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# CHARACTERISTIC SUBGROUPS OF LATTICE-ORDERED GROUPS(1)

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**Abstract.** Characteristic subgroups of an $l$-group are those convex $l$-subgroups that are fixed by each $l$-automorphism. Certain sublattices of the lattice of all convex $l$-subgroups determine characteristic subgroups which we call socles. Various socles of an $l$-group are constructed and this construction leads to some structure theorems. The concept of a shifting subgroup is introduced and yields results relating the structure of an $l$-group to that of the lattice of characteristic subgroups. Interesting results are obtained when the $l$-group is characteristically simple. We investigate the characteristic subgroups of the vector lattice of real-valued functions on a root system and determine those vector lattices in which every $l$-ideal is characteristic. The automorphism group of the vector lattice of all continuous real-valued functions (almost finite real-valued functions) on a topological space (a Stone space) is shown to be a splitting extension of the polar preserving automorphisms by the ring automorphisms. This result allows us to construct characteristically simple vector lattices. We show that self-injective vector lattices exist and that an archimedean self-injective vector lattice is characteristically simple. It is proven that each $l$-group can be embedded as an $l$-subgroup of an algebraically simple $l$-group. In addition, we prove that each representable (abelian) $l$-group can be embedded as an $l$-subgroup of a characteristically simple representable (abelian) $l$-group.

1. **Introduction.** A convex $l$-subgroup $C$ of an $l$-group $G$ is called *characteristic* if $C\tau = C$ for each $l$-automorphism $\tau$ of $G$. In this paper we investigate the characteristic subgroups of $G$ and in the process construct various characteristically simple $l$-groups.

Let $\mathscr{C}(G)$, $\mathscr{L}(G)$ and $\mathscr{K}(G)$ be the lattices of all convex $l$-subgroups, $l$-ideals, and characteristic subgroups of $G$, respectively. Each lattice determines a socle for $G$—namely the cardinal sum of all the atoms. Each of these socles is a characteristic subgroup of $G$. Here the fact that complements of cardinal summands in an $l$-group are unique enables one to obtain much more theory than one gets for groups or

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abelian groups. In §2 we give a construction that produces all of the above socles and more. This leads to various structure theorems for $G$. For example, for an $l$-group $G$ (representable $l$-group $G$) the following are equivalent:

(a) $G$ is characteristically simple and contains a minimal $l$-ideal.

(b) $G$ is a cardinal sum of $l$-isomorphic simple $l$-groups (simple $o$-groups).

(c) $G$ is characteristically simple and completely reducible.

An element $C \in \mathscr{C}(G)$ is called *shifting* if $C\tau = C$ or $C\tau \cap C = 0$ for each $l$-automorphism $\tau$ of $G$. In particular, each characteristic subgroup is shifting and, if $C \in \mathscr{C}(G)$ is minimal with respect to being a convex $l$-subgroup, $l$-ideal, polar, lex-subgroup, or a cardinal summand, then $C$ is an $s$-subgroup. This concept allows us to identify characteristically simple $l$-groups. In fact, a characteristically simple $l$-group $G$ is a cardinal sum of $l$-isomorphic $s$-simple $s$-subgroups. Moreover, these summands together with $G$ and 0 are all the $s$-subgroups of $G$. Thus if $G$ is characteristically simple, then each $s$-subgroup is a cardinal summand, and we say that $G$ is *completely s-reducible.* In Theorem 3.23 we derive eight conditions each of which is equivalent to $G$ being completely $s$-reducible—for example, $\mathscr{K}(G)$ is a complete Boolean algebra; each characteristic subgroup is a cardinal summand; $G$ is a cardinal sum of $s$-simple $s$-subgroups.

In §4 we investigate characteristically simple $l$-groups. In order to identify such groups one needs to know something about the group $A(G)$ of all $l$-automorphisms of $G$. In §6 we investigate $A(G)$ when $G = C(X)$, the vector lattice of all continuous real-valued functions on a topological space $X$ or $G = D(X)$, the vector lattice of all almost finite real-valued functions on a Stone space $X$. In either case $A(G)$ is a splitting extension of the polar preserving automorphisms of $G$ by the ring automorphisms of $G$ and this result allows us to construct characteristically simple vector lattices.

One of the main results about abelian $l$-groups asserts that each such group can be embedded in a vector lattice $V = V(\Lambda, R_\lambda)$ of real-valued functions. In §5 we investigate the characteristic subgroups of $V$. We also determine those $V$'s for which every $l$-ideal is characteristic.

It is known that the category of vector lattices does not admit any injectives. In §7 we show that self-injectives do exist and that an archimedean self-injective vector lattice is characteristically simple. We also take a look at the relationship between hyper-archimedean and self-injective vector lattices.

In §8 we prove that each $l$-group $G$ can be embedded in an algebraically simple $l$-group and hence in a characteristically simple $l$-group. Moreover we show that each representable (abelian) $l$-group can be embedded in a representable (abelian) characteristically simple $l$-group. The last section consists of examples that show the extent as well as the limitations of our theory.

## 2. **The socles of an $l$-group.**

We will denote by $A(G)$ the group of $l$-automorphisms of the $l$-group $G$. A convex $l$-subgroup $C$ of $G$ is said to be *characteristic* if $C\tau = C$ for each $\tau \in A(G)$. $G$ is *characteristically simple* if the only characteristic subgroups of $G$ are $G$ and 0, and $G$ is *simple* if $G$ and 0 are the only $l$-ideals of $G$. If $\{G_\lambda \mid \lambda \in \Lambda\}$ is a collection of $l$-groups, then $\sum G_\lambda$ ($\prod G_\lambda$) denotes the cardinal sum (cardinal product) of the $l$-groups $G_\lambda$. In case $\Lambda = \{1, 2, \ldots, n\}$, we will write $G_1 \boxplus G_2 \boxplus \cdots \boxplus G_n$ instead of $\sum G_\lambda$. If $T$ is a subset of $G$, then $[T]$ denotes the subgroup of $G$ that is generated by $T$ and $T'$ denotes the *polar* of $T$, that is $T' = \{x \in G \mid |x| \wedge |t| = 0 \text{ for all } t \in T\}$. $G$ is said to be *representable* if there is an $l$-isomorphism of $G$ into a cardinal product of totally ordered groups. It is well known that an $l$-group $G$ is representable if and only if each (principal) polar of $G$ is an $l$-ideal. Throughout this paper $\mathscr{C}(G)$ ($\mathscr{L}(G)$, $\mathscr{K}(G)$) will denote the lattice of all convex $l$-subgroups ($l$-ideals, characteristic subgroups) of $G$.

Let $\mathscr{S}$ be a property possessed by certain convex $l$-subgroups of $G$; for example, the property of being normal, characteristic, a polar, etc. Let $\mathscr{S}(G)$ denote the collection of all convex $l$-subgroups with property $\mathscr{S}$. An element of $\mathscr{S}(G)$ will be called an *$\mathscr{S}$-subgroup*. If $\mathscr{S}(G)$ consists only of $G$ and 0, then we shall say that $G$ is *$\mathscr{S}$-simple*. For the remainder of this section we suppose that $\mathscr{S}(G)$ is a complete sublattice of $\mathscr{C}(G)$ containing $G$ and 0 and such that

(i) If $C \in \mathscr{S}(G)$, then $C' \in \mathscr{S}(G)$.

(ii) If $G = A \boxplus B$ and $C \in \mathscr{S}(G)$, then $C \cap A \in \mathscr{S}(A)$.

(iii) If $G = A \boxplus B$ and $A$ is an atom in $\mathscr{S}(G)$, then $A$ is $\mathscr{S}$-simple.

Let $\{C_i \mid i \in I\}$ be the set of all atoms in $\mathscr{S}(G)$. Then $C_i \cap C_j = 0$ for $i \neq j$, and hence $\bigvee C_i = \sum C_i$, and, of course, $\bigvee C_i \in \mathscr{S}(G)$. We shall call $\bigvee C_i$ the *$\mathscr{S}$-socle* of $G$.

PROPOSITION 2.1. *$\mathscr{C}(G)$, $\mathscr{L}(G)$, and $\mathscr{K}(G)$ are complete sublattices of $\mathscr{C}(G)$ containing $G$ and* 0 *and satisfying conditions* (i), (ii), *and* (iii) *above. Moreover*

(a) *Each atom in $\mathscr{C}(G)$ is an archimedean o-group and hence o-isomorphic to a subgroup of the reals.*

(b) *Each atom in $\mathscr{K}(G)$ is a characteristically simple l-group.*

(c) *If $A$ is an atom in $\mathscr{L}(G)$ and a cardinal summand of $G$, then $A$ is a simple l-group. If, in addition, $G$ is representable, then each atom in $\mathscr{L}(G)$ is an o-group.*

**Proof.** Clearly $\mathscr{C}(G)$ and $\mathscr{L}(G)$ satisfy (i), (ii), and (iii). If $C \in \mathscr{K}(G)$ and $\tau \in A(G)$, then $C'\tau = (C\tau)' = C'$ and hence (i) is satisfied. Suppose that $G = A \boxplus B$, let $C \in \mathscr{K}(G)$, and let $\alpha$ be an $l$-automorphism of $A$. Let $\tau$ be the extension of $\alpha$ to $G$ such that $\tau$ induces the identity on $B$. Then $(A \cap C)\alpha = (A \cap C)\tau = A\tau \cap C\tau = A \cap C$. Thus $A \cap C \in \mathscr{K}(A)$ and hence (ii) is satisfied. Since each atom in $\mathscr{K}(G)$ is characteristically simple, (iii) is satisfied.

(a), (b), and the first part of (c) are clear. Suppose that $G$ is representable and let $A$ be an atom in $\mathscr{L}(G)$. If $0 < a \in A$, then $a'$ is an $l$-ideal and $a \notin a'$. Thus $A \cap a' = 0$. It follows that $A$ contains no pair of disjoint elements and is therefore an $o$-group.

THEOREM 2.2. *For an l-group $G$, the following are equivalent*:

(a) *Each $C \in \mathscr{S}(G)$ is a cardinal summand.*

(b) *$G$ is the $\mathscr{S}$-socle of $G$.*

(c) *$G$ is a cardinal sum of $\mathcal{S}$-simple l-groups.*

(d) *$G$ is a join of atoms from $\mathcal{S}(G)$.*

(e) *$\mathcal{S}(G)$ is a complete, atomic, Boolean algebra.*

(f) *$\mathcal{S}(G)$ is a Boolean algebra.*

**Proof.** (a) implies (b). Suppose (by way of contradiction) that $G \neq S$ where $S$ denotes the $\mathcal{S}$-socle of $G$. Let $a \in G \backslash S$ and let $T$ be maximal in $\mathcal{S}(G)$ with respect to $S \subseteq T$ and $a \notin T$; and let $R$ be the intersection of all members of $\mathcal{S}(G)$ that properly contain $T$. Then $R \in \mathcal{S}(G)$. Now $G = T \boxplus K$ and so $R = (R \cap T) \boxplus (R \cap K) = T \boxplus (R \cap K)$. Clearly $(R \cap K)$ must be an atom in $\mathcal{S}(G)$, but this implies that $R \cap K \subseteq S \subseteq T$, a contradiction. Thus $S = G$.

(b) implies (c). This is an immediate consequence of (iii).

(c) implies (d). Suppose that $G = \sum G_\lambda$ $(\lambda \in \Lambda)$, where each $G_\lambda$ is an $\mathcal{S}$-simple $l$-group. If $T \in \mathcal{S}(G)$, then $T = \sum (G_\lambda \cap T)$ $(\lambda \in \Lambda)$. By (ii), $G_\lambda \cap T \in \mathcal{S}(G_\lambda)$ and, since $G_\lambda$ is $\mathcal{S}$-simple, we have for each $\lambda \in \Lambda$ that $G_\lambda \cap T = 0$ or $G_\lambda \cap T = G_\lambda$. Let $\lambda_0 \in \Lambda$ and let $T_{\lambda_0}$ be the $\mathcal{S}$-subgroup of $G$ generated by $G_{\lambda_0}$. Then $T_{\lambda_0} = \sum G_\delta$ $(\delta \in \Delta \subseteq \Lambda)$. We prove that $T_{\lambda_0}$ is an atom in $\mathcal{S}(G)$. Suppose (by way of contradiction) that $C \in \mathcal{S}(G)$ and $0 \neq C \subset T_{\lambda_0}$. Then $C = \sum G_\gamma$ $(\gamma \in \Gamma,\ \Gamma \subseteq \Delta)$ and $\lambda_0 \notin \Gamma$. By (i) $C' = \sum G_\lambda$ $(\lambda \in \Lambda \backslash \Gamma) \in \mathcal{S}(G)$, and so $T_{\lambda_0} \subseteq C'$. From this we conclude that $C \subset C'$, which is a contradiction since $C \neq 0$. Thus $T_{\lambda_0}$ is an atom in $\mathcal{S}(G)$. It follows that $G$ is the join of atoms in $\mathcal{S}(G)$.

(d) implies (e). If $S_i$ and $S_j$ are distinct atoms in $\mathcal{S}(G)$, then $S_i \cap S_j = 0$ and so $G = \sum S_j$ $(j \in J)$, where $\{S_j \mid j \in J\}$ is the set of all atoms in $\mathcal{S}(G)$. If $T \in \mathcal{S}(G)$, then $T = \sum (S_j \cap T)$ $(j \in J)$ and $S_j \cap T = 0$ or $S_j \cap T = S_j$. Thus $\mathcal{S}(G)$ is isomorphic to the set of all subsets of $J$.

The implications (e) implies (f) and (f) implies (a) are trivial.

Note that we do not assume that the summands of part (c) of Theorem 2.2 are elements of $\mathcal{S}(G)$ (see Example 9.2). We leave it to the reader to formulate the special cases of the theorem when $\mathcal{S}(G) = \mathcal{C}(G)$, $\mathcal{L}(G)$, or $\mathcal{K}(G)$. Note that the atoms in each of these three cases are described in Proposition 2.1.

An $l$-group $G$ is said to be *completely reducible* if each $l$-ideal of $G$ is a cardinal summand. Thus we have shown that $G$ is completely reducible if and only if $G$ is a cardinal sum of simple $l$-groups. If this is the case, then $G$ is abelian if and only if $\mathcal{C}(G) = \mathcal{L}(G)$. An abelian $l$-group is completely reducible if and only if it is a cardinal sum of subgroups of the reals. A representable $l$-group is completely reducible if and only if it is a cardinal sum of simple $o$-groups. These facts are, of course, immediate consequences of Proposition 2.1 and Theorem 2.2.

Corollary 2.3. *For an l-group $G$, the following are equivalent:*

(a) *$G$ is characteristically simple and $\mathcal{C}(G)$ contains an atom.*

(b) *$G$ is a cardinal sum of o-isomorphic archimedean o-groups.*

(c) *$G$ is characteristically simple and each convex l-subgroup of $G$ is a cardinal summand.*

**Proof.** (a) implies (b). Let $\mathscr{S}(G)=\mathscr{C}(G)$. If $S$ is the $\mathscr{S}$-socle of $G$, then $S\neq 0$ and is a characteristic subgroup of $G$. Hence $G=S=\sum S_i$, where $\{S_i \mid i\in I\}$ is the set of atoms of $\mathscr{C}(G)$. By Proposition 2.1, each $S_i$ is an archimedean $o$-group and, since $G$ is characteristically simple, $S_i$ is $o$-isomorphic to $S_j$ for $i, j\in I$.

The implications (b) implies (c) and (c) implies (a) follow from the theorem.

COROLLARY 2.4. *For an l-group G (representable l-group G) the following are equivalent:*

(a) *G is characteristically simple and contains a minimal l-ideal.*

(b) *G is a cardinal sum of l-isomorphic simple l-groups (simple o-groups).*

(c) *G is characteristically simple and completely reducible.*

**Proof.** Let $\mathscr{S}(G)=\mathscr{L}(G)$ and proceed as in Corollary 2.3.

COROLLARY 2.5. *For an l-group G, the following are equivalent:*

(a) *G is characteristically simple and contains an atom.*

(b) *G is a cardinal sum of cyclic o-groups.*

**Proof.** (a) implies (b). If $0<x$ is an atom in $G$, then $[x]$ is an atom in $\mathscr{C}(G)$ and hence, by Corollary 2.3, $G=\sum S_i$, where the $S_i$'s are $o$-isomorphic archimedean $o$-groups, and clearly one of them is $[x]$.

