A PAIRING OF A CLASS OF EVOLUTION SYSTEMS WITH A CLASS OF GENERATORS

BY
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Abstract. Suppose that $S$ is a Banach space and that $A$ and $M$ are functions such that if $x$ and $y$ are numbers, $x \geq y$, and $P$ is in $S$ then each of $M(x, y)P$ and $A(y, P)$ is in $S$. This paper studies the relation

$$M(x, y)P = P + \int_x^y A(t, M(t, y)P) \, dt.$$  

Classes $OM$ and $OA$ will be described and a correspondence will be established which pairs members of the two classes which are connected as $M$ and $A$ are by the relation indicated above.

Suppose that $S$ is a Banach space, $A$ is a function such that, if $t$ is a number, then $A(t, \cdot)$ has domain all of $S$ and values in $S$, and $M$ is a function such that, if $x \geq z$, then $M(x, z)$ is a function from $S$ to $S$ satisfying

$$(E) \quad M(x, y)M(y, z)P = M(x, z)P$$

for all $y$ between $x$ and $z$ and all $P$ in $S$. This paper is a study of the relation

$$M(x, y)P = P + \int_x^y A(t, M(t, y)P) \, dt \tag{1}$$

between $A$ and $M$.

In [9] and [10], J. S. Mac Nerney defines classe $OM$ and $OA$ and a one-to-one correspondence $\mathcal{E}$ from $OA$ onto $OM$. Members $M$ of $OM$ have the evolution property $(E)$ and members $V$ of $OA$ have the property that

$$V(x, y)P + V(y, z)P = V(x, z)P$$

for all $y$ between $x$ and $z$ and all $P$ in $S$. The function $\mathcal{E}$ associates members $M$ and $V$ of $OM$ and $OA$ which are related by an equation similar to (1). Important in those papers, but not to be considered here, is the possibility of discontinuities of the solutions $M(\cdot, y)P$. The author's study in [4] and [5] of these discontinuities leads into the analysis in this paper.

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The development of Mac Nerney's (see also [13]) includes the case that, for \( a > b \), the function \( M(a, b) \) is "generated" by a Lipschitz function \( A(t, \cdot) \) from \( S \) to \( S \) and \( \lim_{a \to b} |M(a, b) – 1|P – |M(a, b) – 1|Q| = 0 \). In case \( A(t, \cdot) \) is (not necessarily Lipschitz) continuous and \( \lim_{a \to b} |M(a, b)P – P| = 0 \) and, especially in case \( M \) arises from a one-parameter semigroup of nonlinear functions, many investigations have been made. Some of these are [1], [2], [3], [6], [8], [11], [12], [14], [15], [16], [17], [18], and [19]. However, in none of these more recent papers has the complete pairing of the solutions with their generators been made as provided by \( \eta \) in [9] and [10]. This paper will provide an extension of the function \( \eta \) to a nonlinear analogue of the linear, strong case (see [7, §11.5]).

1. **The main result.** The class \( O_A \) will consist of all functions \( V \) having the property that if \( a \geq b \) then \( V(a, b) \) is a function from \( S \) to \( S \) and

1A. there is a continuous function \( \rho \) which is of bounded variation on each finite interval such that if \( a \geq b \) and \( \rho(a) – \rho(b) < 1 \) then \( 1 – V(a, b) \) has range all of \( S \) and, if \( P \) and \( Q \) are in \( S \), then

\[
\{|1 – [\rho(a) – \rho(b)]|P – Q| \leq |1 – V(a, b)|P – [1 – V(a, b)]Q|,
\]

2A. if \( x \geq y \geq z \) and \( P \) is in \( S \) then \( V(x, y)P + V(y, z)P = V(x, z)P \),

3A. if \( a > b \) and \( B \) is a bounded subset of \( S \) then there is a nondecreasing, continuous function \( \alpha \) such that if \( a \geq x \geq y \geq b \) and \( P \) is in \( B \) then \( |V(x, y)P| \leq \alpha(x) – \alpha(y) \), and

4A. if \( a > b \) then there is a nondecreasing function \( \beta \) such that if \( \epsilon > 0 \) and \( P \) is in \( S \) then \( |Q – P| < \delta \) and \( a \geq x \geq y \geq b \) then \( |V(x, y)P – V(x, y)Q| \leq [\beta(x) – \beta(y)]\epsilon \).

The class \( O_M \) will consist of all functions \( M \) having the property that if \( a \geq b \) then \( M(a, b) \) is a function from \( S \) to \( S \) and

1M. there is a continuous function \( \rho \) which is of bounded variation on each finite interval such that if \( a \geq b \) and \( P \) and \( Q \) are in \( S \) then

\[
|M(a, b)P – M(a, b)Q| \leq \exp(\rho(a) – \rho(b)|P – Q|,
\]

2M. if \( x \geq y \geq z \) and \( P \) is in \( S \) then \( M(x, y)M(y, z)P = M(x, z)P \),

3M. if \( a > b \) and \( B \) is a bounded subset of \( S \) then there is a nondecreasing, continuous function \( \alpha \) such that if \( a \geq x \geq y \geq b \) and \( P \) is in \( B \) then \( |M(x, y)P – P| \leq \alpha(x) – \alpha(y) \), and

4M. if \( a > b \) then there is a nondecreasing function \( \beta \) such that if \( \epsilon > 0 \) and \( P \) is in \( S \) then \( |Q – P| < \delta \) and \( a \geq x \geq y \geq b \) such that \( x – y < d \) then

\[
|[M(x, y) – 1]P – [M(x, y) – 1]Q| \leq [\beta(x) – \beta(y)]\epsilon.
\]

The main result of this paper is the following:

**Theorem.** There is a reversible function \( \eta \) from \( O_A \) onto \( O_M \) such that if \( V \) is \( O_A \) and \( M \) is in \( O_M \) then these are equivalent:

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A PAIRING OF A CLASS OF EVOLUTION SYSTEMS

(a) \( M = \mathcal{E}(V) \),
(b) if \( a \geq b \) and \( P \) is in \( S \) then \( M(a, b)P = \sum_{n=1}^{b} [1 - V]^{-1}P \),
(c) if \( a \geq b \) and \( P \) is in \( S \) then \( V(a, b)P = \sum_{n=1}^{b} [M - 1]P \), and
(d) if \( a \geq b \) and \( P \) is in \( S \) then \( M(a, b)P = P + \int_{a}^{b} V[M(\cdot), b]P \).

