated differentiation. The role of these estimates in our formulation actually makes it more special than the classical one, but clearly in other respects our scheme is far more general even in this case involving a single parameter. Nonetheless, we believe that our formulation is the simplest possible which can deal satisfactorily with such a series as (2.2.9) the terms of which have no simple distinguishing algebraic properties (as in the case of formal power series) and which lacks uniqueness in any natural termwise sense. Moreover we believe that the reader who has some acquaintance with the mystique of generalizing the notion of asymptotic expansion will find that our definitions do not suffer from the easy defect of excessive generality.

We now turn to some more detailed technical considerations which are also most easily formulated in terms of a differential ring defined by a priori estimates. We find it essential to introduce more delicate estimates which take into account very special features of our problem. For this reason we prefer to regard this ring (to be defined below) as a useful intermediate device rather than a basis for defining another notion of asymptotic character, although it would be possible to do so. Also the above definitions apply in full force since this new ring will be a subring of $B_0(\Omega(t_0, \varepsilon_1))$ for ε_1 sufficiently small. This new ring will be the simplest differential ring which embodies accurate derivative estimates for the logarithmic derivative of the discriminant Δ. We now proceed to its definition.

We again refer to the formal factorization $\hat{\Delta} = \hat{P}_0\hat{U}_0$, the roots $\{\hat{t}\}$ of $\hat{P}_0$, and the truncated roots $\{\tau(\varepsilon, \hat{t})\}$ of Definition 2.3.1.

DEFINITION 2.4.1. Let $\sigma(t, \varepsilon)$ be the function

$$\sigma = \left(\sum_{\hat{t}} |t - \tau(\varepsilon, \hat{t})|^{-2}\right)^{-1/2}.$$

Let $A(t_0, \varepsilon_0)$ be the ring of functions from $\Omega = [-t_0, t_0] \times (0, \varepsilon_0]$ to the complex $n \times n$ matrices which for each ε are infinitely differentiable in t and which satisfy the estimates

$$(\sigma D)^k f \in \beta_0([-t_0, t_0] \times (0, \varepsilon_1])$$

where $\| \quad \|$ is some fixed multiplicative matrix norm and $D = d/dt$.

LEMMA 2.4.1. *For ε_1 sufficiently small, $A(t_0, \varepsilon_1)$ is a subring of $B_0([-t_0, t_0] \times (0, \varepsilon_1])$ closed under the action of σD and the function σ is itself an element of $A(t_0, \varepsilon_1)$.*

Proof. Under the hypothesis that turning varieties do not occur, for each truncated root

$$\tau(\hat{t}) = \sum_{k=0}^{N(\hat{t})} c_k(\hat{t}) \varepsilon^{\delta_k(\hat{t})}$$

there is a first subscript $k(\hat{t})$ for which $c_k(\hat{t})$ is complex. It follows that for ε sufficiently small the functions σ, $t - \tau$ and $t - \bar{\tau}$ do not vanish.

We now consider the functions σ, $\sigma/(t-\tau)$ and $\sigma/(t-\bar{\tau})$. For ε sufficiently small these are bounded and therefore generate an algebra of bounded functions. But applying σD to each of these generators we again obtain an element of this algebra. For example

$$\sigma D \sigma = -\frac{1}{2} \sigma^4 D \sigma^{-2} = \frac{1}{2} \sigma \sum_{\hat{t}} \left\{ \frac{\sigma^2}{(t-\tau)^2} \cdot \frac{\sigma}{t-\bar{\tau}} + \frac{\sigma}{t-\tau} \frac{\sigma^2}{(t-\bar{\tau})^2} \right\}.$$

It follows that this algebra and in particular the generator σ belongs to A. The formula above also shows that $\sigma D \sigma \in \sigma A$.

We now show that if $f \in A$, then $\sigma^k D^k f \in A$ for each k. This follows by induction using the identity

$$\sigma^{k+1} D^{k+1} f = \sigma D(\sigma^k D^k f) - k(\sigma D \sigma) \sigma^{k-1} D^k f.$$

Thus if $f \in A$, $\sigma^k D^k f$ is bounded for each k. But for ε sufficiently small

$$\sigma \geqq \left\{ m \max_{t,\hat{t}} \frac{1}{|t-\tau|^2} \right\}^{-1/2} \geqq C \varepsilon^{\max_{\hat{t}} \delta_{k(\hat{t})}}$$

where $k(\hat{t})$ is again the first subscript in $\hat{t}$ for which $c_k(\hat{t})$ is complex. Since we have assumed (recall Definitions 2.3.1 and 2.3.2) that $q - \delta_k - [n(n-1)]^{-1} \gamma_k > 0$ for $k < N(\hat{t})$, it is the case that $\max_{\hat{t}} \delta_{k(\hat{t})} < q$. Thus $\sigma > C\varepsilon^q$ and the boundedness of $\sigma^k D^k f$ implies that of $(\varepsilon^q D)^k f$, that is $f \in B_0([t_0, t_0] \times (0, \varepsilon_1])$ for ε_1 sufficiently small.

As a consequence of this lemma we can apply our previous notions of formal convergence and asymptotic character to series of functions in A. We will adopt the convention which identifies the scalar multiple hI of the identity matrix with the function h.

Our aim is to show that the terms of the series (2.2.9) belong to A and are contained in certain ideals in A. The wellspring of these inclusions is the following.

LEMMA 2.4.2. *Let Δ be the discriminant of the characteristic equation* (2.1.2). *Then for ε_1 sufficiently small,*

$$\sigma D \log \Delta \in A(t_0, \varepsilon_1).$$

Proof. We require a deeper analysis than can be made using the formal factorization $\hat{\Delta} = \hat{P}_0 \hat{U}_0$. Instead we use the Malgrange-Weierstrass preparation theorem for infinitely differentiable functions [6] which ensures that for t and ε sufficiently small

$$\Delta = PU, \tag{2.4.2}$$

where P is a monic polynomial of degree m with coefficients which are infinitely differentiable functions of ε vanishing at $\varepsilon = 0$ and U is a unit in the ring of infinitely differentiable functions. This factorization exists on some subdomain $[-t_0', t_0'] \times (0, \varepsilon_0']$, where $0 < t_0' \leqq t_0$, $0 < \varepsilon_0' \leqq \varepsilon_0$. But if ε_1 is sufficiently small, both Δ and P are units on $\{[-t_0, t_0] - [-t_0', t_0']\} \times (0, \varepsilon_0'']$. Thus on this domain the factorization (2.4.2) is trivial. It follows that for some ε_1, $0 < \varepsilon_1 \leqq \varepsilon_0'$, (2.4.2) is valid on $[-t_0, t_0] \times (0, \varepsilon_1]$, that is to say, the factorization is global in t. We suppose hereafter that $\varepsilon \in (0, \varepsilon_1]$.

It follows from the uniqueness of the formal factorization that $\hat{P} = \hat{P}_0$ and $\hat{U} = \hat{U}_0$. (However, P and U are not uniquely determined by Δ.) To each root $\hat{t}(\varepsilon)$ of $\hat{P}_0$ we assign some infinitely differentiable function of the suitable fractional power of ε, call it $\tilde{t}(\varepsilon)$, having $\hat{t}(\varepsilon)$ as its formal power series. Let the polynomial in t, $P_1(t, \varepsilon)$, be defined by

$$P_1(t, \varepsilon) = \prod_{P_0(\tilde{t}) = 0} [t - \tilde{t}(\varepsilon)].$$

It is easily seen that P_1 has coefficients which are infinitely differentiable functions of ε. Also by its construction P_1 satisfies $\hat{P}_1 = \hat{P}_0 = \hat{P}$. It follows that $P - P_1$ is divisible by any power of ε and we can write

$$P = P_1 + \varepsilon^{mq+1} Q \tag{2.4.3}$$

where Q is also a polynomial in t with infinitely differentiable coefficients.

We now consider the linear factors $t - \tilde{t}(\varepsilon)$ of P_1. The functions $\sigma/(t - \tilde{t}(\varepsilon))$ and $\varepsilon^q/(t - \tilde{t}(\varepsilon))$ are bounded and together with the elements of $A(t_0, \varepsilon_1)$ generate an algebra of bounded functions. Since σD applied to any of these generators again gives an element of this algebra we conclude that the given functions belong to A. Combining (2.4.2) and (2.4.3) we obtain

$$\sigma D \log \Delta = \sum_{\tilde{t}} \frac{\sigma}{t - \tilde{t}} + (\sigma D) \log \left\{ 1 + \varepsilon Q \prod_{\tilde{t}} \frac{\varepsilon^q}{t - \tilde{t}} \right\} + \sigma U^{-1} U'. \tag{2.4.4}$$

The indicated sum belongs to A and $U^{-1} U' \sigma$ belongs to A since σ does by Lemma 2.4.1 and both U^{-1} and U' satisfy the much stronger requirement of being infinitely differentiable functions of (t, ε) on the closure of the underlying domain. The function $h = 1 + \varepsilon Q \prod_{\tilde{t}} (\varepsilon^q/(t - \tilde{t}))$ also belongs to A. Choosing ε_1 smaller if necessary we can suppose that on $[-t_0, t_0] \times (0, \varepsilon_1]$, $1/h$ is bounded. It then follows easily

that $1/h \in A$ and hence that $\sigma D \log h = 1/h \cdot (\sigma Dh) \in A$. Thus each term of the right-hand side of (2.4.4) belongs to A and the proof is concluded.

Our first theorem establishes that the series (2.2.9) is a formal solution of the transformation equation (2.2.7) and that it behaves qualitatively like a formal power series in the quantity $\varepsilon^q \sigma^{-1} \Delta^{-[n(n-1)]^{-1}}$.

THEOREM 1. *If* (2.1.1) *satisfies conditions* I *and* II *of* §2.3, *then for* ε_1 *sufficiently small the recursion scheme* (2.2.8) *defines a sequence* T_k *satisfying*

$$\varepsilon^{kq} T_k(t, \varepsilon) \in \{\varepsilon^q \sigma^{-1} \Delta^{-[n(n-1)]^{-1}}\}^k A(t_0, \varepsilon_1). \tag{2.4.5}$$

Moreover the series $T \doteq \sum_{k=0}^{\infty} \varepsilon^{kq} T_k$ *is a formal solution of the formal equation corresponding to equation* (2.2.7).

We isolate portions of the proof of Theorem 1 in the following two lemmas for which we presuppose the hypotheses of the theorem.

LEMMA 2.4.3. *The sets* $\{\varepsilon^q \sigma^{-1} \Delta^{-[n(n-1)]^{-1}}\}^k A(t_0, \varepsilon_1)$ *are, for sufficiently small* ε_1, *proper ideals in* $A(t_0, \varepsilon_1)$ *closed under the action of* σD.