The implication (b) implies (a) is trivial.

THEOREM 2.6. *If there exists a set of maximal $\mathscr{S}$-subgroups of G which are polars and whose intersection is zero, then there exists an l-isomorphism $\tau$ of G such that*

$$\sum G_\lambda \subseteq G\tau \subseteq \prod G_\lambda$$

*where each $G_\lambda$ is an atom in $\mathscr{S}(G)$. In particular, $\sum G_\lambda$ is the $\mathscr{S}$-socle of G.*

**Proof.** Suppose that $M=M''$ is a maximal $\mathscr{S}$-subgroup. Then $M'\neq 0$ and $M\subset M \boxplus M'\in\mathscr{S}(G)$. Since $M$ is maximal in $\mathscr{S}(G)$, we conclude that $G=M\boxplus M'$. Now let $\{G^\lambda \mid \lambda\in\Lambda\}$ be a collection of maximal $\mathscr{S}$-subgroups such that $G^\lambda=(G^\lambda)''$ and $\bigcap G^\lambda=0$. For each $\lambda\in\Lambda$, $G=G_\lambda\boxplus G^\lambda$ where clearly $G_\lambda$ is an atom in $\mathscr{S}(G)$. Each $g\in G$ has a unique representation of the form $g=g_\lambda+g^\lambda$, where $g_\lambda\in G_\lambda$ and $g^\lambda\in G^\lambda$. Then the mapping $\tau$ given by $g\to g\tau=(\ldots, g_\lambda,\ldots)$ is an $l$-isomorphism of $G$ into $\prod G_\lambda$. If $\alpha$ and $\beta$ are distinct members of $\Lambda$, then $G_\alpha\cap G_\beta=0$ since $G_\alpha$ and $G_\beta$ are distinct atoms in $\mathscr{S}(G)$. Let $g\in G_\alpha$ and $\lambda\in\Lambda$ ($\lambda\neq\alpha$). Then $G_\alpha\cap G_\lambda=0$ implies that $G_\alpha\subseteq G_\lambda'=G^\lambda$. Thus $g=g_\lambda+g^\lambda=0+g$. Therefore $\sum G_\lambda\subseteq G\tau$.

We give a method for producing other $\mathscr{S}$-socles for an $l$-group $G$. Let $S$ be a function that assigns to each subgroup $C$ of $G$ a subgroup $S(C)$ of $A(C)$ such that

(a) If $G=C\boxplus D$ and $\alpha\in S(C)$, then $\alpha$ can be extended to an element of $S(G)$ that is the identity on $D$.

(b) If $G=C\boxplus D$ and $C\tau=C$ for some $\tau\in S(G)$, then $\tau|C\in S(C)$.

PROPOSITION 2.7. *Let S be defined as above and let $\mathscr{S}(G)=\{C\in\mathscr{C}(G) \mid C\tau=C$ for each $\tau\in S(G)\}$. Then $\mathscr{S}(G)$ is a complete sublattice of $\mathscr{C}(G)$ that contains G and* 0 *and satisfies* (i), (ii), *and* (iii). *Moreover, if* $S_1$ *and* $S_2$ *satisfy* (a) *and* (b) *above and if* $\mathscr{S}_i(G)=\{C \in \mathscr{C}(G) \mid C\tau = C \text{ for each } \tau \in S_i(G)\}$ $(i=1, 2)$, *then* $\mathscr{S}_1(G) \cap \mathscr{S}_2(G)$ *satisfies the conclusions of the proposition.*

**Proof.** Note that $\mathscr{K}(G) \subseteq \mathscr{S}(G)$, hence $G$ and $0$ are elements of $\mathscr{S}(G)$. If $\{C_i \mid i \in I\} \subseteq \mathscr{S}(G)$ and $\tau \in S(G)$, then $(\bigvee C_i)\tau = \bigvee (C_i\tau) = \bigvee C_i$ and $(\bigwedge C_i)\tau = \bigwedge (C_i\tau) = \bigwedge C_i$. Therefore $\mathscr{S}(G)$ is a complete sublattice of $\mathscr{C}(G)$.

If $C \in \mathscr{S}(G)$, then $C\tau = C$ for all $\tau \in S(G)$ and so $C'\tau = (C\tau)' = C'$ for all $\tau \in S(G)$ and hence $C' \in \mathscr{S}(G)$. Let $G = A \boxplus B$, $C \in \mathscr{S}(G)$, and $\alpha \in S(A)$. Then, by (a), $\alpha$ can be extended to an element $\tau \in S(G)$ that is the identity on $B$. Thus $(C \cap A)\alpha = (C \cap A)\tau = C\tau \cap A\tau = C \cap A$. Therefore $C \cap A \in \mathscr{S}(A)$. Suppose that $G = A \boxplus B$ and that $A$ is an atom in $\mathscr{S}(G)$. Let $0 \neq D \in \mathscr{S}(A)$ and $\tau \in S(G)$. Then $\alpha = \tau|A \in S(A)$ and so $D\tau = (D \cap A)\tau = (D \cap A)\alpha = D\alpha \cap A\alpha = D \cap A = D$. Thus $D \in \mathscr{S}(G)$ and it follows that $D = A$. Therefore $A$ is $\mathscr{S}$-simple.

Clearly $\mathscr{S}_1(G) \cap \mathscr{S}_2(G)$ is a complete sublattice of $\mathscr{C}(G)$ containing $0$ and $G$ and satisfying (i), (ii), and (iii).

Note that if $S(G) = A(G)$, then $\mathscr{S}(G) = \mathscr{K}(G)$; if $S(G) = I(G)$, the group of inner automorphisms of $G$, then $\mathscr{S}(G) = \mathscr{L}(G)$; and if $S(G)$ consists only of the identity of $A(G)$, then $\mathscr{S}(G) = \mathscr{C}(G)$. Another example of $S(G)$ that satisfies (a) and (b) is

$$P(G) = \{\alpha \in A(G) \mid x \wedge y = 0 \text{ implies } x \wedge y\alpha = 0 \text{ for all } x, y \in G\}.$$

This is the group of polar preserving automorphisms of $G$ which we will investigate at some length in §6.

In the following proposition, we use the notation established above.

PROPOSITION 2.8. *If* $S(G) = P(G)$, *then the following are equivalent*:

(1) *$G$ is the $\mathscr{S}$-socle of $G$.*

(2) *$G$ is the cardinal sum of characteristically simple o-groups.*

(3) *$\mathscr{S}(G)$ consists of polars.*

(4) *$\mathscr{S}(G)$ is a complete, atomic, Boolean algebra.*

**Proof.** (1) implies (2). $G = \sum G_i$ $(i \in I)$ where each $G_i$ is an atom in $\mathscr{S}(G)$. Thus each atom is a polar and hence a minimal polar, but a minimal polar in an $l$-group is an $o$-group. Thus each $G_i$ is an $o$-group. Each characteristic subgroup of $G_i$ belongs to $\mathscr{S}(G)$ and so $G_i$ is characteristically simple.

(2) implies (3). If $H$ is an $o$-group, then $P(H) = A(H)$. Thus a characteristically simple $o$-group is $\mathscr{S}$-simple. By Theorem 2.2, each $C$ in $\mathscr{S}(G)$ is a cardinal summand and hence a polar.

(3) implies (4). $\mathscr{S}(G)$ is the set of all polars and also a complete sublattice of $\mathscr{C}(G)$. But the collection of polars is a Boolean algebra, and hence, by Theorem 2.2, $\mathscr{S}(G)$ is a complete, atomic, Boolean algebra.

(4) implies (1). This is immediate from Theorem 2.2.

Note that if (1) through (4) hold, then each polar of $G$ is a cardinal summand and so the polars form a sublattice of $\mathscr{L}(G)$. The following example shows that this condition does not, in general, imply that the polars form a complete sublattice of $\mathscr{L}(G)$. Let $G=\prod G_\lambda$ $(\lambda \in \Lambda)$, where $\Lambda$ is an infinite set and each $G_\lambda$ is a nonzero $o$-group. For each $\lambda \in \Lambda$, let

$$C_\lambda = \{g \in G \mid g_\gamma = 0 \text{ for } \gamma \neq \lambda\}.$$

Then $\bigvee C_\lambda = \sum G_\lambda$ and is not a polar, and so the polars do not form a complete sublattice of $\mathscr{L}(G)$. Consider a polar $T$ and let

$$\Delta = \{\lambda \in \Lambda \mid \text{the projection of } T \text{ into } G_\lambda \text{ is not zero}\}.$$

Then $T \subseteq \{g \in G \mid g_\lambda = 0 \text{ for all } \lambda \in \Lambda \backslash \Delta\} = (\bigvee C_\delta)''$ $(\delta \in \Delta)$. If $0 < t \in T$ with $t_\delta > 0$, then $\delta \in \Delta$, and since $T$ is convex, $g = (\ldots, 0, t_\delta, 0, \ldots)$ belongs to $T$. Thus $C_\delta = g'' \subseteq T$, hence $T \supseteq \bigvee C_\delta$ and so $T = T'' \supseteq (\bigvee C_\delta)''$. Therefore $T$ is a cardinal summand of $G$.

Proposition 2.9. (1) *If $S(G)$ is a normal subgroup of $A(G)$, then each $l$-automorphism of $G$ permutes the elements in $\mathscr{S}(G)$, and, in particular, the $\mathscr{S}$-socle is characteristic.*

(2) *If each $l$-automorphism of $G$ permutes the elements in $\mathscr{S}(G)$, then $S(G)$ and the normal subgroup of $A(G)$ that is generated by $S(G)$ determine the same $\mathscr{S}(G)$.*

**Proof.** (1) Let $\tau \in A(G)$ and $C \in \mathscr{S}(G)$. If $\sigma \in S(G)$, then $\tau\sigma\tau^{-1} \in S(G)$ and so $C\tau\sigma\tau^{-1} = C$. Therefore $C\tau\sigma = C\tau$ for all $\sigma \in S(G)$ and so $C\tau \in \mathscr{S}(G)$.

(2) If $\sigma \in S(G)$, $\tau \in A(G)$, and $C \in \mathscr{S}(G)$, then $C\tau\sigma\tau^{-1} = C\tau\tau^{-1} = C$ since $C\tau \in \mathscr{S}(G)$ by our hypothesis. Thus each conjugate of an element in $S(G)$ fixes each element of $\mathscr{S}(G)$ and the desired conclusion follows.

Remarks. (1) For all of our examples $S(G)$ is normal in $A(G)$.

(2) If $S(G) = \{\alpha \in A(G) \mid G(g)\alpha = G(g) \text{ for all } g \in G\}$, the group of generalized contractors, then $\mathscr{S}(G) = \mathscr{C}(G)$ and each subgroup of $S(G)$ gives the same $\mathscr{S}(G)$.

We conclude this section with a construction that is essentially given in [19] and will yield a dual Galois correspondence between certain characteristic subgroups of an $l$-group $G$ and the normal subgroups of $A(G)$.

If $H$ is a subgroup of $G$, let

$$H\mu = \{\tau \in A(G) \mid -x + x\tau \in H \text{ for all } x \in G\},$$

and if $K$ is a subgroup of $A(G)$, let

$$K\nu = [\{-x + x\tau \mid x \in G \text{ and } \tau \in K\}],$$

and let $K\rho$ be the convex $l$-subgroup of $G$ generated by $K\nu$. The verification of the next four propositions are similar to those given in [19] and will be omitted.

2.10. *$H\mu$ is a subgroup of $A(G)$.*

2.11. *$K\rho$ is an $l$-ideal of $G$.*

2.12. *If $K$ is a normal subgroup of $A(G)$, then $K\rho$ is a characteristic subgroup of $G$.*

2.13. *If $H$ is a characteristic subgroup of $G$, then $H\mu$ is a normal subgroup of $A(G)$.*

Note that for any subgroup $K$ of $A(G)$, $K\rho\mu \supseteq K$ and, for any convex $l$-subgroup $H$ of $G$, $H\mu\rho \subseteq H$. It follows that $K\rho\mu\rho = K\rho$ and $H\mu\rho\mu = H\mu$. Thus we have the following theorem.

THEOREM 2.14. *There is a one-to-one correspondence between the characteristic subgroups of $G$ of the form $K\rho$ and the normal subgroups of $A(G)$ of the form $H\mu$.*

3. **Shifting subgroups.** A convex $l$-subgroup $C$ of $G$ is called a *shifting subgroup* ($s$-subgroup) if for each $\tau \in A(G)$ either $C\tau = C$ or $C\tau \cap C = 0$. $G$ is said to be *$s$-simple* if $G$ and $0$ are the only $s$-subgroups of $G$. $G$ is *completely $s$-reducible* if each $s$-subgroup of $G$ is a cardinal summand. Clearly any characteristic subgroup is an $s$-subgroup. In addition, if $C \in \mathscr{C}(G)$ and $C$ is minimal with respect to being a convex $l$-subgroup, $l$-ideal, polar, lex-subgroup (defined in §4), or cardinal summand, then $C$ is an $s$-subgroup.

We list below several assertions concerning $s$-subgroups, most of which are easily proven.

3.1. If $C$ is an $s$-subgroup of $G$ and if $\tau \in A(G)$, then $C\tau$ is an $s$-subgroup of $G$.

3.2. If $D$ is an $s$-subgroup of $C$ and $C$ is an $s$-subgroup of $G$, then $D$ is an $s$-subgroup of $G$.

3.3. If $C$ is an $s$-subgroup of $G$, then so is $C''$.

3.4. If $G = A \boxplus B$ and $C$ is an $s$-subgroup of $G$, then $C \cap A$ is an $s$-subgroup of $A$. Thus the set $\mathscr{D}(G)$ of all $s$-subgroups of $G$ satisfies (ii) and (iii) of our definition of $\mathscr{S}(G)$.

3.5. The intersection of an arbitrary collection of $s$-subgroups of $G$ is an $s$-subgroup. Thus $\mathscr{D}(G)$ is a complete lattice with respect to inclusion. In general $\mathscr{D}(G)$ is neither modular nor a sublattice of $\mathscr{C}(G)$.

3.6. If $C$ is a nonzero characteristic subgroup of $G$ and if $D$ is an $s$-subgroup of $G$ containing $C$, then $D$ is characteristic. Thus the collection of nonzero characteristic subgroups of $G$ is a dual ideal of the lattice $\mathscr{D}(G)$.

3.7. If $C$ is an $s$-subgroup of $G$, then $\bigvee \{C\tau \mid \tau \in A(G)\} = \sum C\tau_i$ $(i \in I)$ where $\{\tau_i \mid i \in I\}$ is a system of representatives of the cosets of that subgroup of $A(G)$ consisting of those $l$-automorphisms that fix $C$.

3.8. If $G$ is simple, then $G$ is $s$-simple. If $G$ is $s$-simple, then $G$ is characteristically simple.

**Proof.** If $C$ is a nonzero $s$-subgroup of a simple $l$-group $G$, then $\bigvee \{C\tau \mid \tau \in A(G)\}$ is a characteristic subgroup of $G$. Thus $G = \sum C\tau_i$ (as in 3.7) and so $C$ is a cardinal summand of $G$. Therefore $C$ is normal in $G$ and so $C = G$. The second assertion is clear.

3.9. A characteristically simple $o$-group is $s$-simple and conversely.

3.10. If $G$ is completely reducible and $s$-simple, then $G$ is simple.

**Proof.** Suppose (by way of contradiction) that $A$ is a proper $l$-ideal of $G$ and choose $g \in G \backslash A$. Let $M$ be an $l$-ideal that is maximal with respect to $A \subseteq M$ and $g \notin M$, and let $K$ be the intersection of all $l$-ideals of $G$ that properly contain $M$.

Now $G=M \boxplus L$ and so $K=M \boxplus (K \cap L)$. Clearly $K \cap L$ is a minimal $l$-ideal of $G$ and hence an $s$-subgroup. From this we conclude that $G$ is simple.