**Remark.** If \( a > b \) then a subdivision \( \{s_{j}\} \) of \( \{a, b\} \) is a decreasing sequence such that \( s(0) = a \) and \( s(n) = b \). Also, \( t \) is a refinement of the subdivision \( s \) provided that \( t \) is a subdivision of \( \{a, b\} \) and \( s \) is a subsequence of \( t \). The continuously continued product and sum in (b) and (c) above are defined in [9] and [10]; the integral in (d) is the Riemann-Stieltjes integral.

2. From \( OA \) to \( OM \). In this section, suppose that \( V \) is in \( OA \) and \( \rho \) is as in condition 1A.

**Lemma 2.0.** Suppose that \( a > b \), \( P \) is in \( S \), \( \{s_{j}\} \) is a subdivision of \( \{a, b\} \) such that if \( p \) is an integer in \([1, n] \) then \( \int_{s(p-1)}^{s(p)} |d\rho| \leq \frac{1}{2} \), and \( j \) is an integer in \([1, n] \). Then

\[
\prod_{j=1}^{\infty} \left( 1 - V(s_{p-1}, s_{p}) \right)^{-1}P = P + \sum_{n=1}^{\infty} V(s_{p-1}, s_{p}) \prod_{i=p}^{\infty} \left( 1 - V(s_{i-1}, s_{i}) \right)^{-1}P
\]

and

\[
\left| \prod_{j=1}^{\infty} \left( 1 - V(s_{p-1}, s_{p}) \right)^{-1}P - P \right| \leq \exp \left( 2 \int_{b}^{a} |d\rho| \right) \sum_{n=1}^{\infty} |V(s_{p-1}, s_{p})P|.
\]

**Indication of proof.** With the supposition of the lemma,

\[
\prod_{j=1}^{\infty} \left( 1 - V(s_{p-1}, s_{p}) \right)^{-1}P - P = \sum_{n=1}^{\infty} \left\{ \prod_{i=p}^{\infty} \left( 1 - V(s_{i-1}, s_{i}) \right)^{-1} - 1 \right\} \prod_{i=p+1}^{\infty} \left( 1 - V(s_{i-1}, s_{i}) \right)^{-1}P
\]

Also,

\[
\left| \prod_{j=1}^{\infty} \left( 1 - V(s_{p-1}, s_{p}) \right)^{-1}P - P \right|
\]

\[
= \left| \sum_{n=1}^{\infty} \left\{ \prod_{i=p}^{\infty} \left( 1 - V(s_{i-1}, s_{i}) \right)^{-1}P - \prod_{i=p}^{n-1} \left( 1 - V(s_{i-1}, s_{i}) \right)^{-1}P \right\} \right| \\
\leq \sum_{n=1}^{\infty} \left| \prod_{i=p}^{\infty} \left[ 1 - (\rho(s_{i-1}) - \rho(s_{i})) \right]^{-1} \right| |V(s_{p-1}, s_{p})P|
\]

\[
\leq \exp \left( 2 \int_{b}^{a} |d\rho| \right) \sum_{n=1}^{\infty} |V(s_{p-1}, s_{p})P|.
\]

This last inequality follows since if \( 0 \leq z \leq \frac{1}{2} \) then \( [1 - z]^{-1} \leq 1 + 2z \leq \exp(2z) \).

**Lemma 2.1.** Suppose that \( a > b \), \( \beta \) is as in condition 4A, \( \{R_{\rho}\} \) is a Cauchy sequence with values in \( S \), and \( \epsilon > 0 \). There is a positive number \( \delta \) having the property
that if \( n \) is a positive integer, \( P \) is in \( S \) such that \( |P - R_n| < \delta \), and \( \alpha \geq x \geq y \geq b \) then
\[
|V(x, y)P - V(x, y)R_n| \leq |\beta(x) - \beta(y)| \epsilon.
\]

**Indication of proof.** A proof may be constructed similar to the usual proofs that continuous functions on closed and (sequentially) compact sets are uniformly continuous.

**Remark.** The construction in the proof of the next lemma is similar to that of [6, Lemma 3].

**Lemma 2.2.** Suppose that \( a > b \), \( \beta \) is as in condition 4A, \( \epsilon > 0 \), and \( P \) is in \( S \). There is a subdivision \( \{s_p\}^m_0 \) of \( \{a, b\} \) such that if \( k \) is an integer in \( \{1, m\} \), \( \{t_p\}^n_0 \) is a subdivision of \( \{s_k - 1, s_k\} \), \( j \) is an integer in \( \{1, n\} \), and \( \alpha \geq x \geq y \geq b \) then \( \int_{e(k-1)}^{e(k)} |dp| \leq \frac{1}{2} \) and
\[
|V(x, y) \prod_{p=j}^{n} [1 - V(t_{p-1}, t_p)]^{-1} \prod_{q=k+1}^{m} [1 - V(s_{q-1}, s_q)]^{-1} P - V(x, y) \prod_{q=k+1}^{m} [1 - V(s_{q-1}, s_q)]^{-1} P| \leq |\beta(x) - \beta(y)| \epsilon.
\]