Proof. In the proof of Lemma 2.4.1 we established that ΔP_1^{-1} is a unit in A. Hence the sets in question are $\{\varepsilon^q \sigma^{-1} P_1^{-[n(n-1)]^{-1}}\}^k A$. We consider the function $\sigma^{-n(n-1)} P_1^{-1}$, for each ε, as a rational function of t with poles in the complex t-plane at each of the points $\tau(\varepsilon, \hat{t})$, $\bar{\tau}(\varepsilon, \hat{t})$, $\tilde{t}(\varepsilon, \hat{t})$. For each root $\hat{t}$ we define a circle $\mathscr{C}(\hat{t})$ by

$$\left| t - \sum_{k=0}^{k(\hat{t})} c_k(\hat{t}) \varepsilon^{\delta_k(\hat{t})} \right| = h \varepsilon^{\delta_k(\hat{t})}$$

and let $\bar{\mathscr{C}}(\hat{t})$ be its reflection in the real axis. In terms of the variable s defined by $(2.3.3)_{k(\hat{t})}$ these circles are $|s - c_{k(\hat{t})}| = h$ and $|s - \bar{c}_{k(\hat{t})}| = h$. Since $c_{k(\hat{t})}$ is complex we can choose h so small that none of these intersect the real axis. According to $(2.3.4)_{k(\hat{t})}$, in terms of s, P_1 has the form

$$P_1 = \varepsilon^{\gamma_{k(\hat{t})}} \varphi(s, \hat{t}) + \cdots \tag{2.4.6}$$

where φ is a nonzero polynomial in s having $c_{k(\hat{t})}$ as a root. We can also suppose that h is less than half the distance between any pair of distinct roots of any of the polynomials $\varphi(s, \hat{t})$, $\bar{\varphi}(s, \hat{t})$. It then follows that $\varphi(s, \hat{t})$ does not vanish on $\mathscr{C}(\hat{t}) \cup \bar{\mathscr{C}}(\hat{t})$, which together with (2.4.6) implies that for all ε sufficiently small,

$$|P_1| > M\varepsilon^{+\gamma_{k(\hat{t})}} \quad \text{on } \mathscr{C}(\hat{t}) \cup \bar{\mathscr{C}}(\hat{t}).$$

We now show similarly that on this set $|\sigma^{-2}| > M\varepsilon^{-2\delta_{k(\hat{t})}}$. Now

$$\sigma^{-2} = \sum (t - \tau)^{-1} (t - \bar{\tau})^{-1}.$$

It is easily seen that on $\mathscr{C}(\hat{t}) \cup \bar{\mathscr{C}}(\hat{t})$, any summand corresponding to a root which does not agree with $\hat{t}$ in all terms up to order $\varepsilon^{\delta_{k(\hat{t})}-1}$ will have a stronger bound of the form $M\varepsilon^{2\delta_{k(\hat{t})}-1}$ for all small ε. We therefore restrict our attention to summands corresponding to roots having $\sum_{k=0}^{k(\hat{t})-1} c_k \varepsilon^{\delta_k}$ as an initial section. We now use the

fact (see [5] for the Newton polygon) that the complex numbers which occur among these roots as coefficients of $\varepsilon^{\delta_{k(\hat{t})}}$ are precisely the zeros of $\varphi(s, \hat{t})$. Thus a typical summand on $\mathscr{C}(\hat{t})$ will have the form

$$\frac{\varepsilon^{-\delta_{k(\hat{t})}}}{s+c_{k(\hat{t})}-c+\cdots}\,\frac{\varepsilon^{-\delta_{k(\hat{t})}}}{s+\bar{c}_{k(\hat{t})}-\bar{c}+\cdots}$$

where c is a root of $\varphi(s, \hat{t})$. But $\mathscr{C}(\hat{t})$ has been constructed so that the leading parts in the denominator (and the corresponding ones on $\bar{\mathscr{C}}(\hat{t})$) do not vanish. Hence $|\sigma^{-2}| < M\varepsilon^{-2\delta_{k(\hat{t})}}$. Combining this with the previous estimate for P_1^{-1} we obtain

$$|\sigma^{-n(n-1)}P_1^{-1}| < M\varepsilon^{-n(n-1)\delta_{k(\hat{t})}-\gamma_{k(\hat{t})}}$$

on $\mathscr{C}(\hat{t}) \cup \bar{\mathscr{C}}(\hat{t})$. But for all ε sufficiently small these circles enclose the poles of $\sigma^{-n(n-1)}P_1^{-1}$. Hence by the maximum modulus principle

$$|\sigma^{-n(n-1)}P_1^{-1}| < M\varepsilon^{-\max_{\hat{t}}\{n(n-1)\delta_{k(\hat{t})}+\gamma_{k(\hat{t})}\}}$$

on the region exterior to each of these circles, and in particular on the entire real axis. Thus we have the estimate

$$|\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}| < M\varepsilon^{\min_{\hat{t}}\{q-\delta_{k(\hat{t})}-[n(n-1)]^{-1}\gamma_{k(\hat{t})}\}}.$$

Since the exponent of ε is positive, it follows that $\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}$ is not only bounded, but is actually bounded by a positive power of ε. To establish that $\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}$ belongs to A we proceed by induction, using the identity

$$(\sigma D)^{k+1}\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}} = (\sigma D)^k\{\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}(-[n(n-1)]^{-1}\sigma D\log\Delta - D\sigma)\}.$$

Since $\sigma D\log\Delta \in A$ by Lemma 2.4.2, and

$$D\sigma = -\frac{1}{2}\sigma^3 D\sigma^{-2} = \frac{1}{2}\sum_{\hat{t}}\left(\frac{\sigma}{t-\tau}\right)^2\left(\frac{\sigma}{t-\bar{\tau}}\right)+\left(\frac{\sigma}{t-\tau}\right)\left(\frac{\sigma}{t-\bar{\tau}}\right)^2 \subset A,$$

this identity shows that $(\sigma D)^{k+1}$ acts boundedly if all lower powers do. Also the case $k=1$ shows that

$$(\sigma D)\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}} \in \varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}A,$$

which shows that the ideal on the right, and hence its powers, are closed under the action of σD. This concludes the proof of Lemma 2.4.3.

We next establish that various functions depending on the roots $\{l_k\}$ belong to A. It is here that we will make essential use of the symmetry hypothesis I of §3.

LEMMA 2.4.4. *For ε_1 sufficiently small, the roots l_k of* (1-2), *the matrix S of* (2-3), *and the operator* (ad $\Lambda)^{-1}$ *of* (2-8) *satisfy*

(a) $l_k \in \Delta^{[n(n-1)]^{-1}}A(t_0, \varepsilon_1)$,

(b) $(l_i - l_j)^{-1} \in \Delta^{-[n(n-1)]^{-1}}A(t_0, \varepsilon_1)$,

(c) $S^{-1}S' \in \sigma^{-1}A(t_0, \varepsilon_1)$,

(d) $(\text{ad }\Lambda)^{-1}(I-\text{diag})A(t_0, \varepsilon_1) \in \Delta^{-[n(n-1)]^{-1}}A(t_0, \varepsilon_1)$.

Proof. Since there is no l^{n-1} term in (1-2), $\sum_{k=1}^{n} l_k=0$. This implies $l_k=(1/n)\sum_{j=1}^{n}(l_k-l_j)$. Hence, by the symmetry hypothesis,

$$\frac{l_k}{l_i-l_j}=\frac{1}{n}\sum_{k'=1}^{n}\frac{l_k-l_{k'}}{l_i-l_j}$$

is bounded, and then so is

$$\Delta^{-[n(n-1)]^{-1}}l_k=\left\{\prod_{i\neq j}\frac{l_k}{l_i-l_j}\right\}^{[n(n-1)]^{-1}}.$$

The functions $\Delta^{-k/[n(n-1)]}a_k$, which are the elementary symmetric functions in these quantities, are therefore bounded. We now show that the $\Delta^{-k/[n(n-1)]}a_k$ are not only bounded, but actually belong to A. To establish this it suffices to show that if $f(t,\varepsilon)$ is any function infinitely differentiable on $[-t_0,t_0]\times[0,\varepsilon_0]$ (here $\varepsilon=0$ is included) and for some r, $0<r<1$, $P_1^{-r}f$ is bounded, then $P_1^{-r}f\in A(t_0,\varepsilon_1)$. It follows from [6, §5, pp. 11-08 and 11-09] that since P_1 is regular of order m in t at $(0,0)$ $(P_1(t,0)=t^m)$ f can be represented in the form

$$f(t,\varepsilon)=F(t,\varepsilon)P_1(t,\varepsilon)+Q_1(t,\varepsilon)$$

where F is infinitely differentiable, and Q_1 is a polynomial of degree $m-1$ in t, with coefficients which are infinitely differentiable functions of ε on $[0,\varepsilon_0]$. Thus $P_1^{-r}f=P_1^{1-r}F+P_1^{-r}Q_1$. But induction on the identity

$$(\sigma D)^{k+1}P_1^{\beta}=(\sigma D)^k(\beta P_1^{\beta}\sigma D\log P_1)$$

shows that any positive power of P_1 belongs to A. Hence $P_1^{1-r}F\in A$. It thus remains to show that if $P_1^{-r}Q_1$ is bounded, then $P_1^{-r}Q_1\in A$. Since $P_1^{-r}Q_1$ is analytic in t, and since it is easily seen that for ε sufficiently small the circle of radius $\frac{1}{2}\sigma(t,\varepsilon)$ about the point t in the complex t-plane contains no zeros of P_1, we can represent $P_1^{-r}Q_1$ in the form

$$P_1^{-r}(t,\varepsilon)Q_1(t,\varepsilon)=\frac{1}{2\pi i}\int_{|s-t|=\sigma(t,\varepsilon)/2}P_1^{-r}(s,\varepsilon)Q_1(s,\varepsilon)(s-t)^{-1}\,ds.$$

But it follows from this representation that

$$\sigma^kD^kP_1^{-r}Q_1=\frac{k!}{2\pi i}\sigma^k(t,\varepsilon)\int_{|s-t|=\sigma(t,\varepsilon)/2}P_1^{-r}Q_1(s-t)^{-k-1}\,ds$$

which is bounded if $P_1^{-r}Q_1$ is bounded. Since

$$(\sigma D)^k \text{ is a linear combination of } \sigma D,\ \sigma^2D^2,\ldots$$

with coefficients in A, it follows that $(\sigma D)^kP_1^{-r}Q_1$ is bounded and hence $P_1^{-r}Q_1\in A$. Thus if $\Delta^{-r}f$ is bounded, $\Delta^{-r}f$ must belong to A.

We have now established that the functions $\Delta^{-[n(n-1)]^{-1}}l_k$ are the roots of a monic algebraic equation with coefficients in $A(t_0,\varepsilon_1)$. Moreover the discriminant of this equation is identically 1. The inclusion, $\Delta^{-[n(n-1)]^{-1}}l_k\in A$, which is inclusion

(a) of the lemma, now easily follows by applying $(\sigma D)^k$ to this algebraic equation, imitating the usual procedure for establishing smoothness of solutions of a functional equation by implicit differentiation.