3.11. If $G$ is completely reducible, then $G$ is completely $s$-reducible.

**Proof.** Let $C$ be a proper $s$-subgroup of $G$. Then $K=\bigvee \{C\tau \mid \tau \in A(G)\}=\sum C\tau_i$ is an $l$-ideal of $G$ and hence $G=K \boxplus D$. Thus $C$ is a cardinal summand of $G$.

3.12. A prime $s$-subgroup $C$ is either characteristic or totally ordered. ($M \in \mathscr{C}(G)$ is *prime* if $a, b \in G^+\backslash M$ implies $a \wedge b > 0$.)

**Proof.** If $\tau \in A(G)$ and $C \cap C\tau=0$, then $C\tau \subseteq C'$ and so (see [11, Theorem 2.1]) $C'$ is totally ordered.

An element $C$ of $\mathscr{C}(G)$ is said to be *closed* if for each subset $\{g_i \mid i \in I\}$ of $C$ for which $g=\bigvee g_i$ $(i \in I)$ exists in $G$, it follows that $g \in C$. In particular, each polar is closed. Also if $D \in \mathscr{C}(G)$, then the intersection of all closed subgroups of $G$ containing $D$ is closed, and is called the *closure* of $D$.

3.13. The closure of an $s$-subgroup (characteristic subgroup) is an $s$-subgroup (characteristic subgroup).

**Proof.** Let $C$ be an $s$-subgroup and let $D$ be the closure of $C$. Then $D^+=\{g \in G \mid g=\bigvee g_i$ for some subset $\{g_i \mid i \in I\}$ of $C^+\}$ [7, Lemma 3.2]. Suppose that $\tau \in A(G)$ is such that $D \cap D\tau \neq 0$. If $0<d \in D \cap D\tau$, then there exists $\{c_i \mid i \in I\} \subseteq C^+$ and $\{d_j \mid j \in J\} \subseteq C^+$ such that $d=\bigvee c_i$ $(i \in I)$ and $d=\bigvee (d_j\tau)$ $(j \in J)$. Thus $0<d=d \wedge d=\bigvee (c_i \wedge d_j\tau)$ $(i \in I, j \in J)$. Clearly then $C \cap C\tau \neq 0$, for otherwise $d=0$. Thus $C=C\tau$ and so $D=D\tau$.

THEOREM 3.14. *If $G$ is a characteristically simple $l$-group, then $G=\sum C_\lambda$ $(\lambda \in \Lambda)$ where the $C_\lambda$'s are $l$-isomorphic, $s$-simple $s$-subgroups of $G$. Thus the proper $s$-subgroups are trivially ordered and consist of $\{C_\lambda \mid \lambda \in \Lambda\}$ if $|\Lambda| \geqq 2$.*

**Proof.** If $G$ is $s$-simple, the theorem holds. Otherwise, let $C$ be a proper $s$-subgroup of $G$. Since $\bigvee \{C\tau \mid \tau \in A(G)\}$ is a characteristic subgroup of $G$, $G=\sum C\tau_i$ where $\{\tau_i \mid i \in I\}$ is a subset of $A(G)$. Suppose (by way of contradiction) that $K$ is a proper $s$-subgroup of $G$ that properly contains $C$. If $i \in I$ and if $C\tau_i \cap K \neq 0$, then $K\tau_i \cap K \neq 0$ and so $K=K\tau_i$, whence $C\tau_i \subseteq K$. Now

$$\begin{aligned} K &= \sum (K \cap C\tau_i) && (i \in I) \\ &= \sum (K \cap C\tau_j) = \sum C\tau_j && (j \in J \subseteq I) \end{aligned}$$

where $J=\{i \in I \mid C\tau_i \cap K \neq 0\}$. Since $C \subset K \subset G$, we have $|J| \geqq 2$ and $J \subset I$. Let $j \in J$ and $i \in I\backslash J$. Then the transposition $(i, j)$ induces an $l$-automorphism $\tau$ of $G$ such that $0 \neq K \cap K\tau \neq K$, contradicting the assumption that $K$ is an $s$-subgroup of $G$.

The above argument shows that the proper $s$-subgroups of $G$ are trivially ordered. Thus, because of 3.2, we have that each proper $s$-subgroup is $s$-simple. All of the conclusions of the theorem then follow.

COROLLARY 3.15. *A characteristically simple $l$-group which is cardinally indecomposable is $s$-simple.*

COROLLARY 3.16. *If $G$ is characteristically simple, then $G$ is completely $s$-reducible.*

COROLLARY 3.17. *If $C$ and $D$ are proper s-subgroups of $G$ with $C \subset D$, then $G$ is not characteristically simple. In fact, $\bigvee \{C\tau \mid \tau \in A(G)\}$ is a proper characteristic subgroup of $G$.*

COROLLARY 3.18. *If $G$ is characteristically simple and contains a nonzero abelian s-subgroup $C$, then $G=\sum C_\lambda$, where the $C_\lambda$'s are l-isomorphic, abelian, s-simple l-groups. In particular, $G$ is abelian.*

THEOREM 3.19. *Suppose the collection $\{C_\lambda \mid \lambda \in \Lambda\}$ of all proper s-subgroups of $G$ is trivially ordered and that $|\Lambda| \geqq 2$. Then either $G$ is characteristically simple or $G=C \boxplus D$, where $C$ and $D$ are s-simple characteristic subgroups of $G$ that are not l-isomorphic, and $|\Lambda|=2$.*

**Proof.** For $\gamma, \lambda \in \Lambda$ $(\gamma \neq \lambda)$, $C_\gamma \cap C_\lambda = 0$. Hence $\bigvee \{C_\lambda \mid \lambda \in \Lambda\} = \sum C_\lambda$ $(\lambda \in \Lambda)$. Moreover, $\sum C_\lambda$ is a characteristic subgroup of $G$. Since $|\Lambda| \geqq 2$ and $\{C_\lambda \mid \lambda \in \Lambda\}$ is trivially ordered, we have that $G=\sum C_\lambda$. Suppose that $C \in \{C_\lambda \mid \lambda \in \Lambda\}$ is not characteristic. Then $G=\sum C\tau_i$ where $\{\tau_i \mid i \in I\} \subseteq A(G)$, and it follows that there is a one-to-one correspondence $f$ between $I$ and $\Lambda$ such that $C\tau_i = C_{if}$. Since $C\tau_i$ is not characteristic for each $i \in I$, it follows that $G$ is characteristically simple.

If some $C$ in $\{C_\lambda \mid \lambda \in \Lambda\}$ is characteristic, then from the above each $C_\lambda$ $(\lambda \in \Lambda)$ is characteristic. Since $\{C_\lambda \mid \lambda \in \Lambda\}$ is trivially ordered, $\{C_\lambda \mid \lambda \in \Lambda\} = \{C, D\}$ where $C$ and $D$ are $s$-simple $l$-groups that are not $l$-isomorphic and $G=C \boxplus D$.

THEOREM 3.20. *For each $\lambda$ in an indexing set $\Lambda$, let $G_\lambda$ be a completely s-reducible l-group. Then $G=\sum G_\lambda$ $(\lambda \in \Lambda)$ is completely s-reducible.*

**Proof.** Let $C$ be a nonzero $s$-subgroup of $G$ and let $\Gamma = \{\lambda \in \Lambda \mid G_\lambda \cap C \neq 0\}$. For each $\gamma \in \Gamma$, $C \cap G_\gamma$ is an $s$-subgroup of $G_\gamma$, since each element of $A(G_\gamma)$ can be extended to an element of $A(G)$. Therefore $G_\gamma = (C \cap G_\gamma) \boxplus D_\gamma$ for some $l$-ideal $D_\gamma$ of $G_\gamma$. Thus

$$\begin{aligned} G &= \sum_{(\lambda \in \Lambda \setminus \Gamma)} G_\lambda \boxplus \sum_{(\gamma \in \Gamma)} G_\gamma \\ &= \sum_{(\lambda \in \Lambda \setminus \Gamma)} G_\lambda \boxplus \sum_{(\gamma \in \Gamma)} ((C \cap G_\gamma) \boxplus D_\gamma) = \sum_{\lambda \in \Lambda \setminus \Gamma} G_\lambda \boxplus \sum_{\gamma \in \Gamma} D_\gamma \boxplus C. \end{aligned}$$

Thus $C$ is a cardinal summand of $G$ and hence $G$ is completely $s$-reducible.

THEOREM 3.21. *Let $H$ be a characteristically simple l-group and, for each $\lambda \in \Lambda$, let $H_\lambda = H$. Then $G=\sum H_\lambda$ $(\lambda \in \Lambda)$ is characteristically simple. If $H$ is s-simple and $G$ is not s-simple, then $\{H_\lambda \mid \lambda \in \Lambda\}$ is the collection of proper s-subgroups of $G$. If $|\Lambda| > 1$ and $H$ is simple, then each $H_\lambda$ is an s-subgroup of $G$ and hence $G$ is not s-simple.*

**Proof.** If $K$ is a nonzero characteristic subgroup of $G$, then $K \cap H_\lambda \neq 0$ for some $\lambda \in \Lambda$. Since $K \cap H_\lambda$ is a characteristic subgroup of $H_\lambda$, we have that $K \cap H_\lambda = H_\lambda$. For any $\gamma \in \Lambda$, there exists $\tau \in A(G)$ such that $H_\lambda \tau = H_\gamma$. Thus $H_\gamma \subseteq K$ for all $\gamma \in \Lambda$ and so $G=K$.

Next suppose that $H$ is $s$-simple and that $C$ is a proper $s$-subgroup of $G$. Then $C \cap H_\lambda \neq 0$ for some $\lambda \in \Lambda$ and since $C \cap H_\lambda$ is an $s$-subgroup of $H_\lambda$, we have that $C \cap H_\lambda = H_\lambda$. $C$ can have a nontrivial intersection with only one $H_\lambda$, for otherwise there exists an $l$-automorphism $\tau$ of $G$ such that $0 \neq C\tau \cap C \neq C$. Thus $C = H_\lambda$. Since each permutation of $\Lambda$ induces an $l$-automorphism of $G$, it follows that $\{H_\lambda \mid \lambda \in \Lambda\}$ is the collection of proper $s$-subgroups of $G$.

If $|\Lambda| > 1$ and $H$ is simple, then for each $\tau \in A(G)$, $H_\lambda\tau \cap H_\lambda$ is an $l$-ideal of $H_\lambda$. Thus $H_\lambda\tau \cap H_\lambda = 0$ or $H_\lambda\tau \cap H_\lambda = H_\lambda$ and so each $H_\lambda$ is an $s$-subgroup of $G$.

THEOREM 3.22. *An $l$-group $G$ is completely $s$-reducible if and only if $G = \sum C_\lambda$ ($\lambda \in \Lambda$), where each $C_\lambda$ is an $s$-simple $s$-subgroup of $G$.*

**Proof.** If $G = \sum C_\lambda$ ($\lambda \in \Lambda$) where each $C_\lambda$ is $s$-simple, then by Theorem 3.20, $G$ is completely $s$-reducible.

For the converse, we may suppose that $G$ is not $s$-simple; for otherwise the desired conclusion follows easily. We first show that $G$ must contain proper $s$-simple $s$-subgroups. If $G$ is characteristically simple, then, by Theorem 3.14, $G$ has proper $s$-simple $s$-subgroups. If $G$ is not characteristically simple, then $G$ contains characteristic subgroups $K$ and $L$ such that $0 \neq K \neq G$ and $L$ covers $K$ in the lattice of characteristic subgroups of $G$. Since $G$ is completely $s$-reducible, $G = K \boxplus K_1$ and so $L = K \boxplus (L \cap K_1)$. Clearly $L \cap K_1$ is a proper characteristic subgroup of $G$ and is characteristically simple. If $L \cap K_1$ is $s$-simple, we have a proper $s$-simple $s$-subgroup of $G$. If $L \cap K_1$ is not $s$-simple, let $S$ be a proper $s$-subgroup of $L \cap K_1$. Then, by Theorem 3.14, $S$ is $s$-simple. Clearly $S$ is an $s$-subgroup of $G$.

Now let $\mathscr{M} = \{C_\lambda \mid \lambda \in \Lambda\}$ be the collection of all proper $s$-simple $s$-subgroups of $G$. Then, since each $C_\lambda$ is a cardinal summand of $G$, it follows that $\mathscr{M}$ is a disjoint collection, that is $C_\lambda \cap C_\gamma = 0$ if $\lambda, \gamma \in \Lambda$, $\lambda \neq \gamma$. Thus $M = \bigvee \mathscr{M} = \sum C_\lambda$ ($\lambda \in \Lambda$). Clearly $M$ is a characteristic subgroup of $G$, and so $G = M \boxplus M_1$ where $M_1$ is also characteristic. Since $M_1$ is also completely $s$-reducible, $M_1$ is either $s$-simple or contains proper $s$-subgroups of $G$. The latter case is contradictory and so $M_1$ is $s$-simple. If $M_1 \neq 0$, then $M_1 \in \mathscr{M}$ and so $M_1 \subseteq M$, another contradiction. Thus $M_1 = 0$ and $G = M = \sum C_\lambda$ where each $C_\lambda$ is an $s$-simple $s$-subgroup of $G$.

THEOREM 3.23. *For an $l$-group $G$, the following are equivalent:*

(a) *$G$ is completely $s$-reducible.*

(b) *$G = \sum C_\lambda$ ($\lambda \in \Lambda$), where each $C_\lambda$ is an $s$-simple $s$-subgroup of $G$.*

(c) *Each characteristic subgroup is a cardinal summand.*

(d) *$\mathscr{D}(G)$ is a complemented lattice.*

(e) *$G = \sum K_\delta$, where $\{K_\delta \mid \delta \in \Delta\}$ is the set of minimal characteristic subgroups of $G$.*

(f) *$G$ is a cardinal sum of characteristically simple $l$-groups.*

(g) *$G$ is the join of minimal characteristic subgroups.*

(h) *$\mathscr{K}(G)$ is a complete, atomic, Boolean algebra.*

(i) *$\mathscr{K}(G)$ is a Boolean algebra.*

**Proof.** (a) and (b) are equivalent by Theorem 3.22, and (c), (e), (f), (g), (h), and (i) are equivalent by Theorem 2.2.

(b) implies (c). Let $0 \neq K$ be a characteristic subgroup of $G$. Then $K \cap C_\lambda$ is an $s$-subgroup of the $s$-simple $s$-subgroup $C_\lambda$ for each $\lambda \in \Lambda$, and it follows that $K$ is a cardinal summand of $G$.

(c) implies (d). Let $C$ be a proper $s$-subgroup of $G$ and let $K = \bigvee \{C\tau \mid \tau \in A(G)\} = \sum C\tau_i$. Then $K$ is characteristic and hence $G = K \boxplus D$, where $D$ is also characteristic. If $D \neq 0$, then $D$ is a complement of $C$ in $\mathscr{D}(G)$. If $D = 0$, then $|I| > 1$, as $C$ is a proper $s$-subgroup. In this case, any $C\tau_i \neq C$ is a complement of $C$ in $\mathscr{D}(G)$.

(d) implies (a). Let $C$ be a proper $s$-subgroup of $G$. Let $K = \bigvee \{C\tau \mid \tau \in A(G)\} = \sum C\tau_i$. Since $K \in \mathscr{K}(G) \subseteq \mathscr{D}(G)$, there exists a complement $D$ of $C$ in $\mathscr{D}(G)$. Let $L = \bigvee \{D\tau \mid \tau \in A(G)\}$. Since $K$ is characteristic and $K \cap D = 0$, we have that $K \cap L = 0$. Both $K$ and $L$ are characteristic and hence so is $K \boxplus L$. It follows that $K \boxplus L$ is the join of $K$ and $L$ in $\mathscr{D}(G)$ and hence $G = K \boxplus L$. Thus $C$ is a cardinal summand of $G$.

COROLLARY 3.24. *Each s-subgroup of a completely s-reducible l-group G is either characteristic or an atom in* $\mathscr{D}(G)$.

