A CLASS OF COMPLETE ORTHOGONAL SEQUENCES
OF STEP FUNCTIONS

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Abstract. A class of orthogonal sets of step functions is defined and each member
is shown to be complete in $L_2(0, 1)$. Pointwise convergence theorems are obtained
for the Fourier expansions relative to these sets. The classical Haar orthogonal set
is shown to be a set of this class and the class itself is seen to be a subclass of the
"generalized Haar systems" defined recently by Price.

1. Introduction. Convergence theorems for the Fourier expansions relative to
the classical set of Haar functions have interestingly weak hypotheses. If $f \in L_1(0, 1)$
is the derivative of its indefinite integral at $x \in [0, 1]$, the Haar expansion converges
to $f(x)$ at this point, and, if $f$ is continuous on $[0, 1]$, the convergence is uniform on
this interval.

In this paper we prove that each sequence of points which is dense in $[0, 1]$
determines a complete orthonormal set of step functions whose associated Fourier
expansion has convergence properties similar to the Haar expansion. In fact, the
Haar set is a member of the class of sets defined in §2. This class, in turn, is seen
to be a subclass of the class of "generalized Haar systems" defined by Price [3].

2. Definition of the sequences $\{\theta_n\}$. Suppose that $A = \{a_1, a_2, \ldots \}$ is a sequence
of distinct points in $(0, 1)$ which is dense in $[0, 1]$ and let $\{g_n\}, n = 0, 1, 2, \ldots$, be
the set of unit step functions defined by

\[
\begin{align*}
g_0(x) &= 1 \quad \text{on } [0, 1] \\
g_n(x) &= 0, \quad x \in [0, a_n), \\
&= 1, \quad x \in [a_n, 1].
\end{align*}
\]

Since no two of these functions have discontinuities at the same point, it is clear
that the $g_i$ are linearly independent on $[0, 1]$. Consequently, one can use the
Gram-Schmidt process to construct an orthonormal sequence of functions $\{\theta_n(x)\}$

Received by the editors December 10, 1969 and, in revised form, August 31, 1970.

AMS 1969 subject classifications. Primary 4211, 4215, 4216, 4217; Secondary 4020.

Key words and phrases. Haar functions, Fourier analysis, orthonormal set of functions,
complete in $L_2(0, 1)$, pointwise convergence, generalized Haar system.

(1) The preparation of this paper was sponsored in part by a grant from the North Carolina
Engineering Foundation and in part by the Joint Services Advisory Group under Grant
AFOSR-69-1779.

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such that each \( \theta_n \) is a linear combination of the \( g_k, k \leq n \). Because of the triangular nature of this construction, each \( g_i \) can also be expressed as a linear combination of the \( \theta_k, k \leq i \).

3. **Completeness of \( \{\theta_n\} \).** To obtain convergence theorems for the Fourier expansions relative to the orthonormal sequences defined in §2 one needs a rather obvious property of the sequence \( A \) which is given in Lemma 1. In the statement of this lemma and throughout this paper the term “adjacent points” of a finite subset \( A_N \subset A \) will be used to denote successive elements of this subset when the elements are arranged in order of magnitude; i.e. \( a_m \) and \( a_n \) are adjacent points of \( A_N \) if and only if there is no \( a_k \in A_N \) such that \( a_m < a_k < a_n \) or \( a_n < a_k < a_m \).

**Lemma 1.** Let \( \{a_1, a_2, \ldots\} \) be a sequence of distinct points of \( (0, 1) \) which is dense in \( I = [0, 1] \). Then for each \( \delta > 0 \) there is an integer \( N_\delta \) such that for each \( N > N_\delta \),

(i) any pair of adjacent points \( a_m \) and \( a_n \) in the subset \( A_N = \{a_1, a_2, \ldots, a_N\} \) satisfy \( |a_m - a_n| < \delta \),

(ii) \( d(x, A_N) < \delta \) for all \( x \in I \). (\( d(x, A_N) \) is the distance from \( x \) to the set \( A_N \) defined in the usual manner.)

