ON THE EXISTENCE OF STRONGLY SERIES SUMMABLE MARKUSCHEVICH BASES IN BANACH SPACES

BY

WILLIAM B. JOHNSON

Abstract. The main result is: Let $X$ be a complex separable Banach space. If the identity operator on $X^*$ is the limit in the strong operator topology of a uniformly bounded net of linear operators of finite rank, then $X$ admits a strongly series summable Markushevich basis.

I. Introduction. Let $X$ be a separable Banach space. A biorthogonal sequence $\{x_i, f_i\}_{i=1}^{\infty}$ in $(X, X^*)$ is called a Markushevich basis ($M$-basis) for $X$ provided $\{x_i\}_{i=1}^{\infty}$ is fundamental in $X$ and $\{f_i\}_{i=1}^{\infty}$ is total over $X$. Following Ruckle [8], we say that an $M$-basis $\{x_i, f_i\}_{i=1}^{\infty}$ for $X$ is strongly series summable (s.s.s.) provided there exists a set $\{\lambda_{i,n} : i=1, 2, \ldots, n; n=1, 2, \ldots\}$ of scalars (called a summation matrix for $\{x_i, f_i\}_{i=1}^{\infty}$) such that, for each $x$ in $X$, $x = \lim_{n \to \infty} \sum_{i=1}^{\infty} \lambda_{i,n} f_i(x)x_i$. Note that a Schauder basis is a s.s.s. $M$-basis for which each $\lambda_{i,n}$ can be chosen to be 1.

The results of [8], [9], and [4] indicate that the duality theory of a space which has a s.s.s. $M$-basis is essentially the same as that of a space which admits a Schauder basis. The reason for this appears to be that if $\{x_i, f_i\}_{i=1}^{\infty}$ is a s.s.s. $M$-basis for $X$ with summation matrix $(\lambda_{i,n})$ then, for each $f$ in the coefficient space(1) of the basis, $f$ is the norm limit of $\left\{\sum_{i=1}^{\infty} \lambda_{i,n} f(x_i)f_i\right\}_{i=1}^{\infty}$. Thus the adjoints of the "partial sum" operators defined by $T_n(x) = \sum_{i=1}^{n} \lambda_{i,n} f(x_i)x_i$ also act like partial sum operators. In this respect s.s.s. $M$-bases behave more like Schauder bases than do such weaker structures as generalized summation bases (see [3]).

In this paper we prove the following rather strong existence theorem for s.s.s. $M$-bases:

Theorem 1. Let $X$ be a separable complex Banach space such that $X^*$ has the $\lambda$-metric approximation property for some $\lambda \geq 1$. If $Y$ is a separable subspace of $X^*$, then there exists a strongly series summable Markushevich basis for $X$ whose coefficient space contains $Y$.

If $\lambda \geq 1$, we say that the Banach space $X$ has the $\lambda$-metric approximation property ($\lambda$-m.a.p.) if there is a net $\{S_d : d \in D\}$ of linear operators of finite rank on $X$.

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(1) The coefficient space of an $M$-basis $\{x_i, f_i\}_{i=1}^{\infty}$ for $X$ is the norm closure in $X^*$ of the linear span of $\{f_i\}_{i=1}^{\infty}$.

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uniformly bounded by $\lambda$ which converges pointwise (i.e., in the strong operator topology) to the identity operator on $X$. Equivalently, $X$ has the $\lambda$-m.a.p. provided that, for each finite-dimensional subspace $F$ of $X$ and positive number $\varepsilon$, there is an operator $S$ of finite rank on $X$ such that $\|S\| \leq \lambda$ and $\|S(x) - x\| \leq \varepsilon \|x\|$ for each $x \in F$.