**Proof.** $G = \sum C_\lambda$ ($\lambda \in \Lambda$) where each $C_\lambda$ is an $s$-simple $s$-subgroup of $G$. Suppose that $S$ is a nonzero $s$-subgroup of $G$ and that $S \neq C_\lambda$ for any $\lambda \in \Lambda$. Then $S = \sum (C_\lambda \cap S) = \sum C_\delta$ where $\delta \in \Delta \subseteq \Lambda$ and $|\Delta| \geqq 2$. If $S$ is not characteristic, then there exists $\tau \in A(G)$ such that $C_\delta \tau = C_\lambda$ for some $\lambda \in \Lambda \backslash \Delta$. Then the transposition $(\delta, \lambda)$ induces an $l$-automorphism $\tau_1$ of $G$ such that $0 \neq S \cap S\tau_1 \neq S$, and this is a contradiction.

COROLLARY 3.25. *Let G be a completely s-reducible l-group, let* $\{C_\lambda \mid \lambda \in \Lambda\}$ *be the collection of proper s-subgroups of G, and suppose that* $|\Lambda| > 2$. *Then* $\mathscr{D}(G)$ *is a Boolean algebra if and only if* $\mathscr{D}(G) = \mathscr{K}(G)$.

**Proof.** Suppose that $\mathscr{D}(G)$ is a Boolean algebra, let $C \in \{C_\lambda \mid \lambda \in \Lambda\}$, and let $K = \bigvee \{C\tau \mid \tau \in A(G)\} = \sum C\tau_i$ $(i \in I)$. Then there exists a unique $D \in \mathscr{D}(G)$ such that $G = K \boxplus D$. If $D \neq 0$, then $C \boxplus D$ is a characteristic subgroup of $G$ that contains $C$. Hence $K \subseteq C \boxplus D$ and so $K = K \cap (C \boxplus D) = (K \cap C) \boxplus (K \cap D) = C$. Thus $C \in \mathscr{K}(G)$. Suppose (by way of contradiction) that $D = 0$. Then $K = G$ and $|I| > 1$. If $|I| = 2$, then since $C$ is an atom in $\mathscr{D}(G)$, we have $|\Lambda| = 2$, a contradiction. Thus $|I| > 2$. Let $i, j$, and $k$ be distinct elements of $I$. Then $C\tau_i$ and $C\tau_j$ are distinct complements of $C\tau_k$ in $\mathscr{D}(G)$, a contradiction. The converse is immediate from the theorem.

We conclude this section with the remark that the concept of an $s$-subgroup is lattice theoretic; that is, if $L$ is a lattice with 0, then an element $x$ of $L$ is a shifting element of $L$ if $x = x\pi$ or $x \wedge x\pi = 0$ for each lattice automorphism $\pi$ of $L$. Note that each atom in a lattice is a shifting element.

4. **Characteristically simple $l$-groups.** We derive some results that are, for the most part, corollaries to the theorems in the last two sections. First we supply some needed definitions.

Let $G$ be an $l$-group and let $C \in \mathscr{C}(G)$. $G$ is a *lex-extension* of $C$ provided that $C$ is prime and $g \in G^+ \backslash C$ implies that $g > C$. If, in addition, $G \neq C$, then $G$ is a *proper lex-extension* of $C$. $C$ is a *lex-subgroup* of $G$ if it is a proper lex-extension of some $D \in \mathscr{C}(G)$. If, in addition, $C$ admits no proper lex-extension in $G$, then we say that $C$ is a *maximal* lex-subgroup of $G$. A lex-subgroup $C$ that is not properly contained in any other lex-subgroup is an $s$-subgroup (see [10, Proposition 3.1]).

A polar $C$ of $G$ is called *principal* provided that $C = a''$ for some $a \in G$. An element $s$ of $G$ is *basic* if $s > 0$ and $\{x \in G \mid 0 \leqq x \leqq s\}$ is totally ordered. It follows that $s''$ is a maximal convex $o$-subgroup of $G$ and hence a maximal lex-subgroup. Also $s''$ is a minimal polar and each minimal polar is of this form. A subset $S$ of $G$ is a *basis* for $G$ if $S$ is a maximal set of pairwise disjoint elements and each $s \in S$ is basic.

THEOREM 4.1. *For a characteristically simple $l$-group $G$ ($\neq 0$), the following are equivalent:*

(a) *$G$ has a minimal polar.*

(b) *The collection of lex-subgroups of $G$ contains a maximal element.*

(c) *$G = \sum C_i$ ($i \in I$) where the $C_i$'s are $o$-isomorphic characteristically simple $o$-groups.*

(d) *$G$ has a basis.*

(e) *$G$ has a basic element.*

(f) *Each principal polar is a cardinal summand and $G$ has a closed prime subgroup other than $G$.*

**Proof.** (a) implies (b). Let $P$ be a minimal polar of $G$. By Theorem 3.14, $G = \sum C_\lambda$ ($\lambda \in \Lambda$) where the $C_\lambda$'s are $s$-simple $s$-subgroups of $G$. There exists $\gamma \in \Lambda$ such that $C_\gamma \cap P \neq 0$. Since $P$ is an $s$-subgroup of $G$ and $C_\gamma$ is $s$-simple, we have $C_\gamma \cap P = C_\gamma$ and so $C_\gamma \subseteq P$. Now $C_\gamma$ is a polar and the minimality of $P$ implies that $C_\gamma = P$. Thus $C_\gamma$ is totally ordered. Since a lex-subgroup is cardinally indecomposable, it follows that $C_\gamma$ is maximal in the collection of lex-subgroups of $G$.

(b) implies (c). Let $L$ be maximal in the collection of lex-subgroups. Again $G = \sum C_\lambda$ ($\lambda \in \Lambda$) where the $C_\lambda$'s are $l$-isomorphic $s$-simple $s$-subgroups of $G$. Since $L$ is an $s$-subgroup, $C_\gamma \subseteq L$ for some $\gamma \in \Lambda$; and since $L$ is cardinally indecomposable, $L = C_\gamma$. Thus $L$ is $s$-simple and hence the convex $l$-subgroup of $L$ generated by the nonunits of $L$, the *lex-kernel* of $L$ is trivial [10, Theorem 2.1]. Therefore $L$ is an $o$-group and so condition (c) follows.

(c) implies (d) and (d) implies (e) are trivial.

(e) implies (f). Let $s$ be a basic element of $G$. Again $G = \sum C_\lambda$ ($\lambda \in \Lambda$), where the $C_\lambda$'s are $l$-isomorphic $s$-simple $s$-subgroups of $G$. Clearly there exists $\gamma \in \Lambda$ such that $s \in C_\gamma$. It follows that $s'' = C_\gamma$ and so each of the $C_\lambda$'s is totally ordered. It is then clear that each principal polar is a cardinal summand and that $G$ contains a closed prime subgroup $M \neq G$.

(f) implies (a). Let $M$ $(M \neq G)$ be a closed prime subgroup of $G$. Since each principal polar is a cardinal summand, $G$ is representable. Let $0 < g \in G \backslash M$ and let $C$ be a convex $l$-subgroup of $G$ containing $M$ that is maximal with respect to $g \notin C$. Then $C$ is closed [7, Lemma 3.3] and $C$ contains a unique minimal prime subgroup $N$ and $N = \bigcup b'$ $(b \in G^+ \backslash C)$ [13, Proposition 5.4], and clearly $N = \bigcup b'$ $(b \in G^+ \backslash C$, $b \leqq g)$. Let $C^*$ be the unique convex $l$-subgroup of $G$ that covers $C$. Since $G$ is representable, $C$ is normal in $C^*$ [6, Corollary 3.2]. Let $b \in G^+ \backslash C$, $b \leqq g$. Since $C^*/C$ is an archimedean $o$-group, there exists an integer $n$ such that $C+g < C+nb$. If $x \in b'$, then $x \in (nb)'$ and so $x \in (nb \wedge g)'$. Since $C$ is prime, $nb \wedge g > 0$ and $nb \wedge g \in C+g$. Thus we have $N = \bigcup a'$ $(0 < a \leqq g$ and $a \in C+g)$. Let $d$ $(d > 0)$ be a lower bound for $\{a \mid 0 < a \leqq g \text{ and } a \in C+g\}$ [7, Lemma 3.1]. Then $N = d'$ and so $d''$ is a minimal polar.

THEOREM 4.2. *Let $G$ be a minimal $l$-ideal of an $l$-group $H$ and let $C$ be a proper $s$-subgroup of $G$. Then $G$ is characteristically simple and hence $G = \sum C_\lambda$ $(\lambda \in \Lambda)$, where the $C_\lambda$'s are conjugate subgroups of $C$ in $H$.*

**Proof.** $C$ is an $s$-subgroup of $H$. Let $I(H)$ denote the inner automorphism group of $H$. Then $K = \bigvee \{C\sigma \mid \sigma \in I(H)\}$ is a nonzero $l$-ideal of $H$ contained in $G$. Thus $G = K = \sum C\sigma_i$ where $\{\sigma_i \mid i \in I\} \subseteq I(G)$.

REMARK. In the above theorem, if $C$ is a minimal $l$-ideal of $G$, then since $G = C \boxplus D$, it follows that $C$ is simple.

THEOREM 4.3. *If $G$ is characteristically simple and if $G = A \boxplus B$, where $A$ and $B$ are nonzero and $A$ is cardinally indecomposable, then*

(1) *$A$ is an $s$-simple $s$-subgroup of $G$ and $G = \sum A_\lambda$ $(\lambda \in \Lambda)$, where the $A_\lambda$'s are $l$-isomorphic to $A$.*

(2) *If $G = C \boxplus D$, then $C = \sum A_\delta$ $(\delta \in \Delta)$ for some subset $\Delta$ of $\Lambda$. Hence $C$ is characteristically simple.*

**Proof.** If $\tau \in A(G)$ is such that $A \cap A\tau \neq 0$, then $A = A \cap G = A \cap (A\tau \boxplus B\tau) = (A \cap A\tau) \boxplus (A \cap B\tau) = (A \cap A\tau)$, since $A$ is cardinally indecomposable. Therefore $A$ is a proper $s$-subgroup of $G$ and so $G = \sum A_\lambda$ where the $A_\lambda$'s are $l$-isomorphic to $A$. By Theorem 3.14, $A$ is $s$-simple.

If $C$ is a cardinal summand of $G$ and if $C \cap A_\lambda \neq 0$, then $C \cap A_\lambda = A_\lambda$, for otherwise $A_\lambda$ (and hence $A$) would be cardinally decomposable. It follows that $C = \sum A_\delta$ $(\delta \in \Delta \subseteq \Lambda)$. By Theorem 3.21, $C$ is characteristically simple.

A convex $l$-subgroup $M$ of an $l$-group $G$ that is maximal with respect to not containing some $g \in G$ is called a *value* of $g$ and a *regular subgroup* of $G$. If $M$ is regular, then there is a unique convex $l$-subgroup $M^*$ of $G$ that covers $M$. The pair $(M^*, M)$ is called a *covering pair* of $G$. The covering pair is said to be *normal* if $M$ is a normal subgroup of $M^*$. In this case $M^*/M$ is an archimedean $o$-group.

THEOREM 4.4. *Let $G$ be a characteristically simple $l$-group such that each covering pair is normal. If $G$ has a maximal prime subgroup, then $G$ is a subdirect product of a direct product of subgroups of the reals.*

**Proof.** Since the intersection of all maximal prime subgroups is characteristic, it follows that this intersection is zero. Since $G$ covers each maximal prime subgroup $M$, $M$ is normal in $G$. The map

$$g \to (\ldots, M+g, \ldots)$$

is an $l$-isomorphism of $G$ onto a subdirect product of subgroups of the reals.

COROLLARY 4.5. *Let $G$ be a representable characteristically simple $l$-group. If $G$ contains a maximal prime subgroup or a strong unit, then $G$ is a subdirect product of a direct product of subgroups of the reals.*

**Proof.** If $e$ is a strong unit and $M$ is a value of $e$, then $M$ is a maximal prime subgroup. Also for a representable $l$-group each maximal prime subgroup is normal [6, Corollary 3.2].

An element of an $l$-group $G$ that has exactly one value is said to be *special*. If $0<g\in G$ has only a finite number of values, then $g$ has a unique representation $g=g_1+\cdots+g_n$ where the $g_i$'s are disjoint and special [8, Theorem 3.7]. For an element $g$ in $G$ we shall denote by $G(g)$ the convex $l$-subgroup of $G$ generated by $g$. Then $G(g)=\{x\in G \mid |x|\leqq n|g| \text{ for some } n>0\}$, and is called a *principal convex $l$-subgroup.*

LEMMA 4.6 (MCALLISTER). *For an $l$-group $G$, let*

$$F = \bigvee \{G(g) \mid 0 < g\in G \text{ and } g \text{ is finite valued}\}.$$

*Then $F$ is a characteristic subgroup of $G$ and also the $l$-ideal of $G$ generated by all the special elements of $G$. Moreover*

$$F = \bigcup \{G(g) \mid 0 < g\in G \text{ and } g \text{ is finite valued}\}.$$

**Proof.** If $\tau\in A(G)$ and $0<g\in G$ is finite valued, then so is $g\tau$. Thus $F$ is characteristic. Also $g=g_1+\cdots+g_n$, where the $g_i$'s are disjoint and special and so $G(g)=G(g_1)\boxplus\cdots\boxplus G(g_n)$. It follows that $F$ is the $l$-ideal generated by the special elements. It is easy to verify that if $0<g$, $h\in G$ are finite valued, then so is $g+h$. Thus $F=\bigcup G(g)$.

REMARK. This lemma generalizes to all $g$ with a fixed bound on the cardinality of its values.

THEOREM 4.7. *If $G$ is characteristically simple, contains a special element, and a weak unit, then $G$ is a cardinal sum of a finite number of o-isomorphic characteristically simple o-groups and conversely.*

**Proof.** $F$ is a nonzero characteristic subgroup of $G$. Hence $F=G$. If $g$ is a weak unit, then $g\in G(h)$ for some $0<h\in G$, where $h$ is finite valued. Thus $h$ is also a weak

unit of $G$ and so we may also assume that $g$ is finite valued. Since we have $g'=0$, it follows that $g''=G$. Now $g=g_1+\cdots+g_n$ where the $g_i$'s are disjoint and special. Thus $G$ is the lex-sum of the maximal lex-subgroups $g_1'', \ldots, g_n''$ [10, Corollary II, p. 100]. Therefore $G=A_1 \boxplus \cdots \boxplus A_k$ where each $A_i$ is a lex-subgroup of $G$, and this is the unique decomposition of $G$ into cardinally indecomposable summands. It follows from Theorem 4.3 that the $A_i$'s are $l$-isomorphic $s$-simple $s$-subgroups of $G$. In particular, the lex-kernel of $A_i$ is zero and so $A_i$ is a characteristically simple $o$-group.

LEMMA 4.8. *An $l$-group $G$ is characteristically simple if and only if $0<a, b \in G$ implies $b<n(a\tau_1+\cdots+a\tau_k)$ for some $\tau_1, \ldots, \tau_k \in A(G)$ and some positive integer $n$.*

**Proof.** Clearly the condition is sufficient. If $G$ is characteristically simple, then $b$ must belong to the characteristic subgroup $T$ generated by $a$. Now

$$T = \bigvee \{G(a)\tau \mid \tau \in A(G)\} = \bigvee \{G(a\tau) \mid \tau \in A(G)\}.$$

Thus $b=b_1+\cdots+b_k$ where $b_i \in G(a\tau_i)$ $(i=1, 2, \ldots, k)$ and hence $b_i \leqq |b_i| < n_i a\tau_i$ for some $n_i>0$. Therefore $b<n_1a\tau_1+\cdots+n_ka\tau_k<n_1(a\tau_1+\cdots+a\tau_k)+\cdots+n_k(a\tau_1+\cdots+a\tau_k)=(n_1+\cdots+n_k)(a\tau_1+\cdots+a\tau_k)=n(a\tau_1+\cdots+a\tau_k)$.

Let $\Gamma(G)$ denote the partially ordered set (with respect to inclusion) of all regular subgroups of the $l$-group $G$. Each $\tau \in A(G)$ induces an $o$-automorphism on $\Gamma(G)$. We shall call $G$ *finite valued* if each element of $G$ has only a finite number of values. The next theorem shows that for a finite valued $l$-group $G$, the action of $A(G)$ on $\Gamma(G)$ determines whether or not $G$ is characteristically simple.