It follows from (a) that $\Delta^{-[n(n-1)]^{-1}}(l_i-l_j)\in A$. Since $\prod_{i\neq j}\Delta^{-[n(n-1)]^{-1}}(l_i-l_j)=1$, it follows that each of these is a unit in A, and hence $\Delta^{[n(n-1)]^{-1}}(l_i-l_j)^{-1}\in A$, which establishes (b).

We recall that $S=(l_j^{k-1})$. Hence by (a)

$$S = (\delta_{jk}\Delta^{(k-1)/n(n-1)})(l_j^{k-1}\Delta^{-(k-1)/n(n-1)}) = (\delta_{jk}\Delta^{(k-1)/n(n-1)})S_1,$$

where $S_1\in A$. But the square of the determinant of S_1 is easily seen to be 1. Hence $S_1^{-1}\in A$. Also

$$\begin{aligned}\sigma DS &= \sigma D(\delta_{jk}\Delta^{(k-1)/n(n-1)})S_1\\ &= (\sigma D\log\Delta)\left(\frac{k-1}{n(n-1)}\,\delta_{jk}\Delta^{(k-1)/n(n-1)}\right)S_1+(\delta_{jk}\Delta^{(k-1)/n(n-1)})(\sigma DS_1)\\ &= (\delta_{jk}\Delta^{(k-1)/n(n-1)})S_2\end{aligned}$$

where $S_2\in A$. Thus $S^{-1}(\sigma D)S=S_1^{-1}S_2\in A$ which establishes (c).

The inclusion (d) follows from (b) when we observe that the elements of $(\operatorname{ad}\Lambda)^{-1}(I-\operatorname{diag})(f_{ij})$ are $(l_i-l_j)^{-1}f_{ij}$, if $i\neq j$, and 0 if $i=j$.

Proof of Theorem 1. We now prove the inclusions (2.4.5) by induction. The inclusion is true for $k=0$. Suppose it is true for $i\leqq k$. Then by the recursion formulas (2.2.8),

$$\varepsilon^{(k+1)q}T_{k+1} = \varepsilon^q(\operatorname{ad}\Lambda)^{-1}\left\{D\varepsilon^{kq}T_k+S^{-1}S'\varepsilon^q\cdot\varepsilon^{kq}T_k-\sum_{i+j=k}\varepsilon^{iq}T_i\operatorname{diag}(S^{-1}S'\varepsilon^{jq}T_j)\right\}.$$

Since the diagonal part of the right-hand side is 0 we have

$$\begin{aligned}\varepsilon^{(k+1)q}T_{k+1} \in\ &\varepsilon^q(\operatorname{ad}\Lambda)^{-1}(1-\operatorname{diag})\\ &\cdot\Big\{\sigma^{-1}(\sigma D)[\varepsilon^q\sigma^{-1-[n(n-1)]^{-1}}]^kA+S^{-1}S'[\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}]^kA\\ &\qquad+\sum_{i+j=k}[\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}]^iAS^{-1}S'[\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}]^jA\Big\}.\end{aligned}$$

We now use Lemma 2.4.1 to drop the σD from $(\sigma D)\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}$, and also use inclusions (c) and (d) of Lemma 2.4.4, obtaining

$$\begin{aligned}\varepsilon^{(k+1)q}T_{k+1} \in\ &\varepsilon^q\Delta^{-[n(n-1)]^{-1}}\Big\{\sigma^{-1}[\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}]^kA+\sigma^{-1}[\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}]^kA\\ &\qquad+\sigma^{-1}\sum_{i+j=k}[\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}]^iA[\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}]^iA\Big\}\\ \in\ &[\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}]^{k+1}A.\end{aligned}$$

This establishes (2.4.5). Finally we established in Lemma 2.4.1 that

$$\varepsilon^q\sigma^{-1}\Delta^{-[n(n-1)]^{-1}}\in\varepsilon^\kappa A$$

where

$$\kappa = \min_{\hat{t}}\, q-\delta_{k(\hat{t})}(\hat{t})-[n(n-1)]^{-1}\gamma_{k(\hat{t})}(\hat{t}) > 0.$$

Thus the series $\sum_{k=0}^{\infty} \varepsilon^{kq} T_k$ is formally convergent and, by construction, its substitution into equation (2.2.7) leads to a series formally equivalent to 0.

2.5. **Formal solutions are asymptotic solutions.** We now establish asymptotic significance for the formal solutions of Theorem 1.

THEOREM 2. *Let the system* (2.2.2), $\varepsilon^q w' = \underline{a} w$, *satisfy hypotheses* I, II *and* III *of* §2.3. *Let S be given by* (2.2.3) *and T_k be given by* (2.2.8). *Then for ε_1 sufficiently small and $|t| \leqq t_0$, $0 < \varepsilon \leqq \varepsilon_1$, the system is reduced to diagonal form by a change of variable $w = STv$ where $T \sim \sum_{k=0}^{\infty} \varepsilon^{kq} T_k$.*

Proof. Using the infinitely differentiable version of Proposition 1.3.4, we let $\tilde{T}$ be such that

$$\tilde{T} \sim \sum_{k=0}^{\infty} \varepsilon^{kq} T_k. \tag{2.5.1}$$

It then easily follows from Theorem 1 that for small ε_1, det $\tilde{T}$ is a unit in $A(t_0, \varepsilon_1)$. This easily implies that the series obtained by inverting $\sum_{k=0}^{\infty} \eta^k T_k(t, \varepsilon)$ as an η-formal power series and replacing η by ε^q is an asymptotic expansion for $\tilde{T}^{-1}$.

We now consider the change of variable $w = S\tilde{T}\tilde{v}$, which leads to the transformed system

$$\varepsilon^q \tilde{v}' = (\tilde{T}^{-1}\Lambda\tilde{T} - \varepsilon^q \tilde{T}^{-1}\tilde{T}' - \varepsilon^q \tilde{T}^{-1} S^{-1} S' \tilde{T})\tilde{v}.$$

It follows from (2.5.1) and Theorem 1 that $\tilde{T}$ is an approximate solution of (2.2.7) in the sense that

$$\varepsilon^q \tilde{T}^{-1}\tilde{T}' - \tilde{T}^{-1}(\Lambda - \varepsilon^q S^{-1} S')\tilde{T} + \Lambda - \varepsilon^q \operatorname{diag} S^{-1} S' \tilde{T} + Z = 0$$

where $Z \sim 0$. Hence the system for $\tilde{v}$ has the form

$$\varepsilon^q \tilde{v}' = (\tilde{R} + Z)\tilde{v}, \qquad \tilde{R} = \Lambda - \varepsilon^q \operatorname{diag} S^{-1} S' \tilde{T}, \tag{2.5.2}$$

where we note that $\tilde{R}$ is diagonal.

To establish the theorem it suffices to show that there is a change of variable $\tilde{v} = Vv$ which diagonalizes this system and which satisfies $V \sim I$. Recalling the formal analysis of §2.2, it suffices to find a V satisfying diag $V = I$ and

$$\varepsilon^q V' - (\tilde{R} + Z)V + V(\tilde{R} + \operatorname{diag} ZV) = 0.$$

A simple variant of variation of parameters shows that this equation will be satisfied by any solution of the integral equation

$$\begin{aligned} V(t, \varepsilon) - I - \int_{\pm t_0}^{t} \left(\exp\left(\varepsilon^{-q} \int_s^t \tilde{R}\, d\theta\right)\right)(ZV - V \operatorname{diag} ZV) \\ \cdot \left(\exp\left(+\varepsilon^{-q} \int_s^t \tilde{R}\, d\theta\right)\right) ds = 0, \end{aligned} \tag{2.5.3}$$

where we choose the limit of integration for the (i, j) element of the integrand so

that $\operatorname{Re}\int_s^t (l_i - l_j)\, d\theta \leqq 0$. Hypothesis III ensures that this choice is possible. We will establish the existence of a solution of this integral equation, for ε sufficiently small, by the usual technique of successive approximations. To do this we must establish bounds for the exponential factors appearing in the integral term. These are, in fact, *unbounded* in ε, but we will show that this unboundedness is no worse than a negative power of ε. It follows from (2.5.2) that

$$\tilde{R} \sim \Lambda - \varepsilon^q \operatorname{diag} S^{-1}S' + \cdots \tag{2.5.4}$$

and our previous estimates imply that $\tilde{R} = \Lambda + \varepsilon^q \sigma^{-1}\tilde{R}_1$ where $\tilde{R}_1$ is bounded. It follows that the exponential term appearing in the (i, j) element of the integral has the form

$$\exp\left\{\varepsilon^{-q}\int_s^t (l_i - l_j)\, d\theta + \int_s^t \sigma^{-1} h_{ij}\, d\theta\right\}$$

where h_{ij} is bounded. Since $\operatorname{Re}\varepsilon^{-q}\int_s^t (l_i - l_j)\, d\theta \leqq 0$ by the choice of the domain of integration, an upper bound for the real part of the preceding exponent is $M\int_{-t_0}^{t_0} \sigma^{-1}(s, \varepsilon)\, ds$. This can be estimated in the following way.

$$\begin{aligned}
M\int_{-t_0}^{t_0} \sum_i [(s-\tau)(s-\bar{\tau})]^{-1/2}\, ds &= M\sum_i \int_{-t_0}^{t_0} [(s - \operatorname{Re}\tau)^2 + (\operatorname{Im}\tau)^2]^{-1/2}\, ds \\
&\leqq M\sum_i \int_{-t_0-1}^{t_0+1} [s^2 + (\operatorname{Im}\tau)^2]^{-1/2}\, ds \\
&\leqq M\sum_i \int_0^{(t_0+1)|\operatorname{Im}\tau|^{-1}} (s^2+1)^{-1/2}\, ds \\
&\leqq M_1 + M_2 \sum_i \log|\operatorname{Im}\tau|^{-1} \\
&\leqq M_3 \log(1/\varepsilon).
\end{aligned}$$