The 1-m.a.p. was introduced by Grothendieck [2] under the name metric approximation property. Grothendieck showed that if $X$ is reflexive and has the (topological) approximation property, then in fact both $X$ and $X^*$ have the 1-m.a.p. This result together with Theorem 1 implies that every separable, reflexive complex Banach space which has the approximation property also admits a s.s.s. $M$-basis. Of course, it may be that Theorem 1 is always applicable, for it is not even known that there exists a Banach space which does not have the 1-m.a.p.

We use the following notation: $X$ represents a complex Banach space and $X^*$ is the dual to $X$. The complex assumption is used in an essential way in Lemma 4, and we do not know whether the real version of Theorem 1 is true. $I$ denotes the identity operator on either $X$ or $X^*$. "Operator" means "bounded linear operator". The range space and null space of an operator, $L$, are denoted by, respectively, $\text{R}(L)$ and $\text{ker} L$. If $L$ is an operator on $X$ and $S$ is a subspace of $X$, $L_{|S}$ denotes the restriction of $L$ to $S$. The linear span of a subset, $A$, of a linear space is denoted by $\text{sp} A$. The canonical embedding of $X$ into $X^{**}$ is denoted by $\pi_X$.

II. The existence theorem. Our first lemma is both a generalization and a special case of Helly's theorem [11, p. 103].

**Lemma 1.** Let $F$ be a finite-dimensional Banach space, $S$ a finite-dimensional subspace of $X^*$, $L$ an operator from $X^*$ into $F$, and $\varepsilon > 0$. There exists a weak*-continuous operator $T$ from $X^*$ into $F$ such that $T_{|S} = L_{|S}$ and $\|T\| \leq \|L\| + \varepsilon$.

**Proof.** We use the notation of [10] in this proof. We identify the weak*-continuous operators from $X^*$ to $F$ with $X \otimes F$ and the operators from $X^*$ to $F$ with $X^{**} \otimes F$ [10, p. 30]. Since $F$ is finite dimensional, $X^{**} \otimes F$ is thereby identified with $(X \otimes F)^{**}$. Now $S \otimes F$ is identified with a (finite-dimensional) subspace of $(X \otimes F)^*$, so by Helly's theorem [11, p. 103], there is $T$ in $X \otimes F$ such that $\|T\| \leq \|L\| + \varepsilon$ and $f(T(s)) = f(L(s))$ for each $s \in S$ and $f \in F^*$. Since $F^*$ is total over $F$, $T(s) = L(s)$ for each $s \in S$ and hence $T_{|S} = L_{|S}$. Q.E.D.

A Banach space $X$ is said to have the $\lambda$ duality metric approximation property ($\lambda \geq 1$) provided there is a net $\{S_d : d \in D\}$ of operators of finite rank on $X$ uniformly bounded by $\lambda$ such that $\{S_d : d \in D\}$ converges pointwise to $I$ and $\{S^*_d : d \in D\}$ converges pointwise to $I$. Equivalently, $X$ has the $\lambda$ duality m.a.p. provided that, for each $\varepsilon > 0$ and each pair of finite-dimensional subspaces $E$ of $X$ and $F$ of $X^*$, there is an operator $L$ of finite rank on $X$ such that $\|L\| \leq \lambda$, $\|L(x) - x\| \leq \varepsilon \|x\|$ for each $x \in E$, and $\|L^*(f) - f\| \leq \varepsilon \|f\|$ for each $f \in F$.

**Lemma 2.** Suppose that $X^*$ has the $\lambda$-m.a.p. Then $X$ has the $\lambda$ duality m.a.p.
Proof. Using the hypothesis and Lemma 1, we can construct a net \( \{S_d : d \in D\} \) of operators of finite rank on \( X \) uniformly bounded by \( \lambda \) such that \( \{S^*_d : d \in D\} \) is pointwise convergent on \( X^* \) to \( I \). For each \( x \in X \), the net \( \{S_d(x) : d \in D\} \) weakly converges to \( x \), hence (cf., e.g., [1, p. 477]) there is a net \( \{T_e : e \in E\} \) of operators on \( X \) such that \( \{T_e : e \in E\} \) is pointwise convergent on \( X \) to \( I \); each \( T_e \) is a convex combination of a subset \( \{S_{d(e)} : d \in D\} \); and for each \( d \in D \) there is \( e' \in E \) such that if \( e \geq e' \) then \( e(i) \geq d \) for \( i = 1, 2, \ldots, n_e \). Thus \( \{T_e : e \in E\} \) is uniformly bounded by \( \lambda \) and \( \{T^*_e : e \in E\} \) is pointwise convergent on \( X^* \) to \( I \). Q.E.D.