THEOREM 4.9. *For a finite valued $l$-group $G$, the following are equivalent:*

(a) *$G$ is characteristically simple.*

(b) *If $0<a, b \in G$ are special, then $na\tau>b$ for some $\tau \in A(G)$ and some positive integer $n$.*

(c) *If $A, B \in \Gamma(G)$, then $A\tau \supseteq B$ for some $\tau \in A(G)$.*

**Proof.** Let $A$ and $B$ be the values of $a$ and $b$ respectively and let $A^*$ cover $A$, and $B^*$ cover $B$.

(a) implies (b). By the last lemma, $b<n(a\tau_1+\cdots+a\tau_k)$ for some $\tau_1, \ldots, \tau_k \in A(G)$ and some $n>0$. Thus we may assume that $a\tau_1 \notin B$. If $a\tau_1 \notin B^*$, then the value $A\tau_1$ of $a\tau_1$ contains $B^*$ and this is the only value of $a\tau_1-b$. Moreover, $A\tau_1+a\tau_1> A\tau_1=A\tau_1+b$. If $a\tau_1 \in B^*$, then $B+ma\tau_1>B+b$ for some positive integer $m$. Thus in either case $ma\tau_1>b$ [8, p. 114].

(b) implies (c). If $na\tau>b$, then $na\tau \notin B$. Thus the value $A\tau$ of $a\tau$ contains $B$.

(c) implies (a). Let $K$ be a proper convex $l$-subgroup of $G$. If $0<g \in G\backslash K$, then $K \subseteq M$, a value of $g$. Let $0<k \in K$ and let $N$ be a value of $k$. Then $K \subseteq M \subseteq N\tau$ for some $\tau \in A(G)$ and $k\tau \notin N\tau$. Thus $K\tau \nsubseteq K$ and so $K$ is not characteristic.

COROLLARY 4.10. *An $o$-group $G$ is characteristically simple if and only if, for each $A, B \in \Gamma(G)$, there exists $\tau \in A(G)$ such that $A\tau \supseteq B$.*

5. **The $l$-group $V(\Lambda, R_\lambda)$.** A *root system* is a partially ordered set $\Lambda$ such that no two incomparable elements have a common lower bound. A *root* in $\Lambda$ is a maximal chain. Let $\Lambda$ be a root system and for each $\lambda \in \Lambda$, let $R_\lambda$ be a subgroup of the reals. Let $\Pi = \prod \bigoplus R_\lambda$ $(\lambda \in \Lambda)$ denote the direct product of the $R_\lambda$'s and for $v = (\ldots, v_\lambda, \ldots) \in \Pi$, let $S_v = \{\lambda \in \Lambda \mid v_\lambda \neq 0\}$. Let $V(\Lambda, R_\lambda) = \{v \in \Pi \mid S_v$ satisfies the maximum condition$\}$. For $v \in V(\Lambda, R_\lambda)$, let $\Lambda_v = \{\lambda \in S_v \mid v_\beta = 0$ for all $\beta > \lambda\}$. Then $\lambda \in \Lambda_v$ is called a *maximal component* of $v$. An element $v$ in $V(\Lambda, R_\lambda)$ is positive if $v_\lambda \geqq 0$ for each $\lambda \in \Lambda_v$. With this order $V(\Lambda, R_\lambda)$ is an abelian $l$-group. If each $R_\lambda$ is the group of real numbers, then $V(\Lambda, R_\lambda)$ is a vector lattice. These $l$-groups $V$ are important because each abelian $l$-group and each (real) vector lattice can be embedded in such a $V$. See [14] for proofs of the above remarks.

THEOREM 5.1 (MCCLEARY). *The $l$-group $V = V(\Lambda, R_\lambda)$ is characteristically simple if and only if $V$ is a cardinal sum of a finite number of o-isomorphic characteristically simple o-groups.*

**Proof.** Let $\{c_i \mid i \in I\}$ be a maximal disjoint subset of $V^+$. Then $\bigvee \{c_i \mid i \in I\}$ exists and is a unit. Also it is clear that $V$ contains a special element and hence the theorem follows immediately from Theorem 4.7.

REMARK. If each $R_\lambda$ equals the reals, then the structure of these characteristically simple $o$-groups is described in the remarks before Corollary 5.9.

COROLLARY 5.2. *$V(\Lambda, R_\lambda)$ is s-simple if and only if it is totally ordered and characteristically simple.*

**Proof.** Each totally ordered cardinal summand of an $l$-group is an $s$-subgroup.

An *o-automorphism* of a partially ordered set $\Delta$ is a permutation $\pi$ of $\Delta$ such that both $\pi$ and $\pi^{-1}$ preserve order.

The following results make it clear that the group of $o$-automorphisms of $\Lambda$ plays an important role in the structure theory of $V(\Lambda, R_\lambda)$.

THEOREM 5.3. *For the vector lattice $V = V(\Lambda, R_\lambda)$ the following are equivalent*:

(a) *Each l-ideal of $V$ is characteristic.*

(b) *The only o-automorphism of $\Lambda$ is the identity and $\Lambda$ contains only finitely many roots.*

**Proof.** (a) implies (b). Suppose (by way of contradiction) that $\Lambda$ contains infinitely many roots. Then there exists $v \in V$ such that $|\Lambda_v| = \aleph_0$ and such that if $\lambda \in \Lambda_v$ then $v_\lambda = 1$. Let $\Lambda_v$ be indexed by the natural numbers, say $\Lambda_v = \{\lambda_1, \lambda_2, \ldots\}$. In $R_{\lambda_n}$ consider the $o$-automorphism $x \to nx$. This induces an $l$-automorphism $\tau$ of $V$ such that for the principal $l$-ideal $V(v)$ we have that $V(v\tau) = V(v)\tau \supset V(v)$, a contradiction. Therefore $\Lambda$ contains only finitely many roots. Next suppose (by way of contradiction) that $\pi$ is an $o$-automorphism of $\Lambda$ such that $\lambda\pi \neq \lambda$ for some $\lambda \in \Lambda$. Without loss of generality, we may suppose that $\lambda\pi \nleq \lambda$. Let $v \in V$ be such that $v_\lambda = 1$ and $v_\beta = 0$ for all $\beta \in \Lambda \backslash \{\lambda\}$. If $\tau$ is an $l$-automorphism of $V$ induced by $\pi$, then $V(v)\tau \neq V(v)$, a contradiction. Thus the only $o$-automorphism of $\Lambda$ is the identity.

(b) implies (a). Each $l$-ideal of $V$ is the join of subgroups of the form $V(v)$ and so it suffices to show that $V(v)\tau \subseteq V(v)$ for each $\tau \in A(V)$. Since $0<v\in V$ has only a finite number of values, $V(v)=V(v_1) \boxplus \cdots \boxplus V(v_n)$, where each $v_i$ is special. Thus, without loss of generality, we may suppose that $v$ is special. Now $\tau$ induces the identity $o$-automorphism on $\Lambda$; hence $v\tau$ has the same value as $v$. Therefore $V(v)=V(v\tau)=V(v)\tau$.

THEOREM 5.4. *For the vector lattice $V=V(\Lambda, R_\lambda)$, the following are equivalent*:

(a) *No nonzero principal l-ideal is characteristic.*

(b) *For each finite nonvoid trivially ordered subset $X$ of $\Lambda$, there exists an o-automorphism $\pi$ of $\Lambda$ such that $X\pi \neq X$.*

**Proof.** (a) implies (b). Let $X=\{\lambda_1, \ldots, \lambda_n\}$ be a trivially ordered subset of $\Lambda$ and define $v\in V$ by $v_\lambda=1$ if $\lambda\in X$ and $v_\lambda=0$ otherwise. By (a) there is an $l$-automorphism $\tau$ of $V$ such that $V(v)\tau\neq V(v)$. Thus $\tau$ induces an $o$-automorphism $\pi$ of $\Lambda$ in which $X\pi\neq X$.

(b) implies (a). If $0<v\in V$ has an infinite number of maximal components, then pick a countable subset of these, say $\{\lambda_1, \lambda_2, \ldots\}$, and in $R_{\lambda_n}$ consider the $o$-automorphism $x\to nx$. This induces an $l$-automorphism $\tau$ of $V$ such that $V(v)\tau\neq V(v)$. Suppose that $0<v\in V$ has a finite number of maximal components, $\lambda_1, \ldots, \lambda_n$. Then there exists an $o$-automorphism $\pi$ of $\Lambda$ so that $\{\lambda_1, \ldots, \lambda_n\}\pi\neq\{\lambda_1, \ldots, \lambda_n\}$. This induces an $l$-automorphism $\tau$ of $V$ such that $V(v)\tau\neq V(v)$.

Let $\Gamma$ be a root system and let $\Lambda\subseteq\Gamma$. We say that $\Gamma$ is an *essential* extension of $\Lambda$ if $\Lambda$ is a dual ideal of $\Gamma$ and both $\Gamma$ and $\Lambda$ have the same number of roots.

LEMMA 5.5. *If $\Lambda$ is a finite root system, then there exists a finite essential extension $\Gamma$ of $\Lambda$ such that the o-automorphism group $A(\Gamma)$ of $\Gamma$ is trivial.*

**Proof.** We induct on the number $n$ of roots of $\Lambda$. If $n=1$ or if $A(\Lambda)$ is trivial, the theorem holds. Suppose that $n>1$ and pick a root $\Delta_1$ of $\Lambda$. Let $\Lambda_1=\{\lambda\in\Lambda \mid \lambda \text{ is an element of some root } \Delta_i\neq\Delta_1\}$. By induction there is an essential extension $\Gamma_1$ of $\Lambda_1$ such that $A(\Gamma_1)$ is trivial. Adjoin $\Delta_1\backslash\Lambda_1$ to $\Gamma_1$ and add elements to the tail of $\Delta_1$ until the resulting root system has a trivial $o$-automorphism group.

COROLLARY 5.6. *Each finite-dimensional vector lattice can be embedded in a finite-dimensional vector lattice in which each l-ideal is characteristic.*

It is not difficult to show that each totally ordered set can be embedded in one with a trivial $o$-automorphism group. Hence each root system $\Lambda$ with a finite number of roots can be embedded in a root system $\Gamma$ with the same number of roots and such that $A(\Gamma)$ is trivial. Thus each vector lattice $L$ with a finite basis can be embedded in a vector lattice $V$ in which each $l$-ideal is characteristic and such that a basis for $L$ is also a basis for $V$.

THEOREM 5.7. *If $H$ is the divisible hull of a characteristically simple abelian $l$-group $G$, then $H$ is characteristically simple.*

**Proof.** If $\tau$ is an $l$-automorphism of $G$, then there exists a unique extension of $\tau$ to an $l$-automorphism $\sigma$ of $H$; for if $h \in H$, then $nh \in G$ for some positive integer $n$. Define $h\sigma = (nh\tau)/n$. A routine check shows that $\sigma$ is an $l$-automorphism of $H$. If $0 < a, b \in H$, then $ma, mb \in G$ for some positive integer $m$. Thus by Lemma 4.8, there exists $\tau_1, \ldots, \tau_k \in A(G)$ and a positive integer $n$ such that $mb < n((ma)\tau_1 + \cdots + (ma)\tau_k) = nm(a\sigma_1 + \cdots + a\sigma_k)$, where $\sigma_i$ is the extension of $\tau_i$ to $H$ described above. Therefore $b < n(a\sigma_1 + \cdots + a\sigma_k)$ and so by Lemma 4.8 $H$ is characteristically simple.

*Notation.* Let $\Lambda$ be a root system and let $V = V(\Lambda, R_\lambda)$ where each $R_\lambda$ is the group of reals. Let

$$\Sigma(\Lambda, R_\lambda) = \{v \in V \mid S_v \text{ is finite}\}$$

and

$$F(\Lambda, R_\lambda) = \{v \in V \mid S_v \text{ is contained in a finite number of roots}\}.$$

Note that both $\Sigma(\Lambda, R_\lambda)$ and $F(\Lambda, R_\lambda)$ are finite valued $l$-groups.

THEOREM 5.8. *For a root system $\Lambda$, the following are equivalent:*
(a) *$\Sigma(\Lambda, R_\lambda)$ is characteristically simple.*
(b) *$F(\Lambda, R_\lambda)$ is characteristically simple.*
(c) *For $\alpha, \beta \in \Lambda$, there exists an $o$-automorphism $\pi$ of $\Lambda$ such that $\alpha\pi \geqq \beta$.*

**Proof.** (a) implies (c). Let $\Sigma = \Sigma(\Lambda, R_\lambda)$, let $\alpha, \beta \in \Lambda$, and let $\Sigma_\gamma = \{v \in \Sigma \mid v_\lambda = 0 \text{ for all } \lambda \geqq \gamma\}$ where $\gamma \in \{\alpha, \beta\}$. Then $\Sigma_\alpha$ and $\Sigma_\beta$ are regular subgroups of $\Sigma$. By Theorem 4.9, there exists $\tau \in A(\Sigma)$ such that $\Sigma_\alpha\tau \supseteq \Sigma_\beta$. Then $\tau$ induces an $o$-automorphism $\pi$ of $\Lambda$ such that $\alpha\pi \geqq \beta$.

(c) implies (a). Let $\Sigma_\alpha$ and $\Sigma_\beta$ be as above and let $\pi$ be an $o$-automorphism of $\Lambda$ such that $\alpha\pi \geqq \beta$. Then $\pi$ induces an $l$-automorphism $\tau$ of $\Sigma$ such that $\Sigma_\alpha\tau \supseteq \Sigma_\beta$. Thus by Theorem 4.9, $\Sigma$ is characteristically simple.

The proof of the equivalence of (b) and (c) is similar to the one just given.

In an $l$-group $G$, two elements $a$ and $b$ in $G^+$ are called *a-equivalent* if $a < mb$ and $b < na$ for some positive integers $m$ and $n$. An $l$-group $H$ is said to be an *a-extension* of $G$ if $G$ is an $l$-subgroup of $H$ and if for each $h \in H^+$ there exists $g \in G^+$ such that $g$ and $h$ are $a$-equivalent. $G$ is said to be *a-closed* if it does not admit a proper $a$-extension. An $a$-closed $a$-extension of $G$ is called an *a-closure* of $G$.

If $G$ is an abelian $o$-group, then by Hahn's embedding theorem (see [14] or [15]) we may assume that $G \subseteq V(\Lambda, R_\lambda)$ where $\Lambda$ is totally ordered and each $R_\lambda$ is the group of reals, and $V(\Lambda, R_\lambda)$ is the $a$-closure of $G$. *If $G$ is a-closed, then $G = V(\Lambda, R_\lambda)$ and hence $G$ is characteristically simple if and only if, given $\alpha, \beta \in \Lambda$, then $\alpha\pi \geqq \beta$ for some o-automorphism $\pi$ of $\Lambda$.*

COROLLARY 5.9. *If an abelian o-group is characteristically simple, then so is its a-closure.*

**Proof.** This follows from the above and Theorem 4.9.

6. **The *l*-group $C(X)$.** For a topological space $X$, let $C(X)$ denote the vector lattice of all continuous real-valued functions on $X$, with pointwise order and addition. Many authors (see, for example, [3] or [22]) have shown that an archimedean $l$-group with a strong unit is $l$-isomorphic to an $l$-subgroup of $C(X)$ that contains the constant function 1, where $X$ is a Stone space (i.e. $X$ is extremally disconnected, compact, and Hausdorff). A topological space $X$ is said to be *completely regular* provided it is a Hausdorff space such that whenever $A$ is a closed set and $x \in X\backslash A$, there exists $f \in C(X)$ such that $xf=1$ and $Af=\{0\}$. For each topological space $Y$ there exists a completely regular space $X$ and a continuous map $\pi$ of $Y$ onto $X$ such that $f \to \pi f$ is a ring isomorphism of $C(X)$ onto $C(Y)$ [16, p. 41]. Thus we shall assume throughout this section that $X$ is completely regular.

It is shown in [16, p. 69] that each prime ring ideal of $C(X)$ is a prime subgroup of $C(X)$; thus if $M$ is a maximal ring ideal of $C(X)$, then $C(X)/M$ is an $o$-field. One calls a topological space $X$ *real compact* if whenever $M$ is a maximal ring ideal of $C(X)$ and $C(X)/M$ is $o$-isomorphic to the field of real numbers, then

$$M = C_x = \{f \in C(X) \mid xf = 0\}$$

for some $x \in X$. An *f-ring* is a lattice-ordered ring in which $a \wedge b=0$ and $c \geqq 0$ implies $ca \wedge b = ac \wedge b = 0$.