**Indication of proof.** With the supposition of the lemma, let \( \Delta \) be a function from \( S \) to the positive real numbers such that if \( Q \) is in \( S \) then \( \Delta(Q) \) is the largest number \( \delta \) not exceeding \( 1 \) and having the property that if \( R \) is in \( S \), \( |R - Q| < \delta \), and \( \alpha \geq x \geq y \geq b \) then \( |V(x, y)Q - V(x, y)R| \leq |\beta(x) - \beta(y)| \epsilon \). Let \( D \) be a function such that if \( a > z \geq b \) and \( Q \) is in \( S \) then \( D(z, Q) \) is the largest number \( x \) not exceeding \( a \) and having the property that if \( x \geq y \geq z \) and \( t \) is a subdivision of \( \{y, z\} \) then \( \sum \left| VQ \right| \leq \Delta(Q) \text{exp} \left( 2 \int_{b}^{a} |dp| \right) \) and \( \int_{b}^{a} |dp| \leq \frac{1}{2} \). Let \( u \) be a sequence defined by \( u(0) = b \) and, if \( n \) is a nonnegative integer, then
\[
u(n + 1) = D\left(u(n), \prod_{q=1}^{n} [1 - V(u_{n-q+1}, u_{n-q})]^{-1} P\right).
\]

Suppose that \( u \) is infinite. The sequence \( u \) is increasing and the sequence \( R \) defined by \( R(j) = \prod_{q=1}^{j} [1 - V(u_{j-q+1}, u_{j-q})]^{-1} P, \ j = 1, 2, 3, \ldots \), converges. To see this latter: by Lemma 2.0, there is a bounded set \( B \) which contains \( P \) and the values of \( R \). Let \( \alpha \) be a nondecreasing, continuous function such that if \( a \geq x \geq y \geq b \) and \( Q \) is in \( B \) then \( |V(x, y)Q| \leq \alpha(x) - \alpha(y) \). Let \( m \) and \( n \) be positive integers.

\[
|R_m - R_n| = \left( \prod_{j=1}^{n} [1 - V(u_{m-j+1}, u_{m-j})]^{-1} P \right) \prod_{p=1}^{n} [1 - V(u_{n-p+1}, u_{n-p})]^{-1} P \leq \text{exp} \left( 2 \int_{b}^{a} |dp| \right) \sum_{j=1}^{n} \left| V(u_{m-j+1}, u_{m-j}) \prod_{p=1}^{m-n} [1 - V(u_{n-p+1}, u_{n-p})]^{-1} P \right| \leq \text{exp} \left( 2 \int_{b}^{a} |dp| \right) [\alpha(u_m) - \alpha(u_n)].
\]

The convergence of \( R \) now follows from the continuity of \( \alpha \). By Lemma 2.1, there is a positive number \( \delta \) such that if \( n \) is a positive integer then \( \Delta(R_n) \geq \delta \). By the
uniform continuity of $\alpha$ on $[b, a]$, there is a number $d$ such that if $a \geq x \geq y \geq b$ and $x - y < d$ then $\alpha(x) - \alpha(y) < \delta/\exp\left(2 \int_{a}^{b} |dp|\right)$ and so, if $n$ is an integer, then $D(y, R_n) > x$. This contradicts the assumption that $u$ is infinite. Let $m$ be the least integer such that $u(m) = a$ and define $s(p)$ to be $u(m - p)$ for $p = 0, 1, 2, \ldots, m$. If $k$ is an integer in $[1, m]$, then $\prod_{q=k+1}^{n} \left[1 - V(s_{q-1}, s_q)\right]^{-1} P = R_{m-k}$ and

$$\left| \prod_{p=j}^{n} \left[1 - V(t_{p-1}, t_p)\right]^{-1} R_{m-k} - R_{m-k} \right| \leq \exp \left(2 \int_{b}^{a} |dp| \right) \sum_{p=j}^{n} \left| V(t_{p-1}, t_p) R_{m-k} \right| \leq \Delta(R_{m-k}).$$

Hence the conclusion of the lemma.

Remark. The proof of the following theorem is similar to the proof of Theorem 1 of [19].

**Theorem 2.1.** If $a > b$, $P$ is in $S$, $\beta$ is as in condition 4A and $\varepsilon > 0$ then there is a subdivision $\{s_p\}_0^n$ of $(a, b)$ such that if $p$ is an integer in $[1, m]$ then $\int_{s(p-1)}^{s(p)} |dp| \leq \frac{1}{\varepsilon}$ and if $t$ is a refinement of $s$ then

$$\left| \prod_{p=0}^{n} \left[1 - V(t_{p-1}, t_p)\right]^{-1} R_{m-k} - R_{m-k} \right| \leq \exp \left(2 \int_{b}^{a} |dp| \right) \left[\beta(a) - \beta(b)\right] \varepsilon.$$

**Indication of proof.** Suppose that $\{s_p\}_0^n$ is a subdivision of $(a, b)$ as indicated in Lemma 2.2. If $k$ is an integer in $[1, m]$, then $\{t_p\}_0^n$ is a subdivision of $\{s_{k-1}, s_k\}$, $j$ is an integer in $[1, n]$, and $a \geq x \geq y \geq b$, then

$$\left| V(x, y) \prod_{p=j}^{n} \left[1 - V(t_{p-1}, t_p)\right]^{-1} \prod_{q=k+1}^{m} \left[1 - V(s_{q-1}, s_q)\right]^{-1} P \right| \leq 2 |\beta(x) - \beta(y)| \varepsilon.$$

Let $\{t_p\}_0^n$ be a refinement of $s$, $u$ be an increasing sequence such that $u(0) = 0$ and $t(u(p)) = s(p)$, and define $K$ to be the sequence given by

$$K_p = \prod_{i=1}^{u(p)} \left[1 - V(t_{q-1}, t_q)\right]^{-1}.$$

Then

$$\left| \prod_{q=1}^{n} \left[1 - V(t_{q-1}, t_q)\right]^{-1} P - \prod_{p=1}^{m} \left[1 - V(s_{p-1}, s_p)\right]^{-1} P \right|$$

$$= \left| \prod_{p=1}^{m} K_p P - \prod_{p=1}^{m} \left[1 - V(s_{p-1}, s_p)\right]^{-1} P \right|$$

$$= \left| \sum_{p=1}^{m} \left( \prod_{i=1}^{p} K_i \prod_{q=p+1}^{m} \left[1 - V(s_{q-1}, s_q)\right]^{-1} P - \prod_{i=1}^{p-1} K_i \prod_{q=p}^{m} \left[1 - V(s_{q-1}, s_q)\right]^{-1} P \right) \right|$$

$$\leq \sum_{p=1}^{m} \prod_{i=1}^{u(p-1)} \left(1 - \rho(t_{i-1}) - \rho(t_i)\right)^{-1}.$$
Theorem 2.2. If \( M(x, y)P = xY[y \cdot 1 - V]^{-1}P \) for all \( x \leq y \) and \( P \) in \( S \) then \( M \) is in \( OM \).