The proof of the next lemma is suggested by the proof of Lemma 3.1 of [5].

Lemma 3. Suppose that \( X \) has the \( \lambda \) duality m.a.p., \( E \) is a finite-dimensional subspace of \( X \), \( F \) is a finite-dimensional subspace of \( X^* \), and \( \varepsilon > 0 \). Then there exists an operator \( L \) of finite rank on \( X \) such that \( \|L\| \leq \lambda + \varepsilon \), \( L_{1E} = I_{1E} \), and \( L^*_{1F} = I_{1F} \).

Proof. Let \( n = \dim E \) and \( m = \dim F \). Choose \( 1 > \beta > 0 \) small enough so that \( \beta + \beta m(\lambda + \beta)/(1 - \beta) \leq \varepsilon \) and choose \( 1 > \alpha > 0 \) small enough so that \( (na/(1 - \alpha))^\alpha \leq \beta/2 \).

Let \( M \) be an operator of finite rank on \( X \) such that \( \|M\| \leq \lambda \) and, for each \( x \in E \) and \( f \in F \),

1. \( \|x - M(x)\| \leq \alpha \|x\| \) and
2. \( \|f - M^*(f)\| \leq \beta/2 \|f\| \).

By (1), for each \( x \in E \), \( (1 - \alpha)\|x\| \leq \|M(x)\| \), hence \( M_{1E} \) has an inverse, \( Q \), satisfying \( \|Q\| \leq 1/(1 - \alpha) \). Also, for each \( y \in M[E] \), \( \|Q(y) - y\| \leq (\alpha/(1 - \alpha)) \|y\| \).

Let \( P \) be a projection of \( X \) onto \( M[E] \) such that \( \|P\| \leq n \). Let \( N = QPM + (I - P)M \). Clearly \( N \) has finite rank and \( N_{1E} = I_{1E} \). Now if \( x \in X \),

\[
\|N(x) - M(x)\| = \|QPM(x) - PM(x)\| \leq \frac{\alpha}{1 - \alpha} \|P\| \|M\| \|x\| \leq \frac{n\alpha \lambda}{1 - \alpha} \|x\|.
\]

Thus \( \|N - M\| \leq n\alpha \lambda/(1 - \alpha) \), from which it follows that \( \|N^*\| \leq \lambda + \beta/2 \) and \( \|N^* - M^*\| \leq \beta/2 \). This last inequality and (2) imply that, for each \( f \in F \),

\[
\|N^*(f) - f\| \leq \beta/2 \|f\|.
\]

As in the first part of the proof, we have that \( N^*_F \) is an isomorphism with inverse, \( Q' \), satisfying \( \|Q'(f) - f\| \leq (\beta/(1 - \beta)) \|f\| \) for each \( f \in N^*[F] \). Let \( P' \) be a projection of \( X^* \) onto \( N^*[F] \) such that \( \|P'\| \leq m \) and let \( L^* = Q'P^*N^* + (I - P')N^* \). (Note that \( L^* \) is indeed weak*-continuous because \( N^* \) is weak*-continuous and has finite rank.) Then \( L^*_{1F} = I_{1F} \) and, for each \( f \in X^* \),

\[
\|L^*(f) - N^*(f)\| \leq \frac{\beta}{1 - \beta} \|P'\| \|N^*\| \|f\| \leq \frac{\beta m(\lambda + \beta)}{1 - \beta} \|f\|.
\]

Thus \( \|L^*\| \leq \lambda + \beta + \beta m(\lambda + \beta)/(1 - \beta) \leq \lambda + \varepsilon \).