LEMMA 6.1. *If $G$ is an f-ring and $M$ is a minimal prime subgroup of $(G, +)$, then $M$ is a ring ideal.*

**Proof.** Let $0<x \in M$ and $0<g \in G$. Since $M$ is a minimal prime subgroup of $(G, +)$, there exists $0<a \in G\backslash M$ such that $a \wedge x=0$. Thus $a \wedge xg=0$ and so $xg \in M$. Similarly $gx \in M$.

THEOREM 6.2. *For a topological space $X$ and $C=C(X)$, the following are equivalent:*
(a) *If $M$ is a maximal group l-ideal of $C$, then $M=C_x$ for some $x \in X$.*
(b) *$X$ is real compact.*

**Proof.** (a) implies (b) is trivial.

(b) implies (a). Let $M$ be a maximal group $l$-ideal and let $N$ be a minimal prime subgroup of $C$ contained in $M$. By the preceding lemma, $N$ is a ring ideal and hence is contained in a maximal ring ideal $J$ of $C$. Thus $N \subseteq J \subseteq M$. If $J \neq M$, then $C/J$ is a nonarchimedean $o$-field with a maximal convex $o$-subgroup $M/J$, but this is impossible.

In [16] the following results are proven:
(a) Each compact space is real compact (p. 71).
(b) Each metrizable space of nonmeasurable cardinals is real compact (p. 232).

(c) A discrete space is real compact if and only if its cardinal number is nonmeasurable (p. 163).

(d) Each Lindelöf space is real compact (p. 115).

(e) Each subspace of a Euclidean space is real compact (p. 115).

THEOREM 6.3. *If the topological space $X$ is real compact and $C=C(X)$, then $A(C)$ is a splitting extension of the group $P(C)$ of polar preserving l-automorphisms of $C$ by the group $H$ of ring l-automorphisms of $C$.*

**Proof.** Let $A=A(C)$ and $P=P(C)$. It is shown in [12] that the group $P$ is $\{\tau \in A \mid \text{there exists } g \in C \text{ with } xg>0 \text{ for all } x \in X \text{ and } f\tau=fg \text{ for all } f \in C\}$. Each ring automorphism is an $l$-automorphism [16, p. 13] and $H=\{\tau \in A \mid \text{there exists a homeomorphism } \pi \text{ of } X \text{ such that } f\tau=\pi f \text{ for all } f \in C\}$.

If $\tau \in A$, then $\tau$ induces a permutation of the set $\mathscr{M}$ of maximal group $l$-ideals of $C$. Each maximal group $l$-ideal is of the form $C_x$ for some $x \in X$. Define $\pi: X \to X$ by $x\pi=y$, where $C_x\tau=C_y$. Then $\pi$ is a permutation of $X$. We wish to show that $\pi$ is continuous. Suppose (by way of contradiction) that there exists a net $\{x_\lambda \mid \lambda \in \Lambda\}$ such that $x_\lambda \to x$ and such that $x_\lambda\pi$ lies outside of a given $\varepsilon$-neighborhood of $x\pi$ for each $\lambda \in \Lambda$. Pick $f\tau \in C$ such that $(x_\lambda\pi)f\tau=0$ for all $\lambda \in \Lambda$, but $(x\pi)f\tau\neq 0$. Observe that for $y \in X$ and $h \in C$, the following are equivalent:

(i) $yh=0$.

(ii) $h \in C_y$.

(iii) $h\tau \in C_y\tau=C_{y\pi}$.

(iv) $(y\pi)h\tau=0$.

Thus we have that $x_\lambda f=0$ for all $\lambda$, but $xf\neq 0$. This contradicts the assumption that $f$ is continuous. Therefore $\pi$ is a homeomorphism.

Define $\sigma$ from $C$ into $C$ by $(x)f\sigma=(x\pi)f$ for all $x \in X$ and all $f \in C$. Then $\sigma \in H$ and hence $\tau\sigma \in A$. Using the conditions (i) through (iv) above, it is easily shown that $\tau\sigma=\rho \in P$. Therefore $\tau=\rho\sigma^{-1} \in PH$.

Clearly $P \cap H$ is the identity subgroup. Let $\tau \in P$ where $\tau$ is given by $f\tau=fg$ for some $g \in C$ and $\sigma \in H$ where $\sigma$ is given by $f\sigma=\pi f$. Then for $f \in C$ and $x \in X$, $xf\sigma\tau\sigma^{-1}=(x\pi^{-1})f\sigma\tau=(x\pi^{-1})f\sigma(x\pi^{-1})g=(xf)(xg\sigma^{-1})$. Since $xg\sigma^{-1}>0$ for all $x \in X$, $\sigma\tau\sigma^{-1} \in P$. Hence $P$ is a normal subgroup of $A$ and so $A$ is a splitting extension of $P$ by $H$.

For $f \in C(X)$ let $K_f$ be the closure of the support of $f$, let $C^*(X)=\{f \in C(X) \mid f \text{ is bounded}\}$, and let $C_K(X)=\{f \in C(X) \mid K_f \text{ is compact}\}$.

COROLLARY 6.4. *If $X$ is real compact, then $C_K(X)$ is a characteristic subgroup of $C(X)$.*

**Proof.** The following properties of $C_K(X)$ are proven in [16]:

(a) $C_K(X)$ is an ideal in $C(X)$ (and $C^*(X)$) (p. 61).

(b) For any topological space $X$, $C_K(X)$ is the intersection of all free ideals in $C(X)$ or $C^*(X)$ (p. 109). If $X$ is real compact then $C_K(X)$ is the intersection of all free maximal ideals of $C(X)$ (p. 123).

We first show that $C_K(X)$ is an $l$-ideal of $C(X)$ for any topological space $X$. If $g \in C_K(X)$, then $K_{g \vee 0}$ is a closed subset of the compact set $K_g$ and hence compact. Thus $g \vee 0 \in C_K(X)$. If $h \in C(X)$ is such that $0 < h < g \in C_K(X)$, then again $K_h$ is compact and so $h \in C_K(X)$.

Now if $X$ is real compact, then by the theorem, $A = PH$ and clearly $C_K(X)$ is mapped to itself under the action of elements from both $P$ and $H$.

THEOREM 6.5. *If $X$ is real compact and the group of homeomorphisms of $X$ acts transitively on $X$, then $C_K(X)$ is the minimal characteristic subgroup of $C(X)$.*

**Proof.** Let $0 < g \in C(X)$, $0 < h \in C_K(X)$, let $x \in X$ be such that $xg > 0$, and let $y \in K_h$. Then there exists a homeomorphism $\pi$ of $X$ such that $y\pi = x$. Now $\pi$ induces an $l$-automorphism $\tau$ of $C(X)$ and $yg\tau = (y\pi)g = xg > 0$. It follows by the compactness of $K_h$ that there exists $\tau_1, \ldots, \tau_n \in A(C(X))$ such that $f = g\tau_1 + \cdots + g\tau_n$ and $zf > 0$ for all $z \in K_h$. Thus a suitable multiple of $f$ exceeds $h$, and so $C_K(X)$ is the minimal characteristic subgroup of $C(X)$.

COROLLARY 6.6. *If $X$ is the set of real numbers, then $C_K(X) = \{f \in C(X) \mid$ there exists $x, y \in X$ with $x < y$ and $zf = 0$ if $z \in X \backslash (x, y)\}$ is the minimal characteristic subgroup of $C(X)$.*

COROLLARY 6.7. *If $X$ is discrete with nonmeasurable cardinality, then $\sum R_x$ $(x \in X)$, where $R_x$ is the reals for each $x \in X$, is the minimal characteristic subgroup of $C(X) = \prod R_x$ $(x \in X)$.*

COROLLARY 6.8. *If $X$ is the unit circle, then $C_K(X) = C(X)$. Hence $C(X)$ is characteristically simple.*

COROLLARY 6.9. *If $X$ is compact and the group of homeomorphisms acts transitively on $X$, then $C(X)$ is characteristically simple.*

COROLLARY 6.10. *Suppose that $X$ is a Stone space such that the group of homeomorphisms acts transitively on $X$. Then $C(X)$ is a complete characteristically simple vector lattice.*

As a corollary to the proof of the theorem we have

COROLLARY 6.11. *If $C = C[0, 1]$ and if $K = \{g \in C \mid$ there exists $x, y \in (0, 1)$ with $x < y$ such that $zg = 0$ if $z \in [0, 1] \backslash (x, y)\}$, then $K$ is the minimal characteristic subgroup of $C$.*

Using [1, Theorem 11 and Theorem 12], one can construct uncountably many totally ordered compact spaces where each pair of closed intervals are $o$-isomorphic. If the endpoints of such a space are identified, one obtains a compact space $X$ with a transitive group of homeomorphisms. Thus $C(X)$ will be characteristically simple. Note that Corollary 6.8 is a special case of this, where we start with [0, 1].

THEOREM 6.12. *If $X$ is compact, then $C(X)$ is characteristically simple if and only if, for each $0<f\in C(X)$ and $x\in X$, there exists $y$ in the support of $f$ and a homeomorphism $\pi$ of $X$ such that $x\pi=y$.*

**Proof.** Suppose that $C=C(X)$ is characteristically simple and let $0<f\in C$. Then the characteristic subgroup $K$ generated by $f$ is $\bigvee C(f)\tau=\bigvee C(f\tau)$ ($\tau\in A(C)$). If there exists $x\in X$ such that $xf\tau=0$ for all $\tau\in A(C)$, then clearly $K\neq C$. Thus given $x\in X$ there exists $\tau\in A(C)$ such that $xf\tau>0$. Now, without loss of generality, $\tau$ is induced by a homeomorphism $\pi$ of $X$ and so $x\pi=y$ for some $y$ in the support of $f$.

Conversely suppose that $0<f$ belongs to a characteristic subgroup $K$ of $C$. For $x\in X$ there exists $y\in X$ and a homeomorphism $\pi$ of $X$ such that $yf>0$ and $x\pi=y$. Without loss of generality $yf>1$ and hence there exists an $l$-automorphism $\tau$ of $C$ such that $f\tau>1$ in some neighborhood of $x$. By the compactness, it follows that the constant function 1 is in $K$ and so $K=C$.

THEOREM 6.13. *Suppose that $G$ is an $l$-subgroup of $C(X)$ containing* 1, *where $X$ is a Stone space. Then $G$ is characteristically simple if and only if for each $0<g\in G$ and $x\in X$ there exists an $l$-automorphism $\tau$ of $G$ such that $xg\tau>0$.*

**Proof.** If $G$ is characteristically simple and if $0<g\in G$ and $x\in X$, then by Lemma 4.8, $1<n(g\tau_1+\cdots+g\tau_k)$ where $\tau_1,\ldots,\tau_k\in A(G)$ and $n$ is a positive integer. Thus $xg\tau_i>0$ for some $i$.

Conversely let $0<g\in G$ and for each $x\in X$ choose $\tau_x\in A(G)$ and a positive integer $n_x$ such that $n_x(x)\tau_x>1$. Let $T_x=\{y\in X\mid n_x(y)g\tau_x>1\}$. Then the $T_x$'s form an open cover for $X$ and hence there is a finite subcover. Therefore $1<n_{x_1}g\tau_{x_1}+\cdots+n_{x_k}g\tau_{x_k}<m(g\tau_{x_1}+\cdots+g\tau_{x_k})$ where $m=\max\{n_{x_1},\ldots,n_{x_k}\}$. Thus the characteristic subgroup of $G$ generated by $g$ contains 1 and hence must be $G$.

We note that in the preceding theorem, we use only the fact that $X$ is compact. Now let $D(X)$ denote the ring of almost finite real-valued continuous functions on the Stone space $X$, and let $P$ be the group of polar preserving $l$-automorphisms of $D(X)$ and $H$ the group of ring $l$-automorphisms of $D(X)$.

THEOREM 6.14. *If $X$ is a Stone space, then $A(D(X))$ is a splitting extension of $P$ by $H$.*

**Proof.** Let $D=D(X)$, $A=A(D(X))$ and for $x\in X$, $D_x=\{f\in D\mid xf=0\}$. Let $\tau\in A$ and $f=1\tau$. We shall show that $f$ has a multiplicative inverse. Suppose (by way of contradiction) that the support $S_f$ of $f$ is a proper subset of $X$. Then $X\setminus S_f$ is clopen and hence the characteristic function $g$ on $X\setminus S_f$ belongs to $D$, and $f\wedge g=0$. Thus $0=1\wedge g\tau^{-1}$, a contradiction. Define $xg=1/xf$ for all $x\in X$. Then $g$ is the inverse of $f$.

Define $\sigma$ from $D$ into $D$ by $h\sigma=hf^{-1}$ for all $h\in D$. Then $\sigma\in P$ and $1\tau\sigma=1$. We now show that $\tau\sigma$ induces a homeomorphism $\pi$ of $X$. Let $x\in X$ and $D_x\tau\sigma=M$. Then $M$ is a value of 1. If $M\subseteq D_y$ for some $y\in X$, then $M=D_y$ since $1\notin D_y$. If

$M \nsubseteq D_y$ for all $y \in X$, then, by the compactness of $X$, it follows that $1 \in M$, a contradiction. Thus $D_x\tau\sigma = D_y$ and we define $x\pi = y$. Clearly $\pi$ is a homeomorphism of $X$. (The argument is the same as that given in the proof of Theorem 6.3.) Define $\rho$ by $h\rho = \pi h$. Then $\rho \in H$. Note that the following are equivalent:

(1) $xh = 0$.

(2) $h \in D_x$.

(3) $h\tau\sigma \in D_{x\pi}$.

(4) $0 = (x\pi)h\tau\sigma = xh\rho\tau\sigma$.

Therefore $\rho\tau\sigma$ is a polar preserving $l$-automorphism of $D$ that maps 1 onto 1 and hence it is the identity. Thus $\tau = \rho^{-1}\sigma^{-1} \in PH$. As in Theorem 6.3, $H \cap P$ is the identity subgroup and $P$ is normal in $A$.

THEOREM 6.15. *If $X$ is a Stone space that satisfies the condition given in Theorem* 6.12, *then $D(X)$ is a characteristically simple, complete, laterally complete vector lattice.*

**Proof.** By the proof of Theorem 6.12, the constant function 1 is in any characteristic subgroup $K$ of $D(X)$. Consider $0 < d \in D(X)$ and let $g = 1 \vee d$. Then $f \to fg$ is an $l$-automorphism of the $l$-group $D(X)$ which maps 1 onto $g$. Thus $g \in K$ and hence $d \in K$. Therefore $K = D(X)$.

7. **Self-injective $l$-groups.** The category of all $l$-groups where the subobjects are $l$-subgroups contains no injectives (see [19]), but as we shall show it does contain self-injectives.

An $l$-group $G$ is said to be *self-injective* if each $l$-homomorphism of an $l$-ideal $L$ of $G$ can be extended to an $l$-endomorphism of $G$. An $l$-ideal $L$ of $G$ is said to be *large* in $G$ if whenever $J$ is an $l$-ideal of $G$ such that $J \cap L = 0$, then $J = 0$.

The proofs of the next three propositions are entirely similar to the corresponding proofs for modules and so we omit them.

7.1. *If $G$ is self-injective and $G = A \boxplus B$, then $A$ is self-injective.*

7.2. *If $L$ is an $l$-ideal of a self-injective $l$-group $G$ and $L$ is isomorphic to $G$, then $G = L \boxplus L'$.*

7.3. *If $G$ is self-injective and $L$ is an $l$-ideal that is not large in any $l$-subgroup of $G$ except itself, then $G = L \boxplus L'$.*

An $l$-group $G$ is said to be *hyper-archimedean* if each $l$-homomorphic image of $G$ is archimedean. It is fairly well known that the following assertions are equivalent (see, for example, [2] or [4]):

(i) $G$ is hyper-archimedean.

(ii) The collection of regular subgroups of $G$ is trivially ordered.

(iii) $G = G(g) \boxplus g'$ for each $0 < g \in G$.