Thus the exponential terms appearing in (2.5.2) are each bounded by a negative power of ε. Since Z goes to zero more rapidly than any power it is clear that the integral term in (2.5.3) is small if ε is small, and we expect that the equation can be solved by successive approximations. An easy way to make this argument precise is the following. We consider the left-hand side of (2.5.3) to be a continuous function $H(V)$ on the Banach space of matrix valued functions on $[-t_0, t_0]\times(0, \varepsilon_1]$ which for each ε are continuous in t, endowed with the norm $|V| = \sup_{|t|\leqq t_0} \|V\|$. We now appeal to Kantorovič's Theorem on Newtonian successive approximations [7] using $V_0 = I$ as an initial approximation. This theorem requires estimates for $H(V_0)$, $H'(V_0)$ and $H''(V)$ where H' and H'' are the first and second Fréchet derivatives of H. Namely if $|H(V_0)| \leqq \eta$, $|H'^{-1}(V_0)| \leqq M$, $|H''(V)| \leqq L$, and if $M^2L\eta < \frac{1}{2}$, then $H(V) = 0$ has a solution satisfying

$$|V - V_0| \leqq (ML)^{-1}(1 - \sqrt{(1 - 2M^2L\eta)}).$$

In our case $|H(I)| < \eta(\varepsilon)$ where $\eta(\varepsilon) < M_k \varepsilon^k$ for each k. The linear operator $H'(I)$ is given by

$$H'(I)W = W - \int_{\pm t_0}^{t} \left(\exp\left(\varepsilon^{-q} \int_s^t \tilde{R}\, d\theta\right)\right)\{ZW - W \operatorname{diag} Z - \operatorname{diag} ZW\} \cdot \left(\exp\left(-\varepsilon^q \int_s^t \tilde{R}\, d\theta\right)\right) ds$$

and hence differs from the identity operator by a linear integral operator which has small norm if ε_1 is small. Hence $H'(I)^{-1}$ exists for small ε_1 and we can suppose $|H'(I)^{-1}| \leqq 2$. The quantity $|H''(V)|$ is independent of V (since our problem is quadratic) and is the norm of the bilinear function

$$H''(V)W_1W_2 = -\int^t \left(\exp\left(\varepsilon^{-q} \int_s^t \tilde{R}\, d\theta\right)\right)\{W_1 \operatorname{diag} ZW_2 + W_2 \operatorname{diag} ZW_1\} \cdot \left(\exp\left(-\varepsilon^{-q} \int_s^t \tilde{R}\, d\theta\right)\right) ds$$

which is also small if ε is small. Thus we can suppose that $|H''(V)| \leqq \frac{1}{2}$. Choosing ε_1 so small that $(2)^2 \cdot \frac{1}{2} \cdot \eta(\varepsilon) < \frac{1}{2}$, we conclude that $H(V) = 0$ has a continuous solution satisfying $|V - I| < 1 - \sqrt{(1 - 4\eta(\varepsilon))}$. This in turn implies that

$$V - I \in \varepsilon^k B_0([-t_0, t_0] \times (0, \varepsilon_1])$$

for each k. It must be emphasized that this in itself is not sufficient to conclude that $V \sim I$. However the functional equation (2.5.3) now readily implies that V satisfies this more stringent requirement involving derivative estimates.

The transformation $w = S\tilde{T}Vv$ now satisfies the requirement of the theorem and the proof is complete.

In the following corollary we employ the conventional abuse of the relation $\sim$, using it to indicate a formula which can be easily rearranged to give a precise asymptotic formula.

COROLLARY. *Under the hypotheses of Theorem* 2 *the system* (2.2.2), $\varepsilon^q w' = \underline{a} w$, *has a fundamental matrix solution* W *satisfying*

$$W \sim \left(S \sum_{k=0}^{\infty} \varepsilon^{kq} T_k\right) \cdot \exp\left(\varepsilon^{-q} \int_{-t_0}^{t} \left(\Lambda - \varepsilon^q \operatorname{diag} S^{-1}S' - \varepsilon^q \sum_{k=1}^{\infty} \operatorname{diag} S^{-1}S' \varepsilon^{kq} T_k\right) ds\right). \tag{2.5.5}$$

2.6. **Applications.** We first observe that the symmetry hypothesis, 2.3.1, is trivially verified for second order equations. We have given a detailed analysis of the problem

$$\varepsilon^{2q} y'' = a(t, \varepsilon) y$$

in [1] with particular emphasis on examples and on the computation of concrete

asymptotic formulas. In this case our analysis can be regarded as an elaboration and refinement of the theory of the well-known asymptotic formula

$$y \sim a^{-1/4} \exp\left(\varepsilon^{-q} \int_0^t a^{1/2}\, ds\right),$$

with the proviso that "a" refers here to $a(t, \varepsilon)$ and not $a(t, 0)$ (as it does in the simplest cases). The easiest problem which illustrates our methods and hypotheses is

$$\varepsilon^{2q} y'' + (t^2 g(t) + \varepsilon) y = 0, \qquad |t| \leqq t_0, \quad 0 < \varepsilon \leqq \varepsilon_0,$$

where g is positive and infinitely differentiable. We use this example to explain our ideas briefly, referring to [1] for a more elaborate illustration.

The formal analysis of §2.3 takes the following form. The discriminant of the problem is just $-4(t^2 g + \varepsilon)$ which has the formal fractional ε-power series roots

$$\hat{t} = \pm i g(0)^{-1/2} \varepsilon^{1/2} + \cdots.$$

The factorization of the formal power series of the discriminant has the form $(t^2 + g(0)^{-1}\varepsilon + \cdots)(g(0) + \cdots)$. (As we have remarked, this factorization is primarily a theoretical tool and it is not essential to compute it here.) The Newton polygon consists of the initial vertical side and a side joining the vertices (2, 0) and (0, 1). Corresponding to this latter side is the change of variable $t = \varepsilon^{1/2} s$. The truncated formal roots of the discriminant are determined by $N(\hat{t})$ of Definition 2.3.1 which satisfies

$$\begin{aligned} N(\hat{t}) &= 0, \qquad q \leqq 1 \\ &\geqq 1, \qquad q > 1. \end{aligned}$$

Thus

$$\begin{aligned} \tau(\hat{t}) &= 0, & q \leqq 1 \\ &= \pm i g(0)^{-1/2} \varepsilon^{1/2} + \cdots, & q > 1. \end{aligned}$$

If $q \leqq 1$, $t = 0$ is a turning variety according to our definition, but if $q > 1$ there are no turning varieties. However in either case $t = 0$ is a turning point in the conventional sense. In the latter case we have a uniform asymptotic formula for a fundamental pair of solutions given by

$$y \sim (t^2 g + \varepsilon)^{-1/4} \exp\left(\pm i \varepsilon^{-q} \int_0^t (s^2 g + \varepsilon)^{1/2}\, ds\right). \tag{2.6.1}$$

This formula cannot be simplified in any essential way without sacrificing its uniformity. But if we restrict the range of t, simpler conventional formulas can be derived from these. These have three forms, depending roughly on whether $t = O(1)$ or $t = O(\varepsilon^{1/2})$, and the sign of t.

As an example of a higher order equation which satisfies our hypotheses we mention the third order problem $\varepsilon^2 y''' + t^2 a(t, \varepsilon) y' + b(t, \varepsilon) y = 0$, where $a(t, 0) > 0$ and $b(0, 0) > 0$.

PART 3. PARTIALLY UNIFORM ASYMPTOTIC SOLUTIONS OF $(t\,d/dt)^2y+(t^2-\mu^2\phi)y = 0$ AS t AND μ TEND TO INFINITY

3.1. **Formal solutions.** In this part we study a problem involving two parameters. Specifically we apply the ideas of Part 1 to the problem

$$(t\,d/dt)^2y+[t^2-\mu^2\phi(1/t,\,1/\mu)]y = 0 \tag{3.1.1}$$

where we suppose that ϕ is analytic at $(0, 0)$ and $\phi(0, 0)=1$. The simplest representative of this problem is, of course, Bessel's equation of order μ. To the extent that (3.1.1) yields to our methods, there is not much difference between the general case and this special case. However, it is more instructive to analyze this generalization because it is more typical from a technical standpoint.

It is well known that problems involving several parameters fragment into subcases depending crucially on relative growth rates for the parameters. For example it is known (Iwano [8]) that Bessel's equation can be considered to give rise to five such subcases. We will obtain partially uniform asymptotic solutions which include in a single expansion three of these five cases.

The following change of variables transforms (3.1.1) into an equation which is well suited to our method. Let

$$t = \lambda x^{-1}, \qquad r(x) = -\mu^{-2/3}ty^{-1}\,dy/dt. \tag{3.1.2}$$

Then r satisfies the Ricatti equation:

$$\mu^{-2/3}xr'+r^2-\mu^{-4/3}x^{-2}[\mu^2x^2\phi(x/\lambda,\,1/\mu)-\lambda^2] = 0. \tag{3.1.3}$$

Applying the formal device which leads from (1.1.1) to (1.1.3) we obtain a preformal solution

$$\text{(a)} \qquad r = \sum_{k=0}^{\infty} r_k,$$

$$\text{(b)} \qquad r_0 = \pm\mu^{-2/3}x^{-1}h^{1/2}(x,\,\lambda,\,\mu) \quad \text{where } h = \mu^2x^2\phi-\lambda^2, \tag{3.1.4}$$

$$\text{(c)} \quad r_{k+1} = -\frac{1}{2r_0}\left(\mu^{-2/3}xr_k'+\sum_{i+j=k+1;\,i,j>0} r_ir_j\right), \qquad k > 0.$$

The turning variety of (3.1.3) is evidently the analytic variety $h(x, \lambda, \mu)=0$.

We will establish formal properties for series (3.1.4a) on a subset Ω, yet to be defined, of $|x|\leqq\lambda t_0^{-1}$, $|\lambda|\geqq\lambda_0$, $|\mu|\geqq\mu_0$, which is sufficiently far away from the turning variety. Again recalling §1.3, let $\beta(\Omega)$ be the ring of complex valued functions on Ω which are analytic functions of x on each (λ, μ) cross section of Ω. Let β_1 be the subring of bounded functions in β. Let formal convergence and asymptotic character be induced by the ideal $\zeta_1=\lambda^{-1}\beta_1+\mu^{-1}\beta_1$ in β_1. Let B_0 be the ring of functions satisfying the derivative estimates $\zeta_1^k(\mu^{-2/3}x\,d/dx)^kf\subset\beta_1$. Let $Z=\lambda^{-1}B_0+\mu^{-1}B_0$ and finally let B be the subring of $\beta(\Omega)$ consisting of functions such that

$Z^{k(f)}f \subset B_0$. We observe that in this example B is a differential ring with derivation $D = \mu^{-2/3}x\, d/dx$.

The following theorem establishes a domain on which the series $\sum_{k=0}^{\infty} r_k$ is formally convergent in the above sense.

THEOREM 3. *Let* $\Delta(\lambda, \mu) = |\lambda|^{-\rho} + |\mu|^{-\sigma}$ *where* $0 < \rho$, $0 < \sigma < \frac{2}{3}$. *Let* Ω *be a subset of* $|x| \leqq |\lambda|/t_0$, $|\lambda| \geqq \lambda_0$, $|\mu| \geqq \mu_0$, *on which*

$$|x \pm (\lambda/\mu)| \geqq |\lambda|\, |\mu|^{-5/3}\Delta^{-1}. \tag{3.1.5}$$

Then if t_0, λ_0 *and* μ_0 *are sufficiently large, the scheme* (3.1.4) *defines a formal solution of* (3.1.3).

Proof. We will use the symbol D for $\mu^{-2/3}x\, d/dx$.