**Theorem 1.** The orthonormal sequence of functions \( \{\theta_n\} \) is complete in \( L_2(0, 1) \).

**Proof.** Suppose that \( f \in L_2(0, 1) \) and \( \int_0^1 f \theta_n \, dx = 0 \) for all \( n \). Then since each \( g_n \) is a linear combination of the \( \theta_n, i \leq n \),

\[
\int_0^1 f g_n \, dx = \int_0^1 f \, dx = 0 \quad \text{for all } n,
\]

and since \( A \) is dense in \([0, 1]\),

\[
\int_x^1 f \, dx = 0 \quad \text{for all } x \in [0, 1];
\]

i.e.

\[
f^{n=0} = 0 \quad \text{on } [0, 1].
\]

4. **Pointwise convergence of the Fourier \( \{\theta_n\} \) expansion.** Since the orthonormal sequence \( \{\theta_n\} \) is complete in \( L_2(0, 1) \), any function \( f \) in this space has the norm-convergent Fourier expansion

\[
f(x) \sim \sum b_n \theta_n(x), \quad \text{where } b_n = \int_0^1 f \theta_n \, dx.
\]

In this section we obtain sufficient conditions for the convergence of this expansion in the pointwise sense to \( f(x) \). The main result may be stated as follows:

**Theorem 2.** The Fourier-\( \theta_n \) expansion of a function \( f \in L_1(0, 1) \) converges to \( f(x) \) at each point \( x \in [0, 1] \) at which \( f \) is the derivative of the indefinite integral \( F \) of \( f \). This holds, in particular, (a) almost everywhere, (b) at every point of continuity of \( f \).
**Proof.** Fix $N > 0$ and consider the partial sum $S_N$ of $f$. Let $0 < a_1 < a_2 < \ldots < a_N < 1$ be the points $0, a_1, a_2, \ldots, a_N, 1$ arranged in increasing order. Since each $\theta_i, i \leq N$, is a linear combination of the $g_i, i \leq N$, $S_N$ is a step function whose intervals of constancy are $[0, a_1), [a_1, a_2), \ldots, [a_N, 1]$. We first show that for $x$ in any such interval $I$, $S_N$ is equal to the average of $f$ over $I$; i.e.

$$S_N(x,f) = \frac{1}{|I|} \int_I f \, dt \quad \text{if } x \in I.$$ 

Suppose first that $f \in L^2(0, 1)$ and let $K_0, K_1, \ldots, K_N$ denote the characteristic functions of the intervals $[0, a_1), [a_1, a_2), \ldots, [a_N, 1]$. Then

$$S_N = \sum_{0}^{N} b_n \theta_n = \sum_{0}^{N} B_n K_n,$$

where the $B$'s are constants. Clearly $B_i$ is the value $S_N$ takes in the interval associated with $K_i$. Now if $T_N$ is any linear combination of the $\theta_k, k \leq N$, it is well known that $\int_0^1 (f - T_N)^2 \, dx$ assumes its minimum value when $T_N$ is the partial sum $S_N$. Thus the $B$'s must have values which minimize the integral $\int_0^1 (f - \sum B_n K_n)^2 \, dx$ and, equating the partial derivatives with respect to the $B$'s to 0, we obtain, for each $m = 0, 1, 2, \ldots, N$,

$$\int_0^1 f K_m \, dx = B_m \int_0^1 K_m^2 \, dx.$$ 

If $f$ is merely in $L^1(0, 1)$, there is a sequence $\{f_k\}$ of functions in $L^2(0, 1)$ such that $\|f - f_k\|_1 \to 0$. It is immediate that each Fourier coefficient of $f$ is the limit of the corresponding coefficient of $f_k$ and we have, for $x \in I$,

$$S_N(x,f) = \lim_{k} S_N(x,f_k) = \lim_{k} \frac{1}{|I|} \int_I f_k \, dx = \frac{1}{|I|} \int_I f \, dx.$$ 