Since \( \|L\| = \|L^*\| \), it remains to be seen only that \( L_{1E} = I_{1E} \). Let \( x \in E \) and suppose that \( f \in X^* \). Then using the fact that \( x = N(x) \), we have

\[
f(L(x)) = L^*(f)(x) = L^*(f)(N(x)) = N^*Q'P^*N^*(f)(x) + N^*(I - P')N^*(f)(x) = P^*N^*(f)(x) + f(N(N(x))) - P^*N^*(f)(N(x)) = f(x).
\]

Since \( X^* \) is total over \( E \), \( L(x) = x \). Q.E.D.
LEMMA 4. Let \( \{x_i, f_i\}_{i=1}^n \) be a finite biorthogonal set in \((X, X^*)\), let \( T \) be an operator of finite rank on \( X \) such that \( T(x_i) = x_i \) and \( T^*(f_i) = f_i \) for \( i = 1, 2, \ldots, n \), and let \( \epsilon > 0 \). Then there exists a finite biorthogonal set \( \{x_i, f_i\}_{i=1}^{n+m+1} \) in \((X, X^*)\) and a set \( \{\lambda_i\}_{i=1}^{n+1} \) of complex numbers such that \( \{x_i, f_i\}_{i=1}^{n+m+1} \) is biorthogonal and \( \|L - T\| \leq \epsilon \), where \( L \) is the operator on \( X \) defined by

\[
L(x) = \sum_{i=1}^{n} f_i(x)x_i + \sum_{i=n+1}^{n+m} \lambda_{i-n} f_i(x)x_i.
\]

Proof. Define a projection \( U \) on \( X \) by \( U(x) = \sum_{i=1}^{n} f_i(x)x_i \) and let \( X_0 = \mathcal{R}(I - U) \). Note that \( TU = UT = U \), so \( T[X_0] \subseteq X_0 \) and \( ker T \subseteq X_0 \). Let \( P \) be a projection of finite rank on \( X_0 \) such that \( PT(I - U) = TP(I - U) = T(I - U) \). (For example, choose \( ker P \) to be a closed complement in \( ker T \) to \( \mathcal{R}(T) \cap ker T \) and choose \( \mathcal{R}(P) \) to be a complement in \( X_0 \) to \( ker P \) which contains \( \mathcal{R}(T) \cap X_0 \).)

Let \( m = \dim \mathcal{R}(P) \) and choose a basis \( \{z_i\}_{i=1}^m \) for \( \mathcal{R}(P) \) such that the matrix representation \( (\alpha_{ij})_{i,j=1}^m \) of \( T|_{\mathcal{R}(P)} \) with respect to \( \{z_i\}_{i=1}^m \) is lower triangular—i.e., \( \alpha_{ij} = 0 \) if \( j > i \). Now pick a sequence \( \{\lambda_i\}_{i=1}^m \) of pairwise distinct complex numbers sufficiently close to \( \{\alpha_{ij}\}_{i,j=1}^m \) so that

\[
\|Q - T|_{\mathcal{R}(P)}\| \leq \epsilon\|P\|\|I - U\|,
\]

where \( Q \) is the operator on \( \mathcal{R}(P) \) whose matrix representation, \( (\beta_{ij}) \), with respect to \( \{z_i\}_{i=1}^m \) is given by

\[
\beta_{ij} = \lambda_i \quad \text{if} \quad i = j,
\]

\[
= \alpha_{ij} \quad \text{if} \quad i \neq j.
\]