1. We first establish the purely algebraic result that r_k has the form

$$r_k = (\mu^{2/3}xh^{-1/2})^{k-1}h^{-k}P_k(h, Dh, \ldots, D^{k-1}h) \tag{3.1.6}$$

where $P_k(u_1, \ldots, u_k)$ is a homogeneous polynomial of degree k with coefficients which are bounded functions of μ. Since $r_0 = \pm\mu^{-2/3}x^{-1}h^{1/2}$, $P_0 = \pm 1$, the assertion is true for $k=0$. Assuming it for $j \leqq k$ we obtain from (3.1.4c)

$$\begin{aligned} r_{k+1} &= -\tfrac{1}{2}(\mu^{2/3}xh^{-1/2})\Big\{D(\mu^{2/3}xh^{-1/2})^{k-1}h^{-k}P_k \\ &\qquad + \sum_{i+j=k+1;\, i,j>0} (\mu^{2/3}xh^{-1/2})^{i-1}h^{-i}P_i(\mu^{2/3}xh^{-1/2})^{j-1}h^{-j}P_j\Big\} \\ &= (\mu^{2/3}xh^{-1/2})^k h^{-k-1}\Big\{-\tfrac{1}{2}hDP_k + \frac{3k-1}{4}(Dh)P_k \\ &\qquad - \frac{k-1}{2}\mu^{-2/3}hP_k - \tfrac{1}{2}\sum_{i+j=k+1;\, i,j>0} P_iP_j\Big\} \end{aligned}$$

which has the required form.

2. We now suppose that t_0 and μ_0 are so large that $\phi(z_1, z_2)$ is analytic for $|z_1/u| \leqq 1/t_0$, $|z_2| \leqq 1/\mu_0$. We observe that for $j > 0$, $D^j h = \mu^{4/3}x^2 H_j(x/\lambda, \mu)$ where the H_j are given by

$$\begin{aligned} H_{j+1}(z, \mu) &= \mu^{-2/3}\{2H_j(z, \mu) + z\, \partial H_j(z, \mu)/\partial z\}, \\ H_1(z, \mu) &= 2\phi(z, 1/\mu) + z\phi_z(z, 1/\mu). \end{aligned}$$

A simple induction argument shows that $D^k H_j(x/\lambda, \mu)$ is bounded on Ω. Hence it is certainly the case that $H_j \in B_0$ and we obtain

$$D^j h \in \mu^{4/3}x^2 B_0. \tag{3.1.7}$$

Combining this with (3.1.6) and using the homogeneity of P we obtain the decisive estimate

$$r_k = (\mu^{2/3}xh^{-1/2})^{k-1}\pi_k(\mu^{4/3}x^2h^{-1}), \tag{3.1.8}$$

where π_k is a polynomial with coefficients in B_0.

3. It is now clear from (3.1.8) that to establish formal convergence of the sequence r_k, we need only show that some power of $\mu^{4/3}x^2h^{-1}$ belongs to Z.

We must locate the turning variety, that is the zeros of $h=\mu^2x^2\phi(x/\lambda, 1/\mu)-\lambda^2=0$. Let $z=\mu x/\lambda$. Then $h=0$ becomes $z^2\phi(z/\mu, 1/\mu)-1=0$, where (z, μ) ranges over $|z/u|\leqq 1/t_0$, $|\mu|\geqq 1/\mu_0$. For t_0, μ_0 sufficiently large this implies that this equation has precisely two zeros $z^{\pm}(\mu)$ of the form $z^{\pm}(\mu)=\pm 1+b_{\pm}(\mu)/\mu$ where $b_{\pm}$ is bounded. Thus the zeros of h lie in discs of the form $|x\pm\lambda/\mu|\leqq|\lambda/\mu|C/|\mu|$. The inequality (3.1.5) can be written $|x\pm\lambda/\mu|\geqq|\lambda/\mu|/|\mu|^{2/3}\Delta$. Comparing these inequalities we see that if μ_0 is sufficiently large the inequality (3.1.5) excludes the zeros of h from Ω. By the Weierstrass preparation theorem $z^2\phi(z/\mu, 1/\mu)-1$ can be factored into the form $(z-z_+(\mu))(z-z_-(\mu))\psi(z, 1/\mu)$ where $\psi(0, 0)=1$. Then

$$\mu^{4/3}x^2h^{-1} = \mu^{-2/3}x^2\left(x-\frac{\lambda}{\mu}z_+\right)^{-1}\left(x-\frac{\lambda}{\mu}z_-\right)^{-1}\psi^{-1}\left(\frac{x}{\lambda}, \frac{1}{\mu}\right).$$

But $\psi^{-1}(x/\lambda, 1/\mu)$ is easily seen to be bounded under the action of D^k if t_0 and μ_0 are sufficiently large, and hence $\psi^{-1}(x/\lambda, 1/\mu)\in B_0$. It thus suffices to show that some power of

$$Q = \mu^{-2/3}x^2(x-\lambda z_+/\mu)^{-1}(x-\lambda z_-/\mu)^{-1}$$

belongs to Z. Since for each (λ, μ) Q is analytic outside the circles $|x\pm\lambda/\mu|=|\lambda/\mu|\,|\mu|^{-2/3}\Delta^{-1}$, Q can be bounded by its bound on these circles. Recalling that $\Delta=|\mu|^{-\sigma}+|\lambda|^{-\rho}$, where $0<\sigma<\frac{2}{3}$, we see that if μ_0 is sufficiently large, then $|\mu|^{2/3}\Delta>2$. On the circles $|x\pm(\lambda/\mu)|=|\lambda/\mu|/|\mu|^{2/3}\Delta$ we have the estimates:

(i)
$$|x| \leqq 2\left|\frac{\lambda}{\mu}\right|,$$

(ii)
$$\begin{aligned}\left|x-\frac{\lambda}{\mu}z_{\pm}\right| &\geqq \left|x\pm\frac{\lambda}{\mu}\right|-\left|\frac{\lambda}{\mu}(1-z_{\pm})\right| \\ &= \left|\frac{\lambda}{\mu}\right|\left(\frac{1}{|\mu|^{2/3}\Delta}-\frac{b_{\pm}}{\mu}\right),\end{aligned}$$

which for large μ_0 implies

(iii)
$$\begin{aligned}\left|x-\frac{\lambda}{\mu}z_{\pm}\right| &\geqq \frac{1}{2}\left|\frac{\lambda}{\mu}\right|\frac{1}{|\mu|^{2/3}\Delta}, \\ \left|x-\frac{\lambda}{\mu}z_{\mp}\right| &\geqq \left|x\mp\frac{\lambda}{\mu}\right|-\left|\frac{\lambda}{\mu}(1-z_{\mp})\right| \\ &\geqq 2\left|\frac{\lambda}{\mu}\right|-\left|\frac{\lambda}{\mu}\right|\frac{1}{|\mu|^{2/3}\Delta}-\left|\frac{\lambda}{\mu}\right|\frac{b_{\pm}}{|\mu|} \\ &\geqq \left|\frac{\lambda}{\mu}\right|.\end{aligned}$$

Combining these we obtain

$$|Q| \leqq |\mu|^{-2/3}4\left|\frac{\lambda}{\mu}\right|^2\left(\frac{1}{2}\left|\frac{\lambda}{\mu}\right|\frac{1}{\mu^{2/3}\Delta}\right)^{-1}\left|\frac{\lambda}{\mu}\right|^{-1} \leqq 8\Delta.$$

Thus $\Delta^{-1}Q$ is bounded. We next show that

$$Q^{-1}DQ = \mu^{-2/3}\left(2-\frac{x}{x-\lambda z_+/\mu}-\frac{x}{x-\lambda z_-/\mu}\right)$$

belongs to Z. The functions $\mu^{-2/3}x/(x-\lambda z_\pm/\mu)$ are bounded by 4Δ, and hence generate an algebra of bounded functions over the ring of bounded functions of μ on Ω. But

$$D\frac{\mu^{-2/3}x}{x-\lambda z_\pm/\mu} = \mu^{-2/3}\frac{\mu^{-2/3}x}{x-\lambda z_\pm/\mu}-\left(\frac{\mu^{-2/3}x}{x-\lambda z_\pm/\mu}\right)^2.$$

Hence this algebra is closed under the action of D. In particular $D^k(Q^{-1}DQ)$ is bounded for each k.

Finally, induction on the identity

$$D^kQ = D^{k-1}Q(Q^{-1}DQ) = \sum_{i+j=k-1}(D^iQ)D^j(Q^{-1}DQ)$$

shows that D^kQ is bounded for each k, which implies that $Q\in\Delta B_0$. Combining this estimate with (3.1.8) we finally obtain $r_k\in\Delta^{k-1}B_0$. Since $\Delta^{2/\rho+2/\sigma}\in Z$, this establishes the formal character of the solution and the proof of the theorem is complete.

We remark that this result is true without any restrictions on the relative size of λ and μ. We also note that the inequalities $|x|\leqq|\lambda|/t_0$ and

$$\left|x\pm\frac{\lambda}{\mu}\right|\geqq\left|\frac{\lambda}{\mu}\right||\mu|^{-2/3}\Delta^{-1}$$

are compatible if μ_0 is sufficiently large.

3.2. **Asymptotic solutions.** We now establish that the formal solutions of Theorem 3 are actually asymptotic solutions of the Riccati equation (3.1.3) on certain subdomains of Ω, if the parameters t_0, λ_0, μ_0 defining Ω are sufficiently large. The analysis of these subdomains is the true remaining intrinsic difficulty to be overcome. However, in order to show their meaning, we anticipate their properties in this section and use them to complete the analytic part of our analysis.

By Proposition 4 of §1.3 we can choose a function $\tilde{r}(x,\lambda,\mu)$ such that $\tilde{r}\sim\sum_{k=0}^{\infty}r_k$. Moreover, since $r_0^{-1}\in\Delta^{-1}B_0$, this implies that $\tilde{r}$ has the form

$$\tilde{r} = r_0(1+\Delta b),\qquad b\in B_0. \tag{3.2.1}$$

We transform (3.1.3) by the change of dependent variable $r=\tilde{r}+v$, obtaining $Dv+2\tilde{r}v=\alpha-v^2$. Suppose we restrict (x,λ,μ) to a subdomain Ω' of Ω such that each cross section $\Omega'_{\lambda\mu}$ is a simply connected region having $x=0$ as boundary point, on which

$$\lim_{\xi\to 0}\exp\left(-2\mu^{2/3}\int_\xi^x\tilde{r}(s)\frac{ds}{s}\right) = 0\quad\text{as } \xi\to 0 \text{ in } \Omega'.$$

Then v will be a solution of the above differential equation if it satisfies the integral equation

$$v(x)-\mu^{2/3}\int_0^x \exp\left(-2\mu^{2/3}\int_\xi^x \tilde{r}(s)\frac{ds}{s}\right)[\alpha(\xi)-v^2(\xi)]\frac{d\xi}{\xi}=0. \tag{3.2.2}$$

Our object is to show that this equation has a solution v such that $v\sim 0$. We will again use Newton's method, beginning with first approximation $v=0$. In order to carry out such an argument we require estimates for the exponential kernel appearing in the integral equation. We note here a way in which this two parameter problem differs qualitatively from the one parameter problem of §2. In §2.5 we found that the essential behavior of the corresponding exponential kernel was determined by the first two terms of the asymptotic expansion (2.5.4), $\tilde{R}\sim\Lambda+\varepsilon R_1+\cdots$, in the sense that the remainder after these two terms made a uniformly bounded contribution to the kernel, whatever the choice of domain. In contrast, the exponent in the preceding integral equation cannot be uniformly approximated by inserting any partial sum of the asymptotic expansion of $\tilde{r}$. For the series $\mu^{2/3}\sum_{k=0}^\infty r_k$ behaves roughly like $\mu^{2/3}\sum_{k=0}^\infty \Delta^{k-1}$. But there is no k for which $\mu^{2/3}\Delta^k$ is uniformly bounded on Ω (simply keep λ fixed and let $\mu\to\infty$). Consequently we cannot duplicate the more delicate analysis of the one parameter case. Instead we must restrict the variables to a domain on which the leading term of the exponent, $-2\mu^{2/3}\int_\xi^x r_0(s)\,ds/s$, not only has a real part which is nonpositive but which is sufficiently negative to dominate the contribution of the remainder, uniformly in all three variables. These considerations lie behind the following definition, which describes domains suitable for solution of the integral equation and hence for the construction of asymptotic solutions.