Indication of proof. For property \( IM \), suppose that \( a > b \), \( P \) and \( Q \) are in \( S \), and \( \{sp\}^m \) is a subdivision of \( \{a, b\} \) such that \( p \) is an integer in \( [1, m] \) and \( p(sp - j) - p(sp) < 1 \). Then

\[
\leq \exp \left( 2 \int_b^a |d\rho| \right) \sum_{p=1}^m \left| K_p \prod_{p+1}^m \left( 1 - \rho(s_{p-1}) - \rho(s_p) \right) \sum_{t=1}^{u(p+1)} \left( 1 - \rho(t_{p-1}) - \rho(t_p) \right) \right| \cdot \prod_{j=p+1}^m \left( 1 - V(s_{j-1}, s_j) \right)^{-1}P
\]

Finally, \( \exp(2|d\rho|) = \exp(\rho(a) - \rho(b)) \). For property \( 2M \), suppose that \( x \geq y \geq z \) and \( P \) is in \( S \). Let \( u \) be a subdivision of \( \{x, z\} \) for which there is an integer \( j \) such that \( u(j) = y \) and such that if \( v \) refines \( u \) then \( |M(x, z)P - \prod_v [1 - V]^{-1}P| < \varepsilon \). Let \( \{t_p\}_y \) be a subdivision of \( \{y, z\} \) such that if \( v(p) = u(p) \) for \( 0 \leq p \leq j \) and \( v(p) = t(p - j) \) for \( j \leq p \leq j + n \) then \( v \) is a refinement of \( u \) and \( |M(y, z)P - \prod_t [1 - V]^{-1}P| < \varepsilon / \exp(\rho(x) - \rho(y)) \). Let \( \{s_p\}_x \) be a subdivision of \( \{x, y\} \) such that if \( v(p) = s(p) \) for \( 0 \leq p \leq m \) and \( v(p) = t(p - j) \) for \( m \leq p \leq m + n \) then \( v \) is a refinement of \( u \) and

\[
|M(x, y)P - \prod_t [1 - V]^{-1}P - \prod_s [1 - V]^{-1}Q| \leq \prod_s [1 + dp]^{-1}|P - Q|.
\]
Then

\[ |M(x, y)M(y, z)P - M(x, z)P| \leq |M(x, y)M(y, z)P - M(x, y) \prod_i [1 - V]^{-1}P| \]

\[ + |M(x, y) \prod_i [1 - V]^{-1}P - \prod_i [1 - V]^{-1} \prod_i [1 - V]^{-1}P| \]

\[ + |\prod_i [1 - V]^{-1} \prod_i [1 - V]^{-1}P - M(x, z)P| < 3\epsilon. \]

For property 3M, if \( a > b \), \( B \) is a bounded subset of \( S \), \( \alpha \) is as indicated in condition 3A, and \( \{s_p\} \) is a subdivision of \( \{x, y\} \) such that \( \int_{s(t)^{-1}} |d\rho| \leq \frac{1}{4} \), then

\[ |\prod_i [1 - V]^{-1}P - P| \exp(2 \int_{t_0}^t |d\rho|)|a(x) - a(y)|. \]

For property 4M, suppose that \( a > b \), \( \beta \) is as in condition 4A, \( \epsilon > 0 \), and \( P \) is in \( S \). Corresponding to \( P \) and \( \epsilon \), let \( \delta \) be as in 4A. Corresponding to the bounded set containing only the point \( P \) let \( \alpha \) be as in condition 3A. Let \( d \) be a positive number such that if \( a \geq x \geq y \geq b \) and \( x - y < d \) then \( \alpha(x) - \alpha(y) < \delta/2 \exp(2 \int_{t_0}^t |d\rho|) \). Let \( Q \) be in \( S \) such that \( |P - Q| < \delta/2 \exp(2 \int_{t_0}^t |d\rho|) \) and \( a \geq x \geq y \geq b \) such that \( x - y < d \). It follows that if \( t \) is a subdivision of \( \{x, y\} \) then \( \prod_i [1 - V]^{-1}Q - \prod_i [1 - V]^{-1}P| < \delta/2 \), \( |\prod_i [1 - V]^{-1}P - P| < \delta/2 \), and \( |\prod_i [1 - V]^{-1}Q - P| < \delta \). Thus

\[ \left| \prod_i [1 - V]^{-1}P - \prod_i [1 - V]^{-1}Q-Q \right| \]

\[ = \sum_{p=1}^n \left\{ V(t_{p-1}, t_p) \prod_{i=p}^n [1 - V(t_{i-1}, t_i)]^{-1}P - V(t_{p-1}, t_p) \prod_{i=p}^n [1 - V(t_{i-1}, t_i)]^{-1}Q \right\} \]

\[ \leq |\beta(x) - \beta(y)|2\epsilon. \]

3. From OM to OA. In this section, suppose that \( M \) is in \( OM \) and \( \rho \) is as in condition 1M.

**Lemma 3.1.** Suppose that \( a > b \), \( \beta \) is as in condition 4M, \( \{R_p\}_{p=1}^\infty \) is a Cauchy sequence with values in \( S \), and \( \epsilon > 0 \). There is a positive number \( \delta \) and a positive number \( d \) having the property that if \( n \) is a positive integer, \( P \) is in \( S \) such that \( |P - R_n| < \delta \), and \( a \geq x \geq y \geq b \) such that \( x - y < d \) then

\[ |[M(x, y) - 1]P - [M(x, y) - 1]R_n| \leq |\beta(x) - \beta(y)|\epsilon. \]

