ENGULFING CONTINUA IN AN n-CELL

BY
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Abstract. In this paper it is shown that there exist open connected subsets $D_1$, $D_2$, and $D_3$ of an n-cell $E$ such that, if $C$ is any proper compact connected subset of $E$ and $C \subseteq U$, $U$ open, then there exists a homeomorphism $h$ of $E$ onto itself such that $C \subseteq h(D_i) \subseteq U$ for some $i$, $1 \leq i \leq 3$.

1. Introduction. The main results of this paper are the following two theorems:

ENGULFING Theorem. Let $E$ be an n-cell, $n \geq 2$. There exist proper domains (a domain is an open, connected subset) $D_1$, $D_2$, and $D_3$ of $E$ such that if $X \subseteq E$ is a proper continuum (a continuum is a compact, connected subset) and $G$ is an open set, $X \subseteq G \subseteq E$, then there is a homeomorphism $h$ of $E$ onto itself which is isotopic to the identity and

(i) $X \subseteq h(D_1) \subseteq G$, $h|\partial E = \text{identity}$, if $X \subseteq \text{int } E$;
(ii) $X \subseteq h(D_2) \subseteq G$, if $X \cap \partial E \neq \emptyset$ and $X \cap \partial E \neq \partial E$; and
(iii) $X \subseteq h(D_3) \subseteq G$, $h|\partial E = \text{identity}$, if $X \cap \partial E = \partial E$.

APPROXIMATION Theorem. Let $E$ be an n-cell, $n \geq 2$. There exists a continuum $C \subseteq E$ with the following properties:

(i) $C$ is homeomorphic to the 1-point compactification of a connected n-manifold with boundary;
(ii) if $D$ is a proper domain of $E$, then $D = \bigcup_{k=1}^{\infty} C_k$, where, for all $k \geq 1$, $C_k \subseteq i C_{k+1}$ (i $C_{k+1}$ denotes the point set interior of $C_{k+1}$ relative to $E$) and $C_k$ is homeomorphic to $C$; and
(iii) if $X$ is a proper continuum in $E$, then $X = \bigcap_{k=1}^{\infty} C_k$, where, for all $k$, $C_{k+1} \subseteq i C_k$ and $C_k$ is homeomorphic to $C$.

In what follows, certain notational conventions and definitions will be used. If $X$ is a topological space and $A$ a subset of $X$, then $i_X A$, $cl_X A$, $fr_X A$, and $ed_X A$ will denote the interior, closure, frontier, and edge of $A$ in $X$ respectively, where $ed_X A = A - i_X A$. If there is no possibility of ambiguity, then the subscript "X" will be omitted. If $M$ is a manifold (all manifolds are assumed to be separable

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metric spaces), then \( \text{int } M \) and \( \text{bd } M \) will denote the manifold interior and boundary of \( M \) respectively. If a space \( X \) is homeomorphic to a space \( Y \), then this will be denoted by \( X \equiv Y \).

2. Preliminary results.

**Definition 2.1.** Let \( L \) and \( M \) be \( n \)-manifolds, \( n \geq 1 \), such that \( \text{bd } L \neq \emptyset \) and \( L \subseteq M \). \( L \) is called a relative \( M \) manifold if

(i) \( L \cap \text{bd } M \neq \emptyset \),

(ii) \( L \cap \text{bd } M \) is an \((n-1)\)-manifold, and

(iii) \( \text{ed } L = L - i L \) is empty or an \((n-1)\)-manifold.

Note that if \( L \) is a relative \( M \) manifold, then \( \text{bd } (L \cap \text{bd } M) = \text{bd } (\text{ed } L) \).

**Definition 2.2.** Let \( L \) and \( M \) be \( n \)-manifolds, \( n \geq 1 \), such that \( L \subseteq M \), \( \text{bd } L \neq \emptyset \), and \( \text{ed } L \neq \emptyset \). If \( L \subseteq \text{int } M \) or if \( L \) is a relative \( M \) manifold, then \( L \) is said to be **collared** in \( M \) if there is an embedding \( h \) of the pair \( (\text{ed } L \times [0, 1), (\text{bd } L \times [0, 1]) \) into the pair \( (M, \text{bd } M) \) such that

(i) \( h(x, 0) = x, x \in \text{ed } L \), and

(ii) \( h(\text{ed } L \times [0, 1)) \cap L = h(\text{ed } L \times 0). \)

The set \( h(\text{ed } L \times [0, 1)) \) is called a **collar** of \( L \) in \( M \) and is denoted by \( cL \). The set \( CL = L \cup cL \) is called a collaring of \( L \) in \( M \).

**Lemma 2.3.** Let \( L \) be a relative \( M \) manifold, \( n \geq 2 \), such that \( \text{ed } L \neq \emptyset \). Then there is an embedding \( h \) of the pair \( (\text{ed } L \times [0, 1), (\text{ed } L \cap \text{bd } M) \times [0, 1]) \) into \( (M, \text{bd } M) \) such that

(i) \( h(\text{ed } L \times [0, 1)] \subseteq L \), and

(ii) \( h(x, 0) = x, x \in \text{ed } L \).