(iv) If $0 < f, g \in G$, then there exists a positive integer $n$ such that

$$f \wedge ng = f \wedge (n+1)g.$$

(v) $G$ is $l$-isomorphic to an $l$-group $H \subseteq \prod R_\lambda$ $(\lambda \in \Lambda)$ where each $R_\lambda$ is the group of real numbers, and such that if $0 < x, y \in H$, then there exists a positive integer $n$ such that $nx_\lambda > y_\lambda$ for all $\lambda \in \Lambda$ with $x_\lambda \neq 0$.

THEOREM 7.4. *For a vector lattice $G$, the following are equivalent*:
(a) *$G$ is self-injective and contains a maximal l-ideal $M$.*
(b) *$G$ is self-injective and hyper-archimedean.*
(c) *$G$ is self-injective and archimedean.*
(d) *$G$ is l-isomorphic to $\sum R_\lambda$ where each $R_\lambda$ is the group of reals.*

**Proof.** Clearly (b) implies (a) and (b) implies (c).

(a) implies (b). If $G$ is not hyper-archimedean, then there exists a regular subgroup $G_\alpha$ such that $G_\alpha \subset G^\alpha \subset G$ where $G^\alpha$ is the $l$-ideal that covers $G_\alpha$. If $0 < a \in G^\alpha \backslash G_\alpha$, then $G^\alpha = G_\alpha \oplus Ra$, where $R$ denotes the real numbers. The projection of $G^\alpha$ onto $Ra$ is an $l$-homomorphism. Now pick $0 < g \in G \backslash M$ and $0 < b \in G \backslash G^\alpha$. Then there exists an $l$-homomorphism $\tau$ of $G^\alpha$ onto $Rg$ such that $a\tau = g$. Let $\sigma$ be an extension of $\tau$ to an $l$-endomorphism of $G$. Then $G_\alpha \subseteq \operatorname{Ker}(\sigma)$. For each positive integer $n$, $G_\alpha + na < G_\alpha + b$, and so $ng = na\sigma < b\sigma$. Therefore $M < n(M+g) < M+b$ for all $n$, but this contradicts the fact that $G/M$ is an archimedean $o$-group.

(c) implies (b). Again, if $G$ is not hyper-archimedean, then there exists a regular subgroup $G_\alpha$ such that $G_\alpha \subset G^\alpha \subset G$. If $0 < b \in G^\alpha \backslash G_\alpha$, then $G^\alpha = G_\alpha \oplus Rb$ and the projection $\tau$ of $G^\alpha$ onto $Rb$ is an $l$-homomorphism. Extend $\tau$ to an $l$-endomorphism $\sigma$ of $G$. If $0 < a \notin G^\alpha$, then $G_\alpha + nb < G_\alpha + a$ and so $nb = nb\sigma < a\sigma$ for all positive integers $n$, a contradiction.

(d) implies (c). This follows from the fact that each $l$-ideal of $G$ is a cardinal summand.

(b) implies (d). It suffices to prove that $G$ has property $(F)$ (see §8) or equivalently, that $G(g)$ has a finite basis for each $0 < g \in G$. Suppose (by way of contradiction) that $\{g_i \mid i = 1, 2, \ldots\}$ is an infinite disjoint subset of $G(g)$. Since $0 < g_i \wedge g \leqq g$ for each $i$, we may assume that $g \geqq g_i$ for each $i$. Moreover, we may multiply each $g_i$ by a suitable real number to obtain $g_i \leqq g$ and $2g_i \nleqq g$ for each $i$. If $x \in \sum G(g_i)$ $(i = 1, 2, \ldots)$, then $x = x_1 + x_2 + \cdots$, where $x_i \in G(g_i)$ $(i = 1, 2, \ldots)$ and all but a finite number of the $x_i$'s are zero. For each $i$, the map $y \to iy$ is an $l$-automorphism of $G(g_i)$ and this induces an $l$-automorphism $\tau$ of $\sum G(g_i)$. $\tau$ can be extended to an $l$-endomorphism $\sigma$ of $G$. Note we may assume that $G \subseteq \prod R_\lambda$ $(\lambda \in \Lambda)$ and that there exists a positive integer $m$ such that $mg_\lambda > (g\sigma)_\lambda$ for all $\lambda$ such that $g_\lambda > 0$, where $g = (\ldots, g_\lambda, \ldots)$. Now $(g_i)_\lambda > 0$ implies $g_\lambda \geqq (g_i)_\lambda > 0$ and so $mg_\lambda > (g\sigma)_\lambda \geqq (g_i\sigma)_\lambda = (ig_i)_\lambda$. Therefore we have that $mg > ig_i$ for a fixed integer $m$ and for all $i$. In particular, $mg > 2mg_{2m}$ which implies that $g > 2g_{2m}$, a contradiction.

REMARK. A vector lattice $G$ that satisfies (d) of the theorem is characteristically simple. We also note that a completely reducible $l$-group is self-injective. We have not been able to characterize nonarchimedean self-injective vector lattices. An example of one that is totally ordered is given in §9.

For the remainder of this section, we shall suppose that $G$ is an $l$-group such that $G$ is $l$-isomorphic to $G(g)$ for each $0<g\in G$. We state the following properties, the proofs of which are straightforward.

7.5. If $G$ is archimedean and has a strong unit, then it is $l$-isomorphic to a subdirect sum of reals. (The above hypothesis is not needed for this.)

7.6. If there exists $0<g\in G$ that is finite valued, then $G$ is an $o$-group.

7.7. If $G$ has a nonunit, then $G$ is cardinally decomposable.

THEOREM 7.8. *If $G$ is a hyper-archimedean $l$-group such that $G$ is $l$-isomorphic to $G(g)$ for each $0<g\in G$, then $G$ is characteristically simple.*

**Proof.** Let $0<a, b\in G$. Now $a=a\wedge b+a_1$ and $b=a\wedge b+b_1$, where $a_1\wedge b_1=0$.

*Case* 1. $a_1\neq 0\neq b_1$. Then $G(a_1)$ is $l$-isomorphic to $G(b_1)$ and $G=G(a_1)\boxplus G(b_1)\boxplus D$ for some $l$-ideal $D$ of $G$. There exists an $l$-automorphism $\tau$ of $G$ that interchanges $G(a_1)$ and $G(b_1)$. Since $a\wedge b\in G(a)$ and $G(a_1)\subseteq G(a)$, we have that

$$b = a\wedge b+b_1\in G(a)+G(a_1)\tau\subseteq G(a)+G(a)\tau.$$

*Case* 2. $b_1=0$. Then $b\leqq a$ and hence $b\in G(a)$.

*Case* 3. $a_1=0$. Then $a\leqq b$ and hence $G(a)\subseteq G(b)$. Therefore $G=G(b)\boxplus b'=G(a)\boxplus D\boxplus b'$ for some $l$-ideal $D$ of $G$. Thus $b=x+y\in G(a)\boxplus D$. If $y=0$, then $b=x\in G(a)$. If $y\neq 0$, then $y>0$ and hence $G=G(a)\boxplus G(y)\boxplus b'$. There is an $l$-automorphism $\tau$ of $G$ interchanging $G(a)$ and $G(y)$. Therefore $b\in G(a)\boxplus G(y)=G(a)\boxplus G(a)\tau$. Thus $G$ is the characteristic subgroup generated by $a$ and so $G$ is characteristically simple.

8. **Embedding in characteristically simple $l$-groups.** In this section we prove that any $l$-group can be embedded in an algebraically simple $l$-group. In addition we prove that a representable (abelian) $l$-group can be embedded in a characteristically simple representable (abelian) $l$-group.

THEOREM 8.1. *Each $l$-group can be embedded as an $l$-subgroup of an algebraically simple $l$-group.*

**Proof.** By [21] we may assume that $G$ is an $l$-subgroup of $\mathscr{P}(F)$, where $\mathscr{P}(F)$ is the $l$-group of all $o$-permutations of a totally ordered field $F$. Thus it suffices to embed $\mathscr{P}(F)$ in an algebraically simple group. Without loss of generality, $F$ contains the rational field. Now $\mathscr{P}(F)$ is doubly transitive on $F$ for if $c<d\in F$ then the map $y\to(d-c)y+c$ maps 0 onto $c$ and 1 onto $d$.

Let $M$ be the field of power series in $x$ with coefficients in $F$, lex-ordered so that

$$\cdots\ll x^{-2}\ll x^{-1}\ll 1\ll x\ll x^2\ll\cdots.$$

Let $U^*=\{m\in M\mid m \text{ exceeds each positive integer}\}$ and let $\pi$ be a one-to-one map of $U^*$ onto a set $U$ such that $U\cap M=\varnothing$. Now $\pi$ induces a total order on $U$ and we shall consider $M\cup U$ as a totally ordered set where $M<U$.

Next we shall show that any two closed intervals of $M \cup U$ are isomorphic. It suffices to show that $[a, b]$ is $o$-isomorphic to $[c, d]$ where $a, b \in M$, $a<b$ and $c, d \in M \cup U$, $c<d$. This is clear if $c, d \in M$ or if $c, d \in U$. Suppose that $c \in M$ and $d \in U$ and let $U^*$ be as above. Let $b_1 > b_2 > \cdots > b_\alpha > \cdots$ be an inversely well-ordered coinitial sequence in $U^*$. Then $n < b_\alpha$ for all integers $n$ and all $\alpha$; and if $n<y$ for all integers $n$, then $b_\alpha < y$ for some $b_\alpha$. Thus $b_\alpha\pi < d$ and $c < x^n$ for some $b_\alpha$ and some positive integer $n$. Since any two intervals of $M$ are $o$-isomorphic, there exists an $o$-isomorphism $f$ of $[c, x^n, x^{n+1}, \ldots)$ onto $[1, 2, 3, \ldots)$ and an $o$-isomorphism $g$ of $(\ldots, b_{\alpha+1}\pi, b_\alpha\pi, d]$ onto $(\ldots, b_{\alpha+2}, b_{\alpha+1}, b_\alpha]$. Since $[c, x^n, x^{n+1}, \ldots) \cup (\ldots, b_{\alpha+1}\pi, b_\alpha\pi, d] = [c, d]$ and $[1, 2, 3, \ldots) \cup (\ldots, b_{\alpha+2}, b_{\alpha+1}, b_\alpha] = [1, b_\alpha]$, $f$ and $g$ induce an $o$-isomorphism of $[c, d]$ onto $[1, b_\alpha]$. Since $[1, b_\alpha]$ is contained in $M$, $[1, b_\alpha]$ is $o$-isomorphic to $[a, b]$. Therefore $[c, d]$ is $o$-isomorphic to $[a, b]$.

Next we repeat a similar construction on the lower end of $M \cup U$ and get a totally ordered set $N = L \cup M \cup U$ where $L < M < U$ and any two closed intervals of $N$ are $o$-isomorphic.

Each $o$-permutation of $F$ can be extended to an $o$-permutation of $M$, and by [21], $\mathscr{P}(F)$ is $l$-isomorphic to an $l$-subgroup of $\mathscr{P}(M)$. $\mathscr{P}(M)$ can be considered as an $l$-subgroup of the $l$-group $\mathscr{B}(N)$ of all $o$-permutations of $N$ having bounded support (that is, all $o$-permutations that are the identity outside of some bounded interval). By [17], $\mathscr{B}(N)$ is algebraically simple. This completes the proof of the theorem.

An $l$-group $G$ satisfies property $(F)$ if each element in $G^+$ exceeds at most a finite number of disjoint elements. In the next two theorems we use the fact that a minimal prime subgroup of a representable $l$-group is an $l$-ideal [6, Theorem 3.1].

THEOREM 8.2. *If $G$ is a representable (abelian) $l$-group that satisfies property $(F)$, then $G$ is $l$-isomorphic to an $l$-subgroup of a representable (abelian) $l$-group that satisfies property $(F)$ and is characteristically simple.*

**Proof.** Let $\{s_\lambda \mid \lambda \in \Lambda\}$ be a basis for $G$. For each $\lambda \in \Lambda$, let $M_\lambda$ be a minimal prime subgroup of $G$ such that $s_\lambda \notin M_\lambda$. We may assume that $G \subseteq \sum G_\lambda$ $(\lambda \in \Lambda)$, where $G_\lambda = G/M_\lambda$ for each $\lambda \in \Lambda$. Note that each $G_\lambda$ is an $o$-group. Define a total order on $\Lambda$ and use this to lexicographically order $\sum \bigoplus G_\lambda$ $(\lambda \in \Lambda)$. Denote this $o$-group by $K$. For each integer $i$, let $K_i = K$ and use the natural order of the integers $Z$ to lexicographically order $\sum \bigoplus K_i$ $(i \in Z)$ and call this $o$-group $L$. For each $\lambda \in \Lambda$, let $H_\lambda = L$ and let $H = \sum H_\lambda$ $(\lambda \in \Lambda)$. Then $H$ is characteristically simple and $G$ is $l$-isomorphic to an $l$-subgroup of $H$.

THEOREM 8.3. *Each representable (abelian) $l$-group $G$ is $l$-isomorphic to an $l$-subgroup of a representable (abelian) characteristically simple $l$-group.*

**Proof.** Let $\{M_\delta \mid \delta \in \Delta\}$ be the collection of minimal prime subgroups of $G$. If $\Delta$ is finite, then Theorem 8.2 applies. Hence we assume that $\Delta$ is infinite.

Let $\mathscr{D}=\{A_\delta \mid \delta \in \Delta\}$ be a partition of $\Delta$ such that $|A_\delta|=|\Delta|$ for each $\delta \in \Delta$, and let $Z$ denote the set of integers. We define a partial order on $\Delta \times Z$ by setting $(\delta_m, m) \leqq (\delta_n, n)$ if and only if $\delta_m=\delta_n$ and $m=n$ or $m<n$ and there exists $\delta_{m+1}, \ldots, \delta_{n-1} \in \Delta$ such that $\delta_m \in A_{\delta_{m+1}}, \delta_{m+1} \in A_{\delta_{m+2}}, \ldots, \delta_{n-1} \in A_{\delta_n}$. For $k \in Z$, we will call $\{(\delta, k) \mid \delta \in \Delta\}$ the *kth level* of $\Delta \times Z$. By definition, distinct elements on the $k$th level are incomparable.

Let $(\delta_m, m), (\delta_n, n)$, and $(\delta_p, p)$ be elements of $\Delta \times Z$ such that $(\delta_m, m)<(\delta_n, n)$ and $(\delta_m, m)<(\delta_p, p)$ and suppose that $n \leqq p$. Then there exists $\delta_{m+1}, \ldots, \delta_{n-1}, \lambda_{m+1}, \ldots, \lambda_{p-1} \in \Delta$ such that $\delta_m \in A_{\delta_{m+1}}, \delta_{m+1} \in A_{\delta_{m+2}}, \ldots, \delta_{n-1} \in A_{\delta_n}$ and $\lambda_m \in A_{\lambda_{m+1}}, \ldots, \lambda_{p-1} \in A_{\lambda_n}$. Since $\mathscr{D}$ is a partition, we have that $\delta_{m+1}=\lambda_{m+1}, \delta_{m+2}=\lambda_{m+2}, \ldots, \delta_n=\lambda_n$. Therefore $(\delta_n, n) \leqq (\delta_p, p)$ and it follows that $\Delta \times Z$ is a root system.

For $(\delta, k) \in \Delta \times Z$, we define the *cone beneath* $(\delta, k)$ to be $\{(\lambda, m) \mid (\lambda, m) \leqq (\delta, k)\}$ and denote this set by $(\delta, k)_*$. It is readily verified for $(\gamma, j), (\delta, k) \in \Delta \times Z$ that $(\gamma, j)_*$ is $o$-isomorphic to $(\delta, k)_*$.