**Indication of proof.** Techniques applicable in the proof of Lemma 2.1 are also applicable here.

**Lemma 3.2.** Suppose that \( a > b \), \( \beta \) is as in condition 4M, \( \epsilon > 0 \), and \( P \) is in \( S \). There is a positive number \( \epsilon \) such that if \( a \geq u \geq v \geq z \geq b \) and \( u - z \leq \epsilon \), then

\[ |[M(u, v) - 1]M(v, z)P - [M(u, v) - 1]P| \leq |\beta(u) - \beta(v)|\epsilon. \]
**Indication of proof.** With the supposition of the lemma, let $\delta$ and $d$ be as indicated in condition $4M$. Corresponding to the bounded set containing only the point $P$, let $\alpha$ be as indicated in $3M$. Let $c$ be a positive number such that if $a \geq x \geq y \geq b$ and $x - y \leq c$ then $\alpha(x) - \alpha(y) < \delta$ and $e$ be the minimum of $c$ and $d$. If $z$ is in $[b, a]$ and $z + e \geq u \geq v \geq z$ then $[(M(u, v) - 1)M(v, z)P - [M(u, v) - 1]P] \leq [\beta(u) - \beta(v)]e$.

**Theorem 3.1.** Suppose that $a > b$, $P$ is in $S$, and $e > 0$. There is a subdivision $s$ of $\{a, b\}$ such that if $t$ refines $s$ then $\sum [M_{s} - M_{s-1}]P \leq [\beta(a) - \beta(b)]e$.

**Indication of proof.** With the supposition of the theorem, let $e$ be as in the previous lemma and $\{s_{p}\}_{0}^{n}$ be a subdivision of $\{a, b\}$ such that $s(0) = a$ and, if $p$ is a positive integer and $s(p - 1) > b$, then $s(p)$ is the maximum of $b$ and $s(p - 1) - e$. Let $t$ be a refinement of $s$ and $u$ be an increasing sequence such that $s(p) = t(u(p))$ for each integer $p$ in $[0, m]$. Then

$$\left| \sum [M_{s} - M_{s-1}]P - \sum [M_{t} - M_{t-1}]P \right| \leq [\beta(a) - \beta(b)]e.$$ 

**Theorem 3.2.** If $V(x, y)P = x\sum [M_{s} - M_{s-1}]P$ for all $x \geq y$ and $P$ in $S$ then $V$ is in $OA$.

**Indication of proof.** That $V$ has properties $2A$, $3A$, and $4A$ is established with nearly the same techniques as used in the corresponding parts of Theorem 2.2. To prove that $V$ has property $1A$, consider the following propositions:

**Proposition 1.** If $a > b$, $\{s_{p}\}_{0}^{n}$ is a subdivision of $\{a, b\}$, $c > 0$, and $P$ and $Q$ are in $S$ then

$$\left| \left[ (1 - c) \sum [M_{s} - M_{s-1}]P - (1 - c) \sum [M_{t} - M_{t-1}]Q \right] P - Q \right| \leq \left| (1 - c) \sum [\exp(-\delta p) - 1] \right| P - Q.$$

To see this, let $A$ be $\{1 - c \sum [M_{s} - M_{s-1}]P\}$ and $B$ be $\{1 - c \sum [M_{t} - M_{t-1}]Q\}$. Then

$$(1 + cn)P = A + c \sum_{p=1}^{n} M(t_{p-1}, t_{p})P \quad \text{and} \quad (1 + cn)Q = B + c \sum_{p=1}^{n} M(t_{p-1}, t_{p})Q.$$ 

Hence

$$(1 + cn)|P - Q| \leq |A - B| + c \sum_{p=1}^{n} \exp(\rho(t_{p-1}) - \rho(t_{p}))|P - Q|$$

or

$$\left| (1 - c) \sum_{p=1}^{n} [\exp(\rho(t_{p-1}) - \rho(t_{p})) - 1] \right| P - Q \leq |A - B|.$$ 

**Proposition 2.** If $a > b$, $c > 0$, and $P$ and $Q$ are in $S$ then

$$\left| [1 - cV(a, b)]P - [1 - cV(a, b)]Q \right| \geq \left| (1 - c[\rho(a) - \rho(b)])P - Q \right|.$$
Proposition 3. If \( a > b \) and \( \rho(a) - \rho(b) < 1 \) then \( 1 - V(a, b) \) has range all of \( S \).

To see this, let \( R \) be in \( S \) and \( A \) and \( B \) be functions from \( S \) to \( S \) defined as follows: \( A(P) = V(a, b)P + R - P \) and \( B(P) = V(a, b)P - P \) for each \( P \) in \( S \). Then

\[
\lim_{h \to 0} \frac{[P - Q] + h[AQ - AQ] - [P - Q]}{h}
= \lim_{h \to 0} \frac{[P - Q] + h[BP - BQ] - [P - Q]}{h}
= \lim_{h \to 0} \frac{[P - Q] + h[V(a, b)P - V(a, b)Q] - [P - Q]}{h}
\leq \{[\rho(a) - \rho(b)] - 1\}|P - Q|.
\]

As in [12] each of these limits exists and by [12, Theorem 1], for each \( P \) in \( E \), there is a function \( U(\cdot)P \) from \([0, \infty)\) into \( S \) such that if \( P \) and \( Q \) are in \( S \) and \( x \) and \( y \) are nonnegative numbers, then

\[
U(0)P = P, \quad U(x)P = P - \int_{0}^{x} A(U(\cdot)P) \, dI, \quad U(x)U(y) = U(x + y),
\]
and

\[
|U(x)P - U(x)Q| \leq \exp \left( ([\rho(a) - \rho(b)] - 1)x |P - Q| \right).
\]

Hence, if \( x > 0 \) then \( U(x) \) is a contraction mapping and there is only one point \( Z_x \) such that \( U(x)Z_x = Z_x \). However, if \( x \) and \( y \) are positive then \( Z_x = Z_y \) for \( U(y)Z_x = U(y)U(x)Z_x = U(x)U(y)Z_x \) so that \( Z_x = U(y)Z_x \) and \( Z_y = Z_x \). Hence, there is only one member \( Z \) of \( S \) such that \( U(x)Z = Z \) for all nonnegative numbers \( x \). Thus \( Z = Z + xA(Z) \) for all \( x \geq 0 \), or \( Z = Z + V(a, b)Z + R - Z \), or \( [1 - V(a, b)]Z = R \).