Now if $x_0$ is in the interval $I = [a, b)$ where $a$ and $b$ are adjacent points of the subset $A_N$,

$$|f(x_0) - S_N(x_0,f)| = \left| f(x_0) - \frac{1}{|I|} \int_I f \, dx \right|$$

$$= \left| \frac{x_0 - b}{|I|} \left[ f(x_0) - \frac{1}{x_0 - a} \int_a^{x_0} f \, dx \right] + \frac{b - x_0}{|I|} \left[ f(x_0) - \frac{1}{b - x_0} \int_x^{b} f \, dx \right] \right|.$$ 

Since by Lemma 1 the length of $I$ can be made arbitrarily small by taking $N$ sufficiently large, the last equation implies that the Fourier expansion of $f$ converges to $f(x_0)$ at any point $x_0 \in [0, 1]$ at which $f$ is the derivative of its indefinite integral.

The expression for $S_N$ obtained in the preceding proof and a theorem from [4, p. 32] can be used to establish the following corollaries to Theorem 2.

**Corollary 1.** If $f \in L^p(0, 1)$, then $\|f - S_N\|_p \to 0$ $(1 < p < \infty)$. 

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Corollary 2. If \( f \in L^p(0, 1) \), then \( \|\text{Sup}_N S_N\| \leq A_p \| f \|_p \) \((1 < p \leq \infty)\) where \( A_p \) is constant for each \( p \).

Theorem 3. The Fourier-\( \theta_n \) expansion of a function \( f \) which is continuous on \([0, 1]\) converges uniformly to \( f(x) \) on \([0, 1]\).

Proof. If \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( |f(x_1) - f(x_2)| < \varepsilon \) when \( x_1, x_2 \in [0, 1] \) and \( |x_1 - x_2| < \delta \). By Lemma 1 an integer \( N_\delta \) exists such that if \( N > N_\delta \), the set \( A_N \) determines a partition of \([0, 1]\) into subintervals of length less than \( \delta \). Thus if \( x_0 \) is in any subinterval \( I = [a, b] \) of this partition we have

\[
\left| f(x_0) - S_N(x_0, f) \right| = \left| f(x_0) - \frac{1}{|I|} \int_I f(x) \, dx \right| = |f(x_0) - f(\xi)| \quad \text{where } \xi \in [a, b]
\]

\[
< \varepsilon.
\]

5. Behavior of the Fourier-\( \theta_n \) expansion at a point of discontinuity of \( f \). Suppose \( f \) has an isolated finite discontinuity at a point \( a_i \) of the set \( A \). Since the step function \( g_i \) has a unit jump at \( a_i \), a function \( G \) which is continuous at \( a_i \) can be constructed by adding a constant multiple of \( g_i \) to \( f \). By Theorem 2, the Fourier expansion of \( G \) converges to \( G(a_i) \) and since the \( \theta_n \) expansion of \( g_i \) is a finite sum, it follows that the Fourier expansion of \( f \) must converge at \( a_i \). In fact, one can readily prove the following

Theorem 4. If \( f \in L^2(0, 1) \) has a finite discontinuity at a point \( a_i \) of the sequence \( A \) (which determines \( \{\theta_n\} \)) the Fourier-\( \theta_n \) expansion of \( f \) converges to \( f(a_i^+) \) at this point.

The fact that the expansion converges to \( f(a_i^+) \) rather than \( \frac{1}{2}[f(a_i^+) + f(a_i^-)] \) or some other value between \( f(a_i^+) \) and \( f(a_i^-) \) is not significant since this value is determined solely by the definition of \( g_i(a_i) \).