DEFINITION. Let $\Omega=\Omega(\mu_0,\lambda_0,t_0)$ be the (x,λ,μ) domain appearing in Theorem 3.1. A subset Ω' of Ω is *admissible* if

(i) each cross section $\Omega'_{\lambda\mu}$ is a simply connected set having $x=0$ as a boundary point,

(ii) a branch of $(\mu^2x^2\phi-\lambda^2)^{1/2}$ is defined on $\Omega'_{\lambda\mu}$,

(iii) for each $x\in\Omega'_{\lambda\mu}$ there is a smooth *admissible curve* $x(s)$ joining 0 and x, parameterized by arc length from 0, such that

$$\begin{aligned} &\text{(a)} && |x(s)|^{-1}s < M\Delta^{-N},\\ &\text{(b)} && \operatorname{Re}\mu^{-2/3}x^{-1}(s)r_0(x(s))x'(s) \geqq C\Delta^p|\mu^{-2/3}x^{-1}(s)r_0(x(s))| \end{aligned} \tag{3.2.3}$$

where M, N, C are positive constants and $0<p<1$.

In condition (iii) it is, of course, crucial that the constants are independent of λ and μ.

The concept of admissible domains in one form or another appears in many asymptotic investigations in the complex domain and appears to be deserving of further study. An interesting contribution to this kind of question appears in

work of Evgrafov and Fedorjuk [13] who study admissible domains for problems of the form

$$w''(z)-\lambda^2 p(z)w = 0$$

where p is either a polynomial or an entire function of a certain special kind. Their object is to exploit the formula

$$w \sim p^{-1/4} \exp\left(\lambda \int_0^z p^{1/2}\, ds\right)$$

for its power to represent solutions asymptotically in λ uniformly on unbounded z-domains. They investigate the complicated domains naturally associated with this kind of asymptotics and establish certain of their properties in-the-large by general function-theoretic methods.

We are now able to state and prove the main result of Part 3.

THEOREM 4. *On an admissible domain the formal solution* (3.1.4) *of equation* (3.1.3) *is an asymptotic solution if* λ_0 *and* μ_0 *are sufficiently large.*

Proof. We express the integral equation in the form

$$v(x)-\mu^{2/3}\int_0^{s(x)} \exp\left(-2\mu^{2/3}\int_{s(\xi)}^{s(x)} \tilde{r}(x(s))\frac{x'(s)}{x(s)}\, ds\right) \cdot [\alpha(x(s))-v^2(x(s))]\frac{x'(s)}{x(s)}\, ds = 0. \tag{3.2.4}$$

We must estimate the integral

$$\int_0^{s(x)} \left|\mu^{2/3} \exp\left(-2\mu^{2/3}\int_{s(\xi)}^{s(x)} \tilde{r}(x(s))\frac{x'(s)}{x(s)}\, ds\right)\frac{x'(s)}{x(s)}\, ds\right|.$$

By (3.2.1) $\tilde{r}=r_0(1+\Delta b)$ where b is bounded. Using condition (iii) of admissibility and assuming that λ_0 and μ_0 are large, we estimate

$$\begin{aligned}
\operatorname{Re} 2\mu^{2/3}\int_{s(\xi)}^{s(x)} \tilde{r}(x(s))\frac{x'(s)}{x(s)}\, ds &\geqq \operatorname{Re} 2\mu^{2/3}\int_{s(\xi)}^{s(x)} r_0(x(s))\frac{x'(s)}{x(s)}\, ds \\
&\quad - M_1\Delta\int_{s(\xi)}^{s(x)} |\mu^{2/3} r_0(x(s))x^{-1}(s)|\, ds \\
&\geqq (2C\Delta^p - M_1\Delta)\int_{s(\xi)}^{s(x)} |\mu^{2/3} r_0(x(s))x^{-1}(s)|\, ds \\
&\geqq C\Delta^p\int_{s(\xi)}^{s(x)} |\mu^{2/3} r_0(x(s))x^{-1}(s)|\, ds \\
&\geqq C\Delta^p\int_{s(\xi)}^{s(x)} |\mu^{2/3} r_0(x(s))|\, \frac{ds}{s}.
\end{aligned}$$

But $|r_0| = |\mu^{-2/3}x^{-1}h^{1/2}| \geqq C\Delta^{-1}$. Hence

$$\operatorname{Re} 2\mu^{2/3} \int_{s(\xi)}^{s(x)} \tilde{r}\,\frac{x'(s)}{x(s)}\,ds \geqq c|\mu|^{2/3}\Delta^{-1+p} \log \frac{s(x)}{s(\xi)}.$$

We now regard the left-hand side of the integral equation (3.2.4) as a mapping $F(v)$ into itself of the Banach space of bounded analytic functions on the interior of $\Omega'_{\lambda\mu}$ endowed with the supremum norm. We again apply Kantorovič's Theorem [7] which requires estimates of the form

$$\|F'(0)^{-1}F(0)\| < \eta, \qquad \|F'(0)^{-1}F''(v)\| < L,$$

and ensures the existence of a solution of $F(v)=0$ satisfying $\|v\| \leqq 1-\sqrt{(1-2L\eta)}/L$, if $\eta L < \frac{1}{2}$. In our case $F'(0)$ is the identity mapping. Moreover

$$\begin{aligned}\|F(0)\| &= \sup_{x\in\Omega_{\lambda\mu}} \left| \int_0^{s(x)} \exp\left(-2\mu^{2/3}\int_{s(\xi)}^{s(x)} \tilde{r}(x(s))\right)\mu^{2/3}\alpha(x(s))\,\frac{x'(s)}{x(s)}\,ds \right| \\ &\leqq |\mu|^{2/3}\|\alpha\| \int_0^{s(x)} \exp\left(-c|\mu|^{2/3}\Delta^{-1+p}\log\frac{s(x)}{s(\xi)}\right)\frac{ds}{|x(s)|}.\end{aligned}$$

Since $s/|x(s)| < M\Delta^{-N}$, this implies

$$\begin{aligned}\|F(0)\| &\leqq M|\mu|^{2/3}\|\alpha\|\Delta^{-N}\int_0^{s(x)} \exp\left(-c|\mu|^{2/3}\Delta^{-1+p}\log\frac{s(x)}{s(\xi)}\right)\frac{ds}{s} \\ &\leqq (M/c)\|\alpha\|\Delta^{-N+1-p}.\end{aligned}$$

Similarly $\|F''(0)\| \leqq (M/c)\Delta^{-N+1-p}$. Hence $\eta L = (M^2/c^2)\|\alpha\|\Delta^{-2N+1-2p}$ which (since $\|\alpha\| \sim 0$) for large λ_0 and μ_0 is less than $\frac{1}{2}$. The estimate

$$\|v\| \leqq (1-\sqrt{(1-(M^2/c^2)\Delta^2\|\alpha\|)})((M/c)\Delta)^{-1}$$

then implies that $\|v\|$ is divisible by any power of Δ since $\|\alpha\|$ is. This, together with the fact that the integral equation ensures that v satisfies the necessary derivative estimates, implies that $v \sim 0$. The solution of (3.1.3), $r = \tilde{r} + v$, then satisfies the requirements of the theorem.

3.3. **Admissible domains.** Theorem 4 of the preceding section offers powerful circumstantial evidence that admissible domains are the natural domains on which to seek asymptotic solutions. In fact, the detailed geometric properties of these domains account for many deeper properties of asymptotic solutions which elude the essentially termwise formal analysis. For example, the boundaries of these domains (which are varieties of real codimension one in (x, λ, μ) space) determine the *Stokes' phenomenon* by which solutions undergo an abrupt transition from one asymptotic form to another. Such geometric considerations occur in most asymptotic investigations in the complex domain and their importance is well known. However, because the present example differs in abundance of sheer detail by an

order of magnitude from any previously studied example known to us (even though it is surely one of the simplest problems containing two parameters which is susceptible to our methods), we are not going to give a detailed description of the admissible domains and the related transition phenomena. Instead we discuss the construction of admissible domains generally, addressing ourselves to those aspects of the construction which differ qualitatively from those appearing in the parameter-free case or in the classical single parameter case (as in Wasow [3, §14.3 or §27.3]). The chief difficulty is that the admissible curves are defined by the inequalities (3.2.3) which must be uniform in λ and μ as λ and μ vary over a badly noncompact set.

As is usual, we must analyze the level curves on $\Omega_{\lambda\mu}$ of the real part of the exponent appearing in the integral equation (3.2.4), which has as the first term of its asymptotic expansion

$$H(x) = \int_x^\xi s^{-2}[\mu^2 s^2 \phi(s/\lambda, \mu) - \lambda^2]^{1/2}\, ds. \tag{3.3.1}$$

The level curves of $\operatorname{Re} H$ are also the trajectories of the differential equation $\arg x^{-2}[\mu^2 x^2 \phi - \lambda^2]^{1/2}\, dx = \arg dH = \pi/2$. We find it convenient to consider $\Omega_{\lambda\mu}$ as a portion of the two sheeted Riemann surface belonging to $[\mu^2 x^2 \phi - \lambda^2]^{1/2}$ which satisfies $|x| \leqq |\lambda|/t_0$, $|x \pm \lambda/\mu| \geqq |\lambda/\mu|\, |\mu|^{-2/3}\Delta^{-1}$. We will visualize this as two plane regions joined along cuts connecting each of the inner boundary circles to the outer boundary circle. We will refer to $\operatorname{Re} H(x, \lambda, \mu)$ as the height of x on $\Omega_{\lambda\mu}$ and will use terms such as *higher than* and *lower than* in a self-evident manner.

The first term of the asymptotic expansion of the exponent in the integral equation (3.2.4) contributes a multiplicative factor to the kernel of modulus $\exp \operatorname{Re}\{H(x) - H(\xi)\}$, where ξ lies between 0 and x on the admissible curve $x(s)$ joining 0 and x. For this factor to be uniformly bounded, it suffices that $x(s)$ be a *falling* curve joining 0 and x. Moreover, the second inequality of the admissibility condition (3.2.3) expresses precisely the condition that this curve is *uniformly falling* in the sense that the angle between $x(s)$ and the curves of steepest descent is bounded away from $-\pi/2$ and $\pi/2$ by an angle which is uniformly of the order of Δ^p, $p < 1$. We will obtain this uniformity by tying our construction to conformal properties of the curves $\arg dH = k$.