4. The one-to-one correspondence. In §2, a mapping is defined from \( OA \) to \( OM \) and, in §3, a mapping is defined from \( OM \) to \( OA \). This section will show that the composite of these mappings is the identity mapping.

Lemma 4.1. Suppose that \( V \) is in \( OA \), \( M \) is in \( OM \), \( V \) and \( M \) are related as in Theorem 2.2, \( a > b \), \( \beta \) is as in condition 4A, \( P \) is in \( S \), and \( \epsilon > 0 \). There is a positive number \( d \) such that if \( a \geq x \geq y \geq b \) and \( x - y < d \) then

\[
|V(x, y)P - M(x, y)P + P| \leq |\beta(x) - \beta(y)|\epsilon.
\]

Indication of proof. With the supposition of the lemma, let \( \delta \) be as indicated in condition 4A. Corresponding to the bounded set containing only \( P \), let \( \alpha \) be as in condition 3A. Let \( d \) be a positive number such that if \( a \geq x \geq y \geq b \) and \( x - y < d \) then \( \alpha(x) - \alpha(y) < \delta/\exp (2 \int_{b}^{a} |d\rho|) \). Let \( x \) and \( y \) be such that \( x - y < d \) and \( \{t_{n}\} \) be a subdivision of \( \{x, y\} \). If \( p \) is an integer in \([1, n]\) then

\[
\left| \prod_{i=p}^{n} [1 - V(t_{i-1}, t_{i})^{-1}P - P] \right| \leq \exp \left( 2 \int_{b}^{a} |d\rho| \right) \sum_{t} |VP| < \delta.
\]
Thus

\[ \prod_{t} \left[ 1 - V \right]^{-1} P - P - V(x, y)P \]

\[ = \sum_{p=1}^{n} \left\{ V(t_{p-1}, t_{p}) \prod_{t_{i}=p}^{n} \left[ 1 - V(t_{i-1}, t_{i}) \right]^{-1} P - V(t_{p-1}, t_{p})P \right\} \]

\[ \leq \left[ \beta(x) - \beta(y) \right] e. \]

**Theorem 4.1.** If \( V \) is in \( OA \), \( M(x, y)P = \prod_{t} \left[ 1 - V \right]^{-1} P \), and \( U(x, y)P = \sum_{t} \left[ M - 1 \right] P \) for all \( x \geq y \) and \( P \) in \( S \), then \( U = V \).

**Indication of proof.** Suppose that \( a > b \), \( P \) is in \( S \), \( e > 0 \), and \( d \) is as in the previous lemma. Let \( \{t_{n}\}_{n}^{m} \) be a subdivision of \( \{a, b\} \) such that if \( p \) is an integer in \( [1, n] \) then \( t_{p-1} - t_{p} < d \). Then

\[ \left| \sum_{t} \left[ M - 1 \right] P - V(a, b)P \right| \leq \left[ \beta(a) - \beta(b) \right] e. \]

**Lemma 4.2.** Suppose that \( M \) is in \( OM \), \( V \) is in \( OA \), \( M \) and \( V \) are related as in Theorem 3.2, \( a > b \), \( P \) is in \( S \), \( \beta \) is as in condition 4M, and \( e > 0 \). There is a subdivision \( \{s_{n}\}_{n}^{m} \) of \( \{a, b\} \) such that if \( k \) is an integer in \( [1, m] \) then

\[ \left| \left[ M(s_{k-1}, s_{k}) - 1 \right] M(s_{k}, b)P - V(s_{k-1}, s_{k})M(s_{k}, b)P \right| \leq \left[ \beta(s_{k-1}) - \beta(s_{k}) \right] e. \]

**Indication of proof.** With the supposition of the lemma, let \( x \) be an increasing sequence defined inductively as follows: \( x(0) = b \) and, if \( n \) is a positive integer such that \( a > x(n-1) \geq b \) then \( x(n) \) is the largest number \( c \) not exceeding \( a \) such that if \( c \geq u \geq v \geq x(n-1) \) then

\[ \left| \left[ M(u, v) - 1 \right] M(v, x_{n-1})M(x_{n-1}, b)P - \left[ M(u, v) - 1 \right] M(x_{n-1}, b)P \right| \leq \left[ \beta(u) - \beta(v) \right] e. \]

The existence of such a number \( c \) follows from Lemma 3.2. Lemma 3.1 can be used to show that the supposition that \( x \) is an infinite sequence leads to a contradiction. Hence, there is an integer \( m \) such that \( x(m) = a \). Let \( s(p) \) be \( x(m-p) \) for \( p \) an integer in \( [0, m] \). Let \( k \) be an integer in \( [1, m] \). If \( t \) is a subdivision of \( \{s_{k-1}, s_{k}\} \) then

\[ \left| \left[ M(s_{k-1}, s_{k}) - 1 \right] M(s_{k}, b)P - \sum_{t} \left[ M - 1 \right] M(s_{k}, b)P \right| \]

\[ = \left| \sum_{p=1}^{n} \left\{ M(t_{p-1}, t_{p}) - 1 \right\} M(t_{p}, b)P - \left[ M(t_{p-1}, t_{p}) - 1 \right] M(s_{k}, b)P \right| \]

\[ \leq \left[ \beta(s_{k-1}) - \beta(s_{k}) \right] e. \]

**Lemma 4.3.** Suppose that \( V \) is in \( OA \), \( M \) is in \( OM \), \( a > b \), \( P \) is in \( S \), \( \beta \) is as in condition 4A, and \( e > 0 \). There is a positive number \( d \) such that if \( a \geq x \geq y \geq b \), \( x - y < d \), and \( a \geq u \geq v \geq b \) then

\[ \left| V(u, v)M(x, b)P - V(u, v)M(y, b)P \right| \leq \left[ \beta(u) - \beta(v) \right] e. \]

**Indication of proof.** A proof may be constructed using the fact \( M(\cdot, b)P \) is continuous on \( [b, a] \) and so \( M([b, a], b)P \) is closed and compact.
Theorem 4.2. If $M$ is in $OM$, $V(x, y)P = x\sum_y [M - 1]P$, and $W(x, y)P = x\prod_y [1 - V]^{-1}P$ for all $x \geq y$ and $P$ in $S$, then $W = M$.