**Proof.** Let \( E = \text{ed } L \). If \( \text{bd } E = \emptyset \), then, since \( E \subseteq \text{bd } L \), the result follows from the main result of [2]. Now suppose that \( \text{bd } E \neq \emptyset \) and let \( Q = L \cap \text{bd } M \). Then \( \text{bd } E = \text{bd } Q \). It follows from [2] that there is an embedding \( g_1 : \text{bd } E \times [-1, 0] \rightarrow E \) such that \( g_1(\text{bd } E \times [-1, 0]) = F_1 \) is closed in \( E \) (that \( g_1 \) can be chosen so that \( F_1 \) is closed is shown possible in the proof of Theorem 2.6) and \( g_1(x, 0) = x \) for all \( x \in \text{bd } E \). Also, there is an embedding \( g_2 : \text{bd } E \times [0, 1) \rightarrow Q \) such that \( g_2(x, 0) = x \) for all \( x \in \text{bd } E = \text{bd } Q \). Let \( P = E \cup g_2(\text{bd } E \times [0, 1]) \). Since \( P \subseteq \text{bd } L \), there is an embedding \( g_3 : P \times [0, 1) \rightarrow L \) such that \( g_3(x, 0) = x, x \in P \), and \( g_3(P \times [0, 1)) \cap \text{bd } L = e_3(P \times 0) \). Let \( f \) be a homeomorphism of \( \text{bd } E \times ((-1, 0] \times [0, 1]) \) onto \( \text{bd } E \times ((-1, 1] \times [0, 1)) \) such that

(i) \( f \) restricted to \( (\text{bd } E \times (-1, 0]) \cup (\text{bd } E \times (-1, 0] \times 0) \) is the identity, and

(ii) \( f \) carries \( \text{bd } E \times 0 \times [0, 1) \) homeomorphically onto \( \text{bd } E \times [0, 1] \times 0 \).

Define \( g : \text{bd } E \times [-1, 1] \rightarrow \text{bd } L \) by

\[
g(x, t) = \begin{cases} g_1(x, t), & t \in [-1, 0], \\
= g_2(x, t), & t \in [0, 1). \end{cases}
\]

Then \( h = (g, \text{id}) f (g_1 \text{id})^{-1}, \text{id} \), where \( \text{id} \) is the identity map on \([0, 1]\), is a homeomorphism of \((g_1(\text{bd } E \times [-1, 0])) \times [0, 1) \) onto \((\text{bd } E \times ([0, 1])) \times [0, 1) \). Let
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\(F_1 = g_3(\partial E \times [-1, 0])\) and \(F_2 = E - g_3(\partial E \times (-1, 0])\). Then \(F_1\) and \(F_2\) are closed in \(E\) and \(F_1 \cap F_2 = F = g_1(\partial E \times -1)\). Since \(h_1(F \times [0, 1])\) is the identity, the map \(h_2: E \times [0, 1) \to P \times [0, 1)\) defined by

\[
\begin{align*}
    h_2(x, t) &= h_1(x, t), & x &\in F_1, \\
    &= (x, t), & x &\in F_2,
\end{align*}
\]

is an onto homeomorphism. Since \(h_1\) carries \(g_3(\partial E \times 0) \times [0, 1)\) homeomorphically onto \(g(\partial E \times [0, 1))\times 0\) and is the identity on \(g_3(\partial E \times 0)\times 0\), \(h_2 = g_3h_2\) is the required embedding of \((\text{ed} L \times [0, 1), (\text{ed} L \cap \partial M) \times [0, 1))\) into \((M, \partial M)\).

**Theorem 2.4.** Let \(L\) and \(M\) be \(n\)-manifolds, \(n \geq 2\), such that \(L\) is a closed subset of \(M\) and \(\text{ed} L \neq \emptyset\). Then \(L\) is collared in \(M\) if

(i) \(L \subset \text{int} M\) and \(P = M - \text{int} L\) is an \(n\)-manifold, or

(ii) \(L\) is a relative \(M\) manifold and \(P = M - \text{int} L\) is a relative \(M\) manifold.

**Proof.** If (i) holds, then \(\text{ed} L \subset \partial P\) and the result follows from [2]. If (ii) holds, then \(\text{ed} L = \text{fr} L = \text{fr} P = \text{ed} P\). Since \(\text{ed} L = \text{ed} P\), the result follows by applying Lemma 2.3 to \(P\).