No general statement can be made concerning the convergence of (1) at a point of discontinuity of \( f \) that is not in \( A \). To show this we consider two expansions corresponding to different \( A \) sequences (hence to different \( \theta_n \) sets) of a simple step function with a discontinuity at a point \( c \in (0, 1) \) which is not in \( A \). Let

\[
g_c(x) = \begin{cases} 0, & x \in [0, c), \\ 1, & x \in [c, 1], \end{cases}
\]

where \( c \notin A \). Suppose that \( N \) is an integer sufficiently large for the subset \( A_N \) to contain points in both \((0, c)\) and \((c, 1)\) and let \( S_N \) denote the partial sum of the Fourier expansion of \( g_c \). If \( a_i(N) \) and \( a_r(N) \) are the points of \( A_N \) adjacent to \( c \) on the left and right, respectively, we find

\[
S_N(c, g_c) = \frac{1}{a_r - a_l} \int_{a_l}^{a_r} g_c \, dx = \frac{a_i(N) - c}{a_r(N) - a_l(N)}.
\]
Equation (2) suggests that the convergence of the Fourier expansion at the point \( c \) depends on the sequence \( A \). To see that this is actually the case, construct a sequence \( A \), by choosing \( a_1 = c/2 \) and \( a_2 = (c+1)/2 \), the respective midpoints of \( (0, c) \) and \( (c, 1) \). These two points along with \( c \) determine four successive subintervals of \( (0, 1) \); let their midpoints (from left to right) be \( a_3, a_4, a_5, a_6 \). Continue this subdivision process (with \( 2^n \) new midpoints at the \( n \)-th stage) to obtain \( A \). In (2), \( a_N(N) - c \) is the distance from \( c \) to the closest point of \( A_N \) on the right and has the form \( (1-c)/2^k \) for some integer \( k \). On the other hand, the distance from \( c \) to the closest point of \( A_N \) on the left will either be \( c/2^k \) or \( c/2^k + 1 \) depending on \( N \). In the first case (2) gives \( S_N = 1 - c \) and in the second \( S_N = 2(1-c)/(2-c) \). Since these expressions are independent of \( N \), we see that the sequence \( \{ S_n(c, g_c) \} \) consists of two distinct constant subsequences and cannot converge.

As a second example we construct the sequence \( A \) as follows: \( a_1 = c/2, \) \( a_2 = (1+c)/2, \) \( a_3 = c/3, \) \( a_4 = 2c/3, \) \( a_5 = (1 + 2c)/3, \) \( a_6 = (2 + c)/3, \ldots \); i.e. the elements of \( A \) are the distinct points which divide the intervals \( (0, c) \) and \( (c, 1) \) into two equal parts, three equal parts, four equal parts, etc. For this sequence, the distance from \( c \) to the closest point of \( A_N \) on the right has the form \( (1-c)/k(N) \) where \( k(N) \) is an integer that depends on \( N \); and the distance from \( c \) to \( a_N(N) \) is either \( c/k(N) \) or \( c/[k(N) + 1] \). In the first case (2) gives \( S_N = 1 - c \) and in the second

\[
S_N = \frac{1-c}{1-c/(k(N)+1)}.
\]

Since \( k(N) \) approaches infinity with \( N \), the limit of the subsequence given by the second equation is also \( 1 - c \). Thus \( \lim S_n(c, g_c) \) exists for this sequence \( A \); i.e. the Fourier expansion of \( g_c \) determined by this particular sequence converges at \( c \) to \( 1 - c \).

6. The structure of \( \{ \theta_n \} \). The orthonormal sequence \( \{ \theta_n \} \) defined in §2 is obtained by applying the Gram-Schmidt orthogonalization process to the linearly independent sequence \( \{ g_i \} \). Since this process gives \( \theta_n \) as a linear combination of the \( g_i, i \leq n \), it is clear that \( \theta_n \) is a step function which is constant on the subintervals \( [0, a_1), [a_1, a_2), \ldots, [a_n, 1) \) of \( [0, 1) \) determined by the successive points of \( A_n \). We shall now see that a precise expression for \( \theta_n \) in terms of the points of the subset \( A_n \) can be obtained by induction.