Let $T$ be a root in $\Delta \times Z$. Then $T=\{(\delta_k, k) \mid k \in Z\}$ for some subset $\{\delta_k \mid k \in Z\}$ of $\Delta$. Let $\Lambda=\bigcup (\delta_k, k)_*$ $(k \in Z)$. We assert that the group $A(\Lambda)$ of $o$-automorphisms of $\Lambda$ acts irreducibly on $\Lambda$, i.e., given $(\gamma, j)$ and $(\lambda, l) \in \Lambda$, there exists $\pi \in A(\Lambda)$ such that $(\gamma, j)\pi \geqq (\lambda, l)$. It suffices to show that given $(\gamma, j) \in \Lambda$ and $(\delta_k, k) \in T$ where $(\gamma, j)<(\delta_k, k)$, there exists $\pi \in A(\Lambda)$ such that $(\gamma, j)\pi=(\delta_k, k)$. To do this it will suffice to find $\pi \in A(\Lambda)$ such that $(\lambda, k-1)\pi=(\delta_k, k)$ where $(\lambda, k-1)$ is the unique element in the $(k-1)$th level between $(\gamma, j)$ and $(\delta_k, k)$. Since there exists $\rho \in A(\Lambda)$ such that $(\lambda, k-1)\rho=(\delta_{k-1}, k-1)$, we may further suppose that $(\lambda, k-1)=(\delta_{k-1}, k-1)$. Let $\pi_{k-1}$ be an $o$-isomorphism mapping $(\delta_{k-1}, k-1)_*$ onto $(\delta_k, k)_*$. We will construct an $o$-isomorphism $\pi_k$ of $(\delta_k, k)_*$ onto $(\delta_{k+1}, k+1)_*$ with the property that $\pi_k|(\delta_{k-1}, k-1)_*=\pi_{k-1}$. Let $\nu$ be a one-to-one mapping of $A_{\delta_k}$ onto $A_{\delta_{k+1}}$ such that $\delta_{k-1}\nu=\delta_k$. If $\gamma \in A_{\delta_k}\backslash\{\delta_{k-1}\}$ let $\pi_\gamma$ be an $o$-isomorphism of $(\gamma, k-1)_*$ onto $(\gamma\nu, k)_*$ and let $\pi_{\delta_k}=\pi_{k-1}$. Let

$$\pi_k = \left(\bigcup \{\pi_\gamma \mid \gamma \in A_{\delta_k}\}\right) \cup \{((\delta_k, k), (\delta_{k+1}, k+1))\}.$$

Clearly $\pi_k$ has the required properties. By induction we obtain a chain of functions

$$\pi_{k-1} \subseteq \pi_k \subseteq \cdots \subseteq \pi_{k+i} \subseteq \cdots$$

such that $\pi_{k+i}$ is an $o$-isomorphism of $(\delta_{k+i}, k+i)_*$ onto $(\delta_{k+i+1}, k+i+1)_*$ and such that

$$\pi_{k+i}|(\delta_{k+i-1}, k+i-1)_* = \pi_{k+i-1} \qquad (i = 0, 1, 2, \ldots).$$

Then $\pi=\bigcup \pi_j$ $(j=k-1, k, k+1, \ldots)$ is the required function. Define a total order on the set $\Delta$ and let $H$ be the direct sum of the $o$-groups $G/M_\delta$ $(\delta \in \Delta)$. For $h \in H$, we define $h>0$ if $h_\delta>0$ where $\delta$ is the largest element in the support of $h$. Then $H$ is an $o$-group. For each $\lambda \in \Lambda$, let $H_\lambda=H$. Then $V(\Lambda, H_\lambda)$ is an $l$-group

[14], and clearly this group is representable. Let

$$L = \{v \in V(\Lambda, H_\lambda) \mid \text{there exists } \delta_1, \ldots, \delta_n \in \Delta \text{ and } k_1, \ldots, k_n \in Z \text{ such that } S_v \subseteq \bigcup (A_{\delta_i} \times \{k_i - 1\})\}.$$

Then $L$ is an $l$-subgroup of $V(\Lambda, H_\lambda)$. Let $J$ be the $l$-subgroup of $L$ consisting of those elements whose support is contained in $A_{\delta_1} \times \{0\}$. Then $J$ is $l$-isomorphic to $\prod H_\delta$ ($\delta \in \Delta$) since $|A_{\delta_1}| = |\Delta|$. If $\delta \in \Delta$, let $\tau_\delta$ be the injection of $G/M_\delta$ into $H_\delta$. For $(\ldots, x_\delta, \ldots) \in \prod G/M_\delta$, let $(\ldots, x_\delta, \ldots)\tau = (\ldots, x_\delta \tau_\delta, \ldots)$. Then $\tau$ is an $l$-isomorphism of $\prod G/M_\delta$ into $\prod H_\delta$. Since $G$ is $l$-isomorphic to an $l$-subgroup of $\prod G/M_\delta$, we have an $l$-isomorphism of $G$ into $L$. Since any element of $A(\Lambda)$ induces an $l$-automorphism of $L$ and $A(\Lambda)$ acts irreducibly on $\Lambda$, we have that $L$ is characteristically simple. Finally, if $G$ is abelian, then so is $L$.

## 9. Examples and open questions.

EXAMPLE 9.1. *A nonarchimedean o-group G such that*

(1) *G is s-simple but not simple.*

(2) *G is self-injective.*

Let $G = V(\Lambda, R_\lambda)$ where $\Lambda$ is the set of integers with the natural order and $R_\lambda$ is the group of real numbers for each $\lambda \in \Lambda$. Clearly $G$ is characteristically simple and, for $o$-groups, this is equivalent to being $s$-simple.

To prove that $G$ is self-injective, we need only to consider the case where $L$ is a proper $l$-ideal of $G$ and $\varphi$ is a nonzero $l$-homomorphism of $L$ into $G$. Since $L$ is a proper $l$-ideal of $G$, $L = G(g)$ for some $0 < g \in G$. Let $n$ be the maximal component of $g$ in $\Lambda$. Each $a \in G$ has a unique representation of the form $a = b + c$ where $b_i = a_i$ for $i \leqq n$ and $b_i = 0$ for $i > n$, $c_i = a_i$ for $i > n$ and $c_i = 0$ for $i \leqq n$. Then $b \in L$. Let $m$ be the maximal component of $g\varphi$ in $\Lambda$. Define $a\psi = b\varphi + c\rho$, where $(c\rho)_i = 0$ for $i \leqq m$ and $(c\rho)_i = c_{n+i-m}$ for $i > m$. A straightforward computation shows that $\psi$ is an $l$-endomorphism of $G$.

Note that the root system $\Lambda$ of this example satisfies condition (c) of Theorem 5.8.

EXAMPLE 9.2. *An example of an l-group G such that*

(1) *G is hyper-archimedean and G is l-isomorphic to G(g) for each* $0 < g \in G$.

(2) *G is s-simple and cardinally decomposable, but not completely reducible.*

(3) *G is l-isomorphic to each nonzero cardinal summand.*

Let $\Lambda$ be the trivially ordered set of positive integers and let $G$ consist of all those functions $v$ in $V(\Lambda, R_\lambda)$ ($R_\lambda$ = reals) which satisfy $v_i$ is an integer for each $i \in \Lambda$ and there exists a positive integer $n = n(v)$ such that $v_i = v_{i+n}$ for all $i \in \Lambda$. It is easy to show that $G$ is an $l$-group and $G = G(g) \boxplus g'$ for each $0 < g \in G$. Thus $G$ is hyper-archimedean. Since the support of each $0 < g \in G$ is infinite, it follows that $G$ is $l$-isomorphic to $G(g)$ for each $0 < g \in G$. Since $G$ has a strong unit, each cardinal summand has a strong unit and so each cardinal summand is isomorphic to $G$.

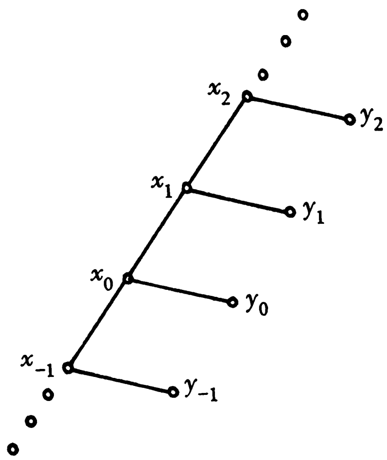
Since $G$ is not an $o$-group, there exists $0 < a, b \in G$ such that $a \wedge b = 0$. Thus $G(a+b) = G(a) \boxplus G(b)$. $G$ is $l$-isomorphic to $G(a+b)$ and therefore cardinally decomposable. This argument is valid for any $l$-group $H$ such that $H$ is $l$-isomorphic to $H(h)$ for each $0<h\in H$ and $H$ is not an $o$-group.

Let $C$ be a proper $l$-ideal of $G$. Then $C$ does not contain a strong unit of $G$. There exists $0<a\in C$ and $0<b\in G\backslash C$ such that $a\wedge b=0$ and $a+b$ is a strong unit. Thus $G=G(a+b)=G(a)\boxplus G(b)$. $a=a_1+a_2$ and $b=b_1+b_2$ where $0<a_1, a_2, b_1, b_2\in G$, $a_1\wedge a_2=0$, $b_1\wedge b_2=0$. Since $b\notin C$ we may suppose (without loss of generality) that $C\cap G(b_2)\neq G(b_2)$. Now $G=G(a_1)\boxplus G(a_2)\boxplus G(b_1)\boxplus G(b_2)$, and $G(a_1)$ is $l$-isomorphic to $G(b_2)$. Any $l$-isomorphism between $G(a_1)$ and $G(b_2)$ induces an $l$-isomorphism $\tau$ of $G$ such that $0\neq C\cap C\tau\neq C$. Thus $G$ is $s$-simple.

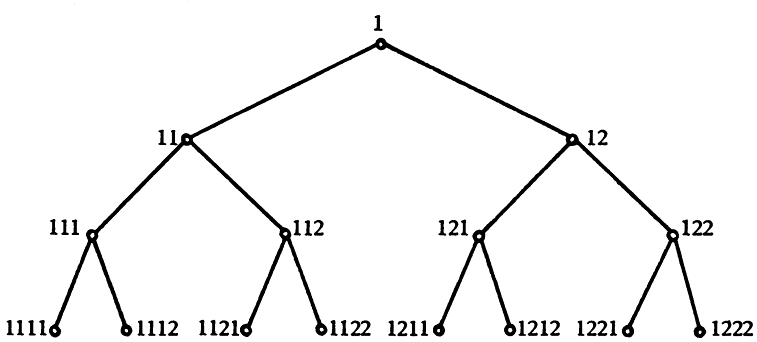
No regular subgroup of $G$ is a cardinal summand and so $G$ is not completely reducible.

EXAMPLE 9.3. *An example of a characteristically simple vector lattice that contains a special element but is not finite valued* (*see Theorem* 4.7).

First we construct a root system $\Lambda$ from the root systems $\Lambda_1$ and $\Lambda_2$ given below. Let $\Lambda_1$ be the root system




and let $\Lambda_2$ be the root system




To each $y_i$ attach a copy of $\Lambda_2$ where we identify the point $y_i$ and 1. Let $\Lambda$ be the resulting root system.

Note that for each $\alpha$ in $\Lambda$, the set $\{\lambda \in \Lambda \mid \lambda \leqq \alpha\}$ is $o$-isomorphic to $\Lambda_2$. Moreover, if $\alpha, \beta \in \Lambda$, then there exists an $o$-automorphism $\pi$ of $\Lambda$ such that $\beta < \alpha\pi$. Let $F$ be the $l$-ideal of $V(\Lambda, R_\lambda)$ that is defined in Lemma 4.6, where $R_\lambda$ is the group of real numbers for each $\lambda \in \Lambda$. Then $F$ has a special element and an element that has an infinite number of values (in fact, infinitely many of both types). To show that $F$ is characteristically simple, it suffices to show that if $0 < a, b \in F$ are special, then there exists $\sigma \in A(F)$ such that $a\sigma > b$. Each of $a$ and $b$ has exactly one maximal component in $\Lambda$, say $\alpha$ and $\beta$ respectively. There exists an $o$-automorphism $\pi$ of $\Lambda$ such that $\alpha\pi > \beta$ and $\pi$ induces an $l$-automorphism $\tau$ of $V(\Lambda, R_\lambda)$. If $\sigma = \tau|F$, then $a\sigma > b$.

EXAMPLE 9.4. *An example of a characteristically simple l-group with a strong unit that is not archimedean.*

Let $\mathscr{P}(R)$ denote the $l$-group of $o$-permutations of the naturally ordered set of real numbers. Let $f \in \mathscr{P}(R)$ be defined by $xf = x+1$ and let $G = \{g \in \mathscr{P}(R) \mid gf = fg\}$. It is known that $G$ is a nonarchimedean simple $l$-group and $f$ is a strong unit in $G$.

EXAMPLE 9.5. *An example of an l-group in which no principal convex l-subgroup is characteristic but which has proper characteristic subgroups.*

For each natural number $n$, let $R_n$ denote the additive group of reals and let $G = \prod R_n$. Then no principal convex $l$-subgroup is characteristic, but $\sum R_n$ is a characteristic subgroup of $G$.

EXAMPLE 9.6. *Examples of characteristic subgroups of an l-group G.*

(a) The radical, ideal radical, and distributive radical of $G$ (see [7]).

(b) The lex-kernel (i.e., the join of all minimal prime subgroups) of $G$.

(c) The subgroup generated by the singular elements of $G$ ($s \in G$ is *singular* if $s > 0$ and if $0 \leqq a < s$, $a \in G$ implies that $a \wedge (s-a) = 0$).

(d) If $S(G)$ is a normal subgroup of $A(G)$, then the $\mathscr{S}$-socle is characteristic.

(e) The $l$-ideal $F$ of Lemma 4.6.

(f) The subgroup generated by the convex $o$-subgroups of $G$.

(g) The intersection of the maximal $l$-ideals or maximal convex $l$-subgroups of $G$.

(h) The convex $l$-subgroup generated by a characteristic subgroup of the group $G$.

(i) If $A$ is an infinite cardinal and if $X$ is the collection of all $g \in G^+$ such that the cardinality of any disjoint subset of $G$ bounded by $g$ is less than $A$, then $[X]$ is characteristic.

EXAMPLE 9.7. *Examples of characteristically simple l-groups.*

(a) The periodic sequences of real numbers.

(b) For each natural number $n$, let $Q_n$ denote the additive group of rationals. Then $G = \prod Q_n / \sum Q_n$ is characteristically simple; in fact $A(G)$ acts transitively on the collection of nonunits of $G$.

(c) $C(X)$ where $X$ is compact with a transitive group of homeomorphisms.

(d) If $\Lambda$ is a root system with property (c) of Theorem 5.8, then $\Sigma(\Lambda, R_\lambda)$ and $F(\Lambda, R_\lambda)$ are characteristically simple.

(e) If $C$ is a maximal characteristic subgroup of $G$, then $G/C$ is characteristically simple.

(f) (Bleier) A free abelian $l$-group of finite rank.

(g) The $l$-group $\mathscr{B}(N)$ of all $o$-permutations of a totally ordered set $N$ having bounded support, where the $o$-permutation group of $N$ acts irreducibly on $N$ [17].

9.8. Let $\mathscr{S}, \mathscr{T}, \mathscr{U}, \mathscr{V}$, and $\mathscr{W}$ denote the classes of simple, $s$-simple, characteristically simple, completely reducible, and completely $s$-reducible $l$-groups respectively. We have noted in 3.8 that $\mathscr{S} \subseteq \mathscr{T} \subseteq \mathscr{U}$, in 3.11 that $\mathscr{V} \subseteq \mathscr{W}$, and it is clear that $\mathscr{S} \subseteq \mathscr{V}$. The $l$-group of Example 9.1 belongs to $\mathscr{T}$ but not $\mathscr{V}$. Let $Q$ and $R$ denote the additive groups of rationals and reals respectively. Then $Q \boxplus R$ belongs to $\mathscr{V}$ and not $\mathscr{U}$. $R \boxplus R$ belongs to $\mathscr{U} \cap \mathscr{V}$ and not $\mathscr{T}$.

We conclude by asking the following questions.

I. *If $G$ is an abelian l-group and if $H$ is its divisible hull, then is $G$ self-injective if and only if $H$ is self-injective*?

II. *Does Theorem* 7.4 *hold for abelian l-groups as well as vector lattices*?

III. *What can be said of s-simple l-groups*?

IV. *If $G = A \boxplus B$ is characteristically simple, then is $A$ characteristically simple*?

V. *If $G$ is an archimedean l-group, can $G$ be embedded in a characteristically simple archimedean l-group*?

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