We observe that the change of variables $\tilde{x} = |\lambda/\mu| x$, $\lambda = |\lambda| e^{i\alpha}$, $\mu = |\mu| e^{i\beta}$ transforms the curves $\arg dH = k$ into

$$\begin{aligned} &\arg d\tilde{H}(\tilde{x}, \alpha, \beta, \mu) = k, \\ &\qquad \text{where } d\tilde{H} = \tilde{x}^{-2} e^{i\beta}[\tilde{x}^2 \phi(e^{-i\alpha}\tilde{x}/|\mu|, 1/\mu) - e^{2i(\alpha-\beta)}]^{1/2}\, d\tilde{x}. \end{aligned} \tag{3.3.2}$$

Thus for each k the trajectories of $\arg dH = k$ correspond to those of $\arg d\tilde{H} = k$. Moreover the curve $x(s)$ satisfies $-\pi/2 + \theta\Delta^p \leqq \arg dH \leqq \pi/2 - \theta\Delta^p$ and $|x(s)|^{-1} s$

$< M\Delta^{-N}$, if and only if the transformed curve $\tilde{x}(\tilde{s})$, again parameterized by arc length, satisfies $-\pi/2+\theta\Delta^p \leqq \arg d\tilde{H} \leqq \pi/2-\theta\Delta^p$ and $|\tilde{x}(\tilde{s})|^{-1}\tilde{s} < M\Delta^{-N}$. Thus to obtain admissible domains it suffices to construct $(\tilde{x}, \alpha, \beta, \mu)$ domains which satisfy the obviously corresponding requirements of admissibility. The advantage of this reformulation can be read in the dependence of $d\tilde{H}$ on (α, β, μ). Since $\tilde{x}e^{-i\alpha}/|\mu|$ is less than $1/t_0$ in modulus, for large t_0, μ_0 we can suppose that $\phi(\tilde{x}e^{-i\alpha}/|\mu|, 1/\mu)$ is uniformly close to $\phi(0, 0)=1$. Also the angles α and β range over compact sets so that we can reasonably expect to establish uniform properties of the curves $\arg d\tilde{H}=k$. Accordingly we take for granted that these trajectories can be determined in detail and concern ourselves mainly with the decisive qualitative aspects of the trajectories.

We next inspect these trajectories near $\tilde{x}=0$, near $\tilde{x}=\pm e^{i(\alpha-\beta)}$, and for $|\tilde{x}|\gg 1$.

Near $\tilde{x}=0$, the equation $\arg d\tilde{H}=k$ has the approximate form

$$\arg e^{i(\pi/2+\alpha-\beta-k)}\tilde{x}^{-1} = \text{const.} \tag{3.3.3}$$

The trajectories of this are the equipotential curves of a dipole at $\tilde{x}=0$ with axis along $\arg \tilde{x}=\pi/2+\alpha-\beta-k$. Similarly near $\tilde{x}=\pm e^{i(\alpha-\beta)}$ we have $\phi\approx 1$. Letting $\tilde{x}=\pm e^{i(\alpha-\beta)}+x_1$ we obtain the approximate form

$$\operatorname{Re}\left(\exp\left(i(5\beta/2-3\alpha/2-k+\pi/4\mp\pi/4)\right)x_1^{1/2}\,dx_1\right) = 0. \tag{3.3.4}$$

The origin $x_1=0$ is here a critical point of index -2, from which emanate 6 trajectories at the angles

$$\arg\{\tilde{x}\mp e^{i(\alpha-\beta)}\} = \alpha-\frac{5}{3}\beta+\frac{2}{3}k-\frac{\pi}{6}\pm\frac{\pi}{6}+\frac{2}{3}j\pi, \qquad j = 0, 1, \ldots, 5. \tag{3.3.5}$$

Finally for large $\tilde{x}$, we have approximately $\operatorname{Re} e^{i(\beta-k)}\tilde{x}^{-1}\,d\tilde{x}=0$, the trajectories of which are typically exponential spirals

$$\log|\tilde{x}|-\tan(\beta-k)\arg\tilde{x} = \text{const.} \tag{3.3.6}$$

Figure 1 exhibits the information contained in (3.3.3) to (3.3.6) for the case $k=\pi/2$, that is for the level curves of Re H, and for $\alpha\approx 0$, $\beta\approx\pi/4$. We will restrict our remaining discussion to this special range of α and β.

In the small insets of Figure 1 we ignore the metric and conformal properties of the trajectory portrait $\arg d\tilde{H}=\pi/2$ in order to bring the main topological features into evidence. We regard the main part of Figure 1 as a single sheet of a two sheeted Riemann surface on which three of the six trajectories emanating from each zero of $\tilde{x}^2\phi-e^{2i(\alpha-\beta)}$ can be seen. We observe that in each triple, two trajectories terminate at $\tilde{x}=0$, and one trajectory terminates on the outer circle. These features are indicated in the insets, which indicate two such sheets cut and joined along the trajectory $a'b'$, and cut (but not joined) along ab. In addition, we cut sheet 2 along a curve connecting the two inner circles in order to make the

resulting domain simply connected. We suppose that the square root in $d\tilde{H}$ has been chosen so that Re $\tilde{H}$ approaches $+\infty$ as $\tilde{x}$ approaches 0 from the right on sheet 1, and we indicate schematically in the insets a few more level curves of this function.