Let \( \theta_0 = g_0 \) and assume that \( \theta_{n-1}, n \geq 1, \) has been determined. Suppose \( a_n \) falls in the interval \( (a, b) \) whose endpoints are successive points of the partition of \( [0, 1) \) determined by \( A_{n-1} \), and let \( f_n \) be given by

\[
f_n(x) = \begin{cases} 
1/(a_n-a), & x \in [a, a_n), \\
-1/(b-a_n), & x \in [a_n, b), \\
0, & \text{otherwise}.
\end{cases}
\]
Since \( a \) and \( b \) are either in \( A_{n-1} \) or are endpoints of \([0,1]\), it is obvious that \( f_n \) is a linear combination of the \( g_i, \ i \leq n \). Furthermore if \( k < n \), \( \theta_k \) is constant on \((a, b)\) so

\[
(\theta_k, f_n) = \int_a^b \theta_k f_n \, dx = c \int_a^b f_n \, dx = 0
\]

and we see that \( f_n \) is orthogonal to each \( \theta_k, k < n \). It follows, of course, that the function \( \theta_n \) given by the Gram-Schmidt process must be \( \pm f_n/\|f_n\| \).

7. Examples and remarks. (A) The set of rationals in \((0,1)\) is countable and different enumerations of this set lead to an infinity of sequences of the type \( A \) described in §2. For example, one can take \( a_1 = 1/2, \ a_2 = 1/3, \ a_3 = 2/3, \ a_4 = 1/4, \ a_5 = 3/4, \ldots \) (where the irreducible fractions with denominator \( 2, 3, 4, \ldots \) are used successively in blocks) and the corresponding \( \{g_i\} \) will be the set of all unit step functions with jumps at the rational points of \((0,1)\). The orthonormal set \( \{\theta_n\} \), in this case, consists of step function with discontinuities at the rationals.

(B) If \( p \) is any prime, the set of all numbers of the form \( k/p^n \) where \( m \) and \( k \) are integers, \( k < p^n \) and \( k \not\equiv 0 \pmod{p} \), is countable since it is a subset of the rationals and is obviously dense in \( I \). Any such set with an appropriate specific enumeration could therefore be used for the sequence \( A \). In fact, if \( A \) is a particular sequence of this type with \( p = 2 \), the corresponding \( g_i \) can be orthonormalized to obtain the familiar Haar functions \( \{\chi_i\} \). Thus if \( a_1 = 1/2, \ a_2 = 1/4, \ a_3 = 3/4, \ a_4 = 1/8, \ a_5 = 3/8, \ldots, \ a_n = (2n+1-2^k)/2^k, \ldots, \) where \( k \) is the smallest integer such that \( 2^k > n \), the corresponding \( \theta_i \) are the classical Haar functions.

Price [3] has defined a class of orthonormal sets of step functions of a more general type. Price’s definition is not sufficiently restrictive to imply the completeness of his sets. It is readily verified that the class of sequences \( \{\theta_n\} \) defined in this paper is a subclass of Price’s and that our requirement that \( A \) be dense in \([0,1]\) (which insures completeness) is the essential difference.

Franklin [1] constructed a complete orthonormal sequence of linear functions related to the Haar functions. It can be readily generalized. Let \( \{g_n\} \) be defined as in §2 and construct a sequence \( \{h_n\} \) on \([0,1]\) with

\[
h_0(x) \equiv 1 \quad \text{and} \quad h_n(x) = \int_0^x g_{n-1}(t) \, dt, \quad n \geq 1.
\]

The sequence \( \{h_n\} \) is linearly independent on \([0,1]\) and the Gram-Schmidt process yields a complete orthonormal sequence of continuous polygonal functions.

8. Acknowledgment. The authors express their gratitude to the referee of the first draft of this paper. His constructive comments provided a stronger and more general Theorem 2.
References


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