Since \( (\beta_{ij}) \) is lower triangular, \( \{\lambda_i\}_{i=1}^m \) is the set of eigenvalues for \( Q \). The \( \lambda_i \)'s are distinct, so there is a basis \( \{x_i\}_{i=n+1}^{n+m+1} \) for \( \mathcal{R}(P) \) such that \( Q(x_i) = \lambda_{i-n} x_i \) for \( i = n+1, \ldots, n+m \). Picking \( \{f_i\}_{i=n+1}^{n+m} \) in \( \mathcal{R}((P(I - U))^*) \) biorthogonal to \( \{x_i\}_{i=n+1}^{n+m+1} \), we have that, for each \( x \in \mathcal{R}(P) \), \( Q(x) = \sum_{i=n+1}^{n+m} \lambda_{i-n} f_i(x)x_i \).

Now \( \{x_i, f_i\}_{i=1}^{n+m+1} \) is biorthogonal and if \( L \) is defined by

\[
L(x) = \sum_{i=1}^{n} f_i(x)x_i + \sum_{i=n+1}^{n+m} \lambda_{i-n} f_i(x)x_i,
\]

then clearly \( L = TU + QP(I - U) \). Thus

\[
\|L - T\| = \|TU + QP(I - U) - TU - TP(I - U)\| \leq \|Q - T|_{\mathcal{R}(P)}\|\|P\|\|I - U\| \leq \epsilon.
\]

Q.E.D.

Proof of Theorem 1. Let \( \lambda \) be such that \( X \) has the \( \lambda \) duality m.a.p. (Lemma 2). Let \( \{z_i\}_{i=1}^n \) be fundamental in \( X \) and let \( \{g_i\}_{i=1}^n \) be a subset of \( X^* \) such that \( Y \) is contained in the closure of the linear span of \( \{g_i\}_{i=1}^n \). Assume, without loss of generality, that \( \|z_i\| = \|g_i\| = g_i(z_i) = 1 \). We define the desired s.s.s. \( M \)-basis \( \{x_i, f_i\}_{i=1}^n \) for \( X \) and a summability matrix \( (\lambda_{i,n}) \) for \( \{x_i, f_i\}_{i=1}^n \) by induction. Set \( k(1) = 1, \ x_1 = z_1, \ f_1 = g_1, \ \lambda_{1,1} = 1 \). Now suppose \( k(m), \ \{x_i, f_i\}_{i=1}^{k(m)} \), and...
{λi, n : i ≤ n; n = 1, 2, ..., k(m)} have been defined. Extend \{x_i, f_i\}_{i=1}^{k(m)} to a biorthogonal set \{x_i, f_i\}_{i=1}^{k(m)+2} so that \(z_{n+1} \in \text{sp} \{x_i\}_{i=1}^{n+1}\) and \(g_{n+1} \in \text{sp} \{f_i\}_{i=1}^{n+1}\) (cf., e.g., the proof of Theorem III.1 in [3]). Now by Lemma 3 and Lemma 4 there are a positive integer \(k(m+1)\), a biorthogonal set \(\{x_i, f_i\}_{i=1}^{k(m+1)}\) in \((X, X^*)\), and complex numbers \(\{\alpha_i\}_{i=1}^{k(m+1)}\) such that \(\{x_i, f_i\}_{i=1}^{k(m+1)}\) is biorthogonal and if \(T\) is defined on \(X\) by \(T(x) = \sum_{i=1}^{k(m+1)} f_i(x) x_i + \sum_{i=k(m+1)}^{k(m+1)} \alpha_i f_i(x) x_i\), then \(\|T\| \leq \lambda + 1/m\). We complete the induction by defining

\[
\lambda_{i, n} = \lambda_{i, k(m)} \quad \text{if } i \leq k(m) < n < k(m+1), \\
= 1 \quad \text{if } i \leq k(m) \text{ and } n = k(m+1), \\
= \alpha_i \quad \text{if } k(m) < i \leq k(m+1) = n.
\]

It is easy to check that \(\{x_i, f_i\}_{i=1}^{k(m)}\) has the desired properties. Q.E.D.