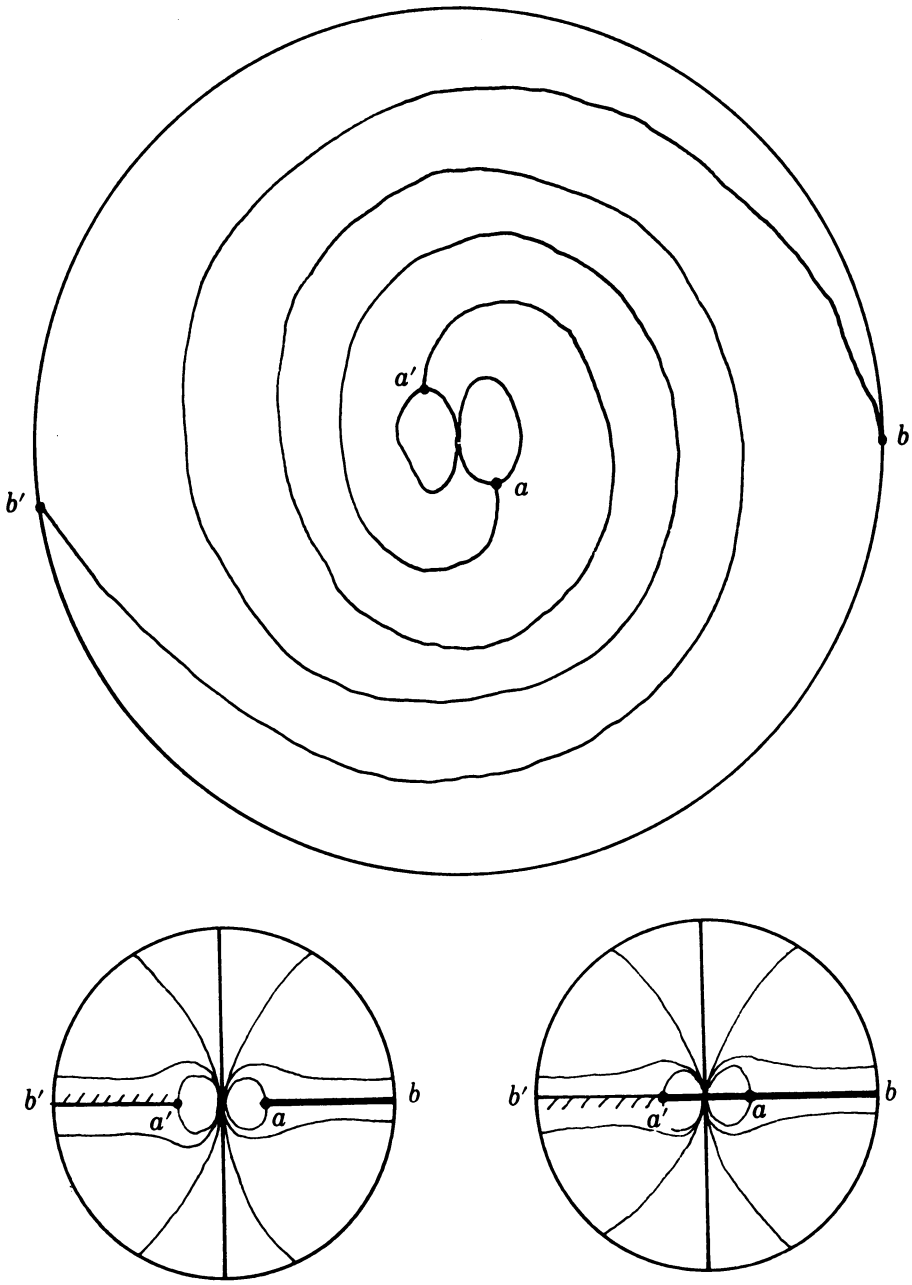

FIGURE 1. $k = \pi/2$, $\alpha \approx 0$, $\beta \approx \pi/4$

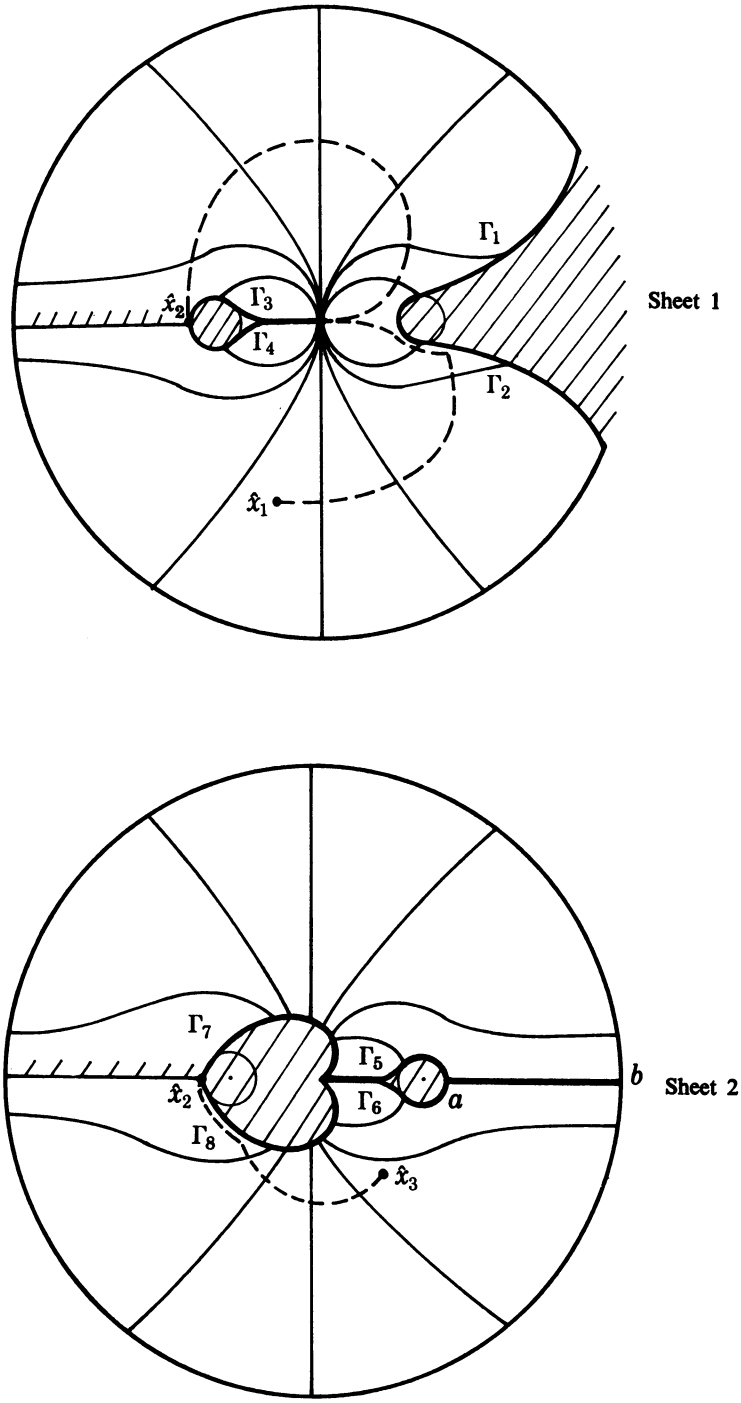
Γ_1
Γ_2
Γ_3
Γ_4
$\hat{x}_2$
$\hat{x}_1$
Sheet 1
Γ_7
Γ_5
Γ_6
Γ_8
$\hat{x}_2$
$\hat{x}_3$
a
b
Sheet 2

FIGURE 2

We now construct a subdomain (which will turn out to be admissible), illustrated by Figure 2, for which in every (α, β) cross section near the depicted section, and for every $\tilde{x}$ in this section, there is a uniformly falling curve joining 0 on sheet 1 to $\tilde{x}$. Our principle of construction is very simple. Since the domain in Figure 1 is simply connected, the height Re $\tilde{H}$ is a globally defined harmonic function. Given any $\tilde{x}$ we envisage constructing an admissible curve for $\tilde{x}$ in the following way. We attempt to construct the curve in reverse by beginning at $\tilde{x}$ and following a path of steepest ascent. This prescription clearly results in curves which are uniformly falling to $\tilde{x}$ in the strongest possible sense at interior points. However, it has two defects. First there are points, such as those near to $\tilde{x}=0$ on the left side of sheet 2, for which the path of steepest ascent leads to $\tilde{x}=0$ on sheet 2 rather than sheet 1. Also there are points such as those near the cut *ab* on sheet 1, for which the path of steepest ascent leads to a portion of the boundary on which Re H is stationary, and for which a path of steepest ascent is not even defined. We therefore modify this region so that those portions of the new boundary on which any interior curves of steepest ascent impinge are also uniformly rising curves which rise toward $\tilde{x}=0$ on sheet 1. We describe this modification in Figure 2.

We open the cut *ab* on sheet 1 by attaching to the right inner boundary circle on sheet 1 the trajectories Γ_1 of $\arg d\tilde{H}=-\pi/2+\theta\Delta^p$ and Γ_2 of $\arg d\tilde{H}=\pi/2-\theta\Delta^p$ which are tangent to the circle in the indicated sectors. (We cannot say, without imposing additional restrictions on λ, μ and Δ whether these trajectories will terminate on one of the outer circles or on the cut *ab* on sheet 2. We have illustrated termination on the outer circle of sheet 1, but fortunately our construction depends only on the fact that this termination is far from $|\tilde{x}|=1$.) The curves Γ_1 and Γ_2, together with the intermediate portion of the inner boundary circle, now form a barrier which deflects a curve of steepest ascent such as that beginning at $\tilde{x}_1$ in sheet 1 of Figure 2 into $\tilde{x}=0$ in the illustrated manner. The resulting curve of steepest ascent is only piecewise smooth, but clearly its corners can be smoothed without destroying the uniformity of ascent. We have depicted the angle between Γ_1 and Γ_2 and the level curves as fairly large in order to clarify the illustration, but it should be kept in mind that this angle is actually small of the order of Δ^p, $0<p<1$. In a similar manner we attach trajectories $\Gamma_3, \ldots, \Gamma_8$ of $\arg d\tilde{H}=\pm(\pi/2-\theta\Delta^p)$ to the remaining inner boundary circles. The curves $\Gamma_3, \ldots, \Gamma_6$, deflect curves of steepest ascent away from the nearly stationary portions of the inner boundary circles, while the curves Γ_7 and Γ_8 form a barrier which deflects curves of steepest ascent (such as $\tilde{x}_3\tilde{x}_2$ on sheet 2 of Figure 2) away from the high side of 0 on sheet 2 and toward 0 on sheet 1. In this way we obtain, for each $\tilde{x}$, a curve which descends uniformly from 0 to $\tilde{x}$. Reversing the transformation from $(\tilde{x}, \alpha, \beta)$ to (x, λ, μ) we obtain the (λ, μ) cross section of a domain containing curves which satisfy condition (3.2.3b) uniformly in $|\lambda|$, $|\mu|$.

We now consider condition (3.2.3a) which, as we will see, imposes no additional

restrictions on the domain. It is the case that all curves of steepest ascent beginning near the boundary approach the interior along the curves Γ_1 and Γ_2 which are approximately of the form (3.3.6) given by

$$\log |\tilde{x}| - \tan (\beta \pm \pi/2 \mp \theta\Delta^p) \arg \tilde{x} = \text{const.}$$

But along an exponential spiral, arc length from the origin is actually a constant multiple of $|\tilde{x}|$. Hence it appears that $\tilde{s}|\tilde{x}(s)|^{-1}$ is bounded along the admissible curves, and the corresponding curves $x(s)$ satisfy the admissibility condition (3.2.3a).

We now draw a conclusion which illustrates the significance of the previous discussion, namely that *Figure* 2 *represents a cross section of an admissible domain uniformly in* $|\lambda|$, $|\mu|$ *for* arg λ *and* arg μ *in the indicated range.*

There is one remaining, less essential, but interesting feature of the cross section which we mention. The curves Γ_1 and the cut *ab* on sheet 2 are, for large $\tilde{x}$, approximately spirals of slightly different pitch given by

$$\log \tilde{x} - \tan (\beta + \pi/2 - \theta\Delta^p) \arg \tilde{x} = \text{const}$$

and

$$\log \tilde{x} - \tan (\beta + \pi/2) \arg \tilde{x} = \text{const}$$

respectively. Their first intersection occurs when arg $\tilde{x}$ is of the order of Δ^{-p}, and hence when $\tilde{x}$ is of the order of $\exp (\tan (\pi/2 - \beta)\Delta^{-p})$. Since the outer boundary circle has radius $|\mu/t_0|$, it is the relative size of $|\mu|$ and $\exp (\tan (\pi/2 - \beta)\Delta^{-p})$ which determines the portion of the boundary on which the curves Γ_1 and Γ_2 terminate. In terms of the original variable x we thus observe that the domains of asymptotic validity have outer dimensions of the order of

$$|\lambda| \min (1, \exp (\tan (\pi/2 - \beta)\Delta^{-p})/|\mu|).$$

However, typically, we cannot draw sharper conclusions without imposing additional conditions on λ, μ and Δ.

3.4. **Asymptotic formulas. Remarks.** We now use the asymptotic solutions $r \sim \sum_{k=0}^{\infty} r_k$ of (3.1.3) to obtain formulas for solutions of (3.1.1) which we express in terms of the variable $x = \lambda/t$ in the form

$$(x\, d/dx)^2 y - x^{-2}[\mu^2 \phi(x/\lambda, 1/\mu)x^2 - \lambda^2]y = 0. \tag{3.4.1}$$

From (3.1.2) we obtain

$$\mu^{-2/3} x \frac{d}{dx} \log y \sim \sum_{k=0}^{\infty} r_k.$$

We observe that if this formula is multiplied by an unbounded function or integrated term by term with respect to x on an (x, λ, μ) domain which does not

have uniformly bounded (λ, μ) cross sections, its strict asymptotic character can be destroyed. However, we again abuse the symbol $\sim$ (as we did in formula (2.5.1)) to express slight rearrangements of strict asymptotic formulas (even though we believe that this would be injudicious in this more complex situation if we wanted to give, say, a more detailed analysis of the connection between our formulas and classical results).

If we now suppose that (x, λ, μ) range over an admissible domain Ω' and $x_0(\lambda, \mu)$ is a function such that $x_0(\lambda, \mu) \in \Omega'_{\lambda\mu}$, then there is a solution y of (3.4.1) satisfying

$$y \sim \exp \mu^{2/3} \int_{x_0}^{x} \xi^{-1} \sum_{k=0}^{\infty} r_k \, d\xi. \tag{3.4.2}$$

If we take into account the specific form of r_0 and r_1 we find that there are solutions y^* such that

$$y^* \sim x^{1/2}[x^2\phi - \lambda^2/\mu^2]^{-1/4} \exp\left\{\pm\mu \int_{x_0}^{x} \xi^{-2}\left(\xi^2\phi - \frac{\lambda^2}{\mu^2}\right)^{1/2} ds + \cdots\right\}.$$

As in the case of the asymptotic formulas (2.6.1) these formulas cannot be further simplified in any significant way without sacrificing their uniformity.

The effectiveness of this formula can be illustrated in the case of Bessel's equation. Formulas of this kind have been found for Bessel functions by special methods using integral representations and the method of saddle points in [14]. This work also contains a deep and detailed study of the transition phenomena for these asymptotic forms using the same methods. The effectiveness of this formula can be appreciated in this case by observing the way in which it includes other important asymptotic forms as special cases [14].

It should be emphasized that if we wish to consider not only the leading terms, but the entire expansion appearing in (3.4.2) we must bear in mind (see the discussion preceding Theorem 3.2) that we cannot, in general, extract a finite leading exponential part from the right-hand side of this formula. This can be regarded as a coarsening of the asymptotic character of (3.4.2) on certain portions of the parameter space (where it still gives asymptotic information). However, it should be clear that the full significance of (3.4.2), the nature of the Stokes' phenomenon, and the relation of our results to more conventional asymptotic formulas depend upon a detailed investigation of the admissible domains which we have by no means obtained.

In conclusion, we wish to claim significance for the preceding analysis mainly as an illustration of a general method. In particular we call attention to the analysis of Part 2 which makes it plain that the approach to multiple parameter asymptotics exhibited here is neither restricted to the complex domain nor to second order equations.

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