Indication of proof. Suppose that $a > b$, $P$ is in $S$, $\varepsilon > 0$, $d$ is as in the previous lemma, and $\{s_p\}_{0}^{n}$ is a subdivision of $\{a, b\}$ having the property indicated in Lemma 4.2 and the property that if $k$ is an integer in $[1, m]$ then $s_{k-1} - s_k < d$. Then

$$\prod_{p=1}^{n} [1 - V(s_p - 1, s_p)]^{-1}P - M(a, b)P$$

$$\leq \exp \left( 2 \int_{b}^{a} |dp| \right) \sum_{p=1}^{n} |M(s_{p-1}, b)P - M(s_{p}, b)P - V(s_{p}, s_{p})M(s_{p}, b)P|$$

$$\leq \exp \left( 2 \int_{b}^{a} |dp| \right) [\beta(a) - \beta(b)]\varepsilon.$$

5. The integral equation. With the usual arguments, it can be shown that if $a > b, f$ is a continuous function from $[b, a]$ to $S$, and $V$ is in $OA$ then the Riemann-Stieltjes integral $\int_{b}^{a} VF$ exists. In this section it will be shown that the member $M$ in $OM$ related to the member $V$ in $OA$ as in Theorems 2.2 and 3.2 is the only member $M$ of $OM$ satisfying $M(x, y)P = P + \int_{x}^{y} VM(\cdot, y)P$ for all $x \geq y$ and all $P$ in $S$.

Theorem 5.1. Suppose that $M$ is in $OM$, $V$ is in $OA$, $M$ and $V$ are related as in Theorem 3.2, $a > b$, and $P$ is in $S$. Then $M(a, b)P = P + \int_{a}^{b} VM(\cdot, b)P$.

Indication of proof. With the supposition of the theorem, suppose that $\beta$ is as in condition 4A and $\varepsilon > 0$. By Lemma 4.2, if $t$ is a subdivision of $\{a, b\}$ then there is a refinement $\{s_p\}_{0}^{n}$ of $t$ such that if $k$ is an integer in $[1, m]$ then

$$| [M(s_{k-1}, s_k) - 1]M(s_k, b)P - V(s_{k-1}, s_k)M(s_k, b)P | \leq [\beta(s_{k-1}) - \beta(s_k)]\varepsilon.$$

Then

$$| M(a, b)P - P - \sum_{s} VM(\cdot, b)P |$$

$$\leq | \sum_{p=1}^{n} M(s_{p-1}, b)P - M(s_{p}, b)P - V(s_{p-1}, s_p)M(s_{p}, b)P |$$

$$\leq [\beta(a) - \beta(b)]\varepsilon.$$

Lemma 5.1. Suppose that $a > b$, $V$ is in $OA$, $\beta$ is as in condition 4A, $M$ is in $OM$, $P$ is in $S$, and $\varepsilon > 0$. There is a positive number $d$ such that if $a \geq x \geq y \geq b$ and $x - y < d$ then $| \int_{x}^{y} VM(\cdot, y)P - V(x, y)P | \leq [\beta(x) - \beta(y)]\varepsilon$.

Indication of proof. With the supposition of the lemma, let $B$ be the bounded set consisting of only the point $P$ and $a$ be as in condition 3M. Corresponding to
{a, b}, P, and e, let δ be as in 4A. There is a positive number d such that if
\[ a \geq x \geq y \geq b \text{ and } x - y < d \text{ then } |M(x, y)P - P| \leq \alpha(x) - \alpha(y) < \delta \] and, hence,
\[ \int_x^y VM(\cdot, y)P - V(x, y)P < \beta(x) - \beta(y)\cdot e. \]

**Theorem 5.2.** If V is in OA, M is in OM, and M(x, y)P = P + \[ \int_x^y VM(\cdot, y)P \] for \( x > y \) and P in S then V and M are related as in Theorem 3.2.

**Indication of proof.** Suppose that P is in S, each of U and V is in OA, M is in OM, and, for \( x \geq y \), both the following hold: \( M(x, y)P = P + \[ \int_x^y VM(\cdot, y)P \] \) and \( U(x, y)P = x \sum_y [M - l]P \). If \( a > b \) and \( t \) is a subdivision of \{a, b\} then
\[ \sum_t [M - l]P - V(a, b)P = \sum_t \int_{[t_p, t_{p+1}]} VM(\cdot, b)P - V(t_p, t_{p+1})P. \]
The previous lemma gives that \( U = V \).

6. **Examples.** In view of the proof for Proposition 3 to Theorem 3.2, an alternate characterization of OA may be obtained by changing 1A in (see also \[12, Example 2\])

1A'. There is a continuous function \( \rho \) which is of bounded variation on each finite interval such that if \( a > b \) and \( c > 0 \) then
\[ \{ 1 - c[\rho(a) - \rho(b)] \} |P - Q| \leq \{ 1 - cV(a, b) \} |P - [1 - cV(a, b)]Q| \]

**Example 1.** The class OA described in this paper contains the continuous members of the class OA described in \[10\]. That is, a sufficient condition that \( U \) be in OA is that if \( a \geq b \) then \( U(a, b) \) is a function from S to S and

1. there is a nondecreasing, continuous function \( \rho \) such that if \( a \geq b \) and \( P \) and \( Q \) are in S, then \( |U(a, b)P - U(a, b)Q| \leq [\rho(a) - \rho(b)]|P - Q| \),
2. if \( x \geq y \geq z \) and \( P \) is in S then \( U(x, y)P + U(y, z)P = U(x, z)P \), and
3. if \( x \geq y \) then \( U(x, y)0 = 0 \).