Let \(X\) be a metric space, \(A\) a subset of \(X\), and \(f, g: A \to R\) bounded continuous functions such that, for all \(a \in A\), \(f(a) \leq g(a)\). Let

\[
P[f, g; A] = \{(x, t) \in X \times R | x \in A \text{ and } f(x) \leq t \leq g(x)\}.
\]

If \(f(a) < g(a)\) for all \(a \in A\), let

\[
TP[f, g; A] = \{(x, t) \in X \times R | x \in A \text{ and } f(x) \leq t < g(x)\}.
\]

**Definition 2.5.** Suppose that \(L\) is collared in \(M\) and that \(B\) is a closed set, perhaps empty, \(B \subset \text{cl}_M (\text{ed}_M L) = C\). Consider \(\text{ed} L \times [0, 1)\) as contained in \(C \times R\) and let \(h: \text{ed} L \times [0, 1) \to M\) give a collar of \(L\) in \(M\). Let \(f\) be continuous, \(f: C \to [0, \frac{1}{2}]\) such that \(f(c) = 0\) if and only if \(c \in (B \cup (C - \text{ed} L))\). Let \(TL\) and \(TL^*\) be defined by

\[
TL = L \cup h(TP[0, \frac{1}{2}; \text{ed} L - B])
\]

and

\[
TL^* = L \cup C \cup h(P[0, \frac{1}{2}; \text{ed} L]).
\]

TL is called a tapered collaring of \(L\) in \(M\) with base \(B\) and support \(F\) if, given an open set \(G\), \(L - B \subset G\), then there is a homeomorphism \(g\) of \(M\) onto itself, which is isotopic to the identity under an isotopy \(H\) such that

(i) \(L - B \subset g(TL) \subset G\),

(ii) if \(L \cup C \subset G\), then \(L \cup C \subset g(TL^*) \subset G\), and

(iii) \(H(x, t) = x\) for all \(x \in (L \cup (M - F)), t \in [0, 1]\).

**Theorem 2.6.** Let \(L\) and \(M\) be \(n\)-manifolds, \(L \subset M\), \(L\) a relative \(M\) manifold or \(L \subset \text{int} M\). If \(L\) is collared in \(M\) and \(B \subset \text{cl}_M (\text{ed}_M L) = C\), then \(L\) has
a tapered collaring TL in M with base B, TL=ι_M L and TL*=ι_M L ∪ C. Furthermore, if C is compact, then the support F of TL may be chosen to be compact also.

**Proof.** The proof is straightforward but tedious, and so only an outline of the proof will be given. Let \( A=\text{ed}_M L \). If \( \text{cl}_M A \) is compact, let \( X=M \); otherwise, let \( X \) be the 1-point compactification of \( M \) with compactification point \( p \), and consider \( M \) as embedded in \( X \). In either case, since \( A \) is locally compact, \( D=\text{cl}_X A-A \) is either empty or compact. If \( B \) is closed, \( B=\text{cl}_M (\text{ed}_M L)=C \), then \( B \cup D \) is closed in \( E=\text{cl}_X A \). Let \( h: A \times [0, 1) \to M \) give a collar \( c \) of \( L \) in \( M \). Since \( L \) is a relative \( M \) manifold or \( L \subset \text{int } M \), it follows from Lemma 2.3 or [2] that there is an embedding \( h_1: (A \times (-1, 0], (A \cap \text{bd } M) \times (-1, 0]) \to (M, \text{bd } M) \cap L \) such that \( h(x, 0)=h_1(x, 0)=x, x \in A \). Since \( B \cup D \) is closed in \( E \) there exist continuous functions \( f: E \to [0, \frac{1}{2}] \) and \( f_1: E \to (-\frac{1}{2}, 0] \) such that \( f(e)=f_1(e)=0 \) if and only if \( e \in B \cup D \). Let \( P=P[f_1, f; E] \). Then \( P \) is compact and there is an embedding \( h_2: P \to X \) defined by

\[
\begin{align*}
  h_2(x, t) &= x, & x \in E, t = 0, \\
  &= h(x, t), & x \in A, 0 \leq t \leq f(x), \\
  &= h_1(x, t), & x \in A, f_1(x) \leq t \leq 0.
\end{align*}
\]

Let \( G \subseteq M \), \( G \) open, and suppose that \( L-B \subseteq G \). Since \( G \) is open in \( M \), \( G \) is open in \( X \) and \( H=(G-(B \cup D)) \) is open in \( X \). Let \( H^{'}=(h_2)^{-1}(H) \). Then \( H^{'} \) is open in \( P \), \( (E-(B \cup D)) \times 0 \subseteq H^{'} \), and \( (B \cup D) \times 0 \subseteq P-H^{'} \). Therefore there exist continuous functions \( g: E \to [0, \frac{1}{2}] \) and \( g_1: E \to (-\frac{1}{2}, 0] \) such that

(i) \( f_1(e) \leq g_1(e) \leq 0 \leq g(e) \leq f(e), e \in E \);
(ii) \( f_1(e) < g_1(e) < 0 < g(e) < f(e), e \in (E-(B \cup D)) \); and
(iii) \( P[g_1, g; E-B \cup D] \subseteq H^{'} \).

Let \( E_1=E \) if \( X=M \) and let \( E_1=E-p \) otherwise. If \( L \cup C \subseteq G \), then \( g \) may be chosen so that \( P[0, g; E_1] \subseteq h_2^{-1}(G) \). Using the results of [8, p. 556], it is easily established that

(i) \( TL=L \cup h(TP[0, \frac{1}{2} f_; \text{ed } L-B]) \) is a tapered collaring in