**Remark 1.** Suppose that \(X\) is separable and \(X^*\) has the \(\lambda\)-m.a.p. for some \(\lambda\). Then \(X\) is separable and \(X^*\) has the \(\lambda\)-m.a.p. for some \(\lambda\). Theorem 1 shows that there are s.s.s. \(M\)-bases for \(X\) whose coefficient spaces are “arbitrarily large”. One might guess that if \(Y\) is a separable subspace of \(X^*\) and \(Y\) contains a subspace which is the coefficient space of some s.s.s. \(M\)-basis for \(X\), then \(Y\) itself is the coefficient space for some s.s.s. \(M\)-basis for \(X\), because the corresponding statement for generalized summation bases is true (cf. [3, proof of Theorem IV.1]). This is not the case: Let \(X = l_1\). It is a rather easy consequence of Theorem 4.3 of [6] that the coefficient space of any s.s.s. \(M\)-basis for \(l_1\) is an \(L_\infty\) space in the sense of [6]. Simply pick \(Y\) to be a separable subspace of \(l_\infty\) \((= l_1^\#)\) which contains \(c_0\) but is not an \(L_\infty\) space. (For example, \(Y\) can be the closed span of \(c_0 \cup K\), where \(K\) is a subspace of \(l_\infty\) isomorphic to \(l_2\). It follows from Theorem 1 of [7] that \(Y\) is isomorphic to \(c_0 \oplus l_2\) and is thus not an \(L_\infty\) space.)

Recall that an \(M\)-basis \(\{x_i, f_i\}_{i=1}^{\infty}\) for \(X\) whose coefficient space is \(X^*\) is called shrinking (see [3]). Let \(\{x_i, f_i\}_{i=1}^{\infty}\) be a s.s.s. \(M\)-basis then the remarks in the introduction show that \(\{f_i, x_i\}_{i=1}^{\infty}\) is a s.s.s. \(M\)-basis for the coefficient space of the basis. Thus a shrinking \(M\)-basis \(\{x_i, f_i\}_{i=1}^{\infty}\) which is s.s.s. also shrinking as a s.s.s. \(M\)-basis in the sense that \(\{f_i, x_i\}_{i=1}^{\infty}\) is a s.s.s. \(M\)-basis for \(X^*\). In view of Theorem 1, we thus have

**Corollary 1.** If \(X^*\) is separable and has the \(\lambda\)-m.a.p. for some \(\lambda \geq 1\), then \(X\) admits a shrinking s.s.s. \(M\)-basis.

Let us say that a s.s.s. \(M\)-basis \(\{x_i, f_i\}_{i=1}^{\infty}\) is **boundedly complete** provided there is a summability matrix \((\lambda_{i,n})\) for \(\{x_i, f_i\}_{i=1}^{\infty}\) such that for every sequence \(\{t_i\}_{i=1}^{\infty}\) of scalars, if \(\sum_{i=1}^{n} t_i \lambda_{i,n} x_i\sum_{i=1}^{\infty} x_i\) is bounded then it is convergent. A simple modification of Theorem II.3 of [3] shows that a s.s.s. \(M\)-basis is boundedly complete if and only if it is boundedly complete as an \(M\)-basis in the sense of [3] (and thus in the above definition of boundedly complete “there is a summability matrix” can be replaced by “for each summability matrix”). Thus using Corollary 1 and the results of [3] we have

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Theorem 2. $X$ admits a boundedly complete s.s.s. $M$-basis if and only if $X$ has the $\lambda$-m.a.p. for some $\lambda \geq 1$ and $X$ is isomorphic to a separable conjugate Banach space.

We conclude with a conjecture which, by Theorem 2, has an affirmative answer if $X$ is a conjugate space:

Conjecture 1. If $X$ is separable and has the $\lambda$-m.a.p. for some $\lambda \geq 1$, then $X$ admits a s.s.s. $M$-basis.

References


University of Houston,
Houston, Texas 77004