**Example 2.** Suppose that \( A \) is a function with values in S and that \( A \) has the following properties: (compare \[18, Theorem 3\])
(a) if \( t \) is a number then \( A(t, \cdot) \) has domain all of S,
(b) if \( P \) is in S then \( A(\cdot, P) \) is continuous,
(c) if \( a > b \) and \( B \) is a bounded subset of S then \( A \) is bounded on \([b, a] \times B\),
(d) if \( a > b \), \( P \) is in S, and \( e > 0 \) then there is a positive number \( \delta \) having the property that if \( a \geq u \geq b \) and \( Q \) is in S such that \( |Q - P| < \delta \) then \( |A(u, Q) - A(u, P)| < e \), and
(e) there is a continuous function \( \rho \) such that if \( t \) is a number, \( P \) and \( Q \) are in S, and \( c > 0 \) then \( |[P - cA(t, P)] - [Q - cA(t, Q)]| \geq [1 - c\rho(t)]|P - Q| \).

**Theorem 6.1.** If \( a > b \) and \( Q \) is in S then \( \int_x^y A(\cdot, Q) \) dI exists and, if \( V(x, y)P \) is defined to be \( \int_x^y A(\cdot, P) \) dI for \( x \geq y \) and \( P \) in S, then \( V \) is in OA.
By the usual arguments, it can be shown that if \( a > b \) and \( P \) is in \( S \) then the Riemann-Stieltjes integral \( \int_{0}^{a} A(\cdot, P) \, dI \) exists. That this integral generates a member of \( OA \) will be proved in the next sequence of lemmas.

Let \( S^* \) be the dual space of \( S \) and \( | \cdot | \) denote the norm on \( S^* \). As in [8] if \( x \) is in \( S \), denote by \( Fx \) the set of all functions \( f \) in \( S^* \) such that \( f(x) = |x|^2 - |f|^2 \). As in [11], if \( x \) is in \( S \), denote by \( Gx \) the set of all functions \( g \) in \( S^* \) such that \( g(x) = |x| \) and \( |g| = 1 \). Note that \( g \) is in \( Gx \) only in case \( |x| \cdot |g| \) is in \( Fx \).

**Lemma 6.1.** If \( x \) and \( y \) are in \( S \) and \( k \) is a number then these are equivalent:
(i) if \( c > 0 \) then \( (1 - ck)|x| \leq |x + cy| \), and
(ii) there is a member \( f \) of \( Fx \) such that \( Re f(y) \geq -k|x|^2 \).

**Indication of proof.** The proof of [8, Lemma 1.1] may be used with only minor modifications.

**Lemma 6.2 (Martin, [12, Remark 4]).** Suppose that \( k \) is a number and \( L \) is a function which is continuous from \( S \) to \( S \). These are equivalent: for all \( P \) and \( Q \)
(1) if \( c > 0 \) then \( |[P - cL(P)] - [Q - cL(Q)]| \geq |1 - ck| |P - Q| \), and
(2) if \( g \) is in \( G(P - Q) \) then \( Re g(L(P) - L(Q)) \leq k|P - Q| \).

**Indication of proof.** With the supposition of the lemma, consider the following statements, for all \( P \) and \( Q \) is \( S \):
(i) if \( c > 0 \) then
\[
|P - Q| - c[L(P) - L(Q)] \geq |1 - ck| |P - Q|,
\]
(ii) \[
\lim_{h \to 0^{-}} \frac{|P - Q| + h[L(P) - L(Q)] - |P - Q|}{h} \leq k|P - Q|,
\]
(iii) \[
\lim_{h \to 0^{+}} \frac{|P - Q| + h[L(P) - L(Q)] - |P - Q|}{h} \leq k|P - Q|,
\]
(iv) if \( g \) is in \( G(P - Q) \) then
\[
Re g(L(P) - L(Q)) \leq k|P - Q|.
\]
That (i) implies (ii) can be seen by rearranging the inequality in (i). That (ii) implies (iii) follows from [12, Remark 4]. That (iii) and (iv) are equivalent follows from [12, Example 1] and [11, Corollary 2.2]. That (iv) implies (i) follows from Lemma 6.1 and the preceding remarks and definitions.

**Indication of proof for Theorem 6.1.** Let \( A \) have the properties (a)--(e) and \( V \) be as defined in the theorem. It is not difficult to show that \( V \) has properties 2A--4A. Suppose that \( a > b \) and \( \{t_p\}^n \) is a nondecreasing sequence such that \( t(0) = b \) and \( t(2n) = a \). Let \( P \) and \( Q \) be in \( S \) and \( g \) be a member of \( G(P - Q) \). Then
\[
Re g \left( \sum_{p=1}^{n} A(t_{2p-1}, P) \cdot (t_{2p} - t_{2p-2}) - \sum_{p=1}^{n} A(t_{2p-1}, Q) \cdot (t_{2p} - t_{2p-2}) \right)
\geq \sum_{p=1}^{n} (t_{2p} - t_{2p-2}) \rho(t_{2p-1}) |P - Q|.
\]
Hence, if $c > 0$ then
\[
\left| \left[ P - c \int_b^a A(\cdot, P) \, dt \right] - \left[ Q - c \int_b^a A(\cdot, Q) \, dt \right] \right| \geq \left| 1 - c \int_b^a \varrho \, dt \right| |P - Q|
\]
or
\[
\left| \left[ 1 - c \varphi(a, b) \right] P - \left[ 1 - c \varphi(a, b) \right] Q \right| \geq \left| 1 - c \int_b^a \varrho \, dt \right| |P - Q|.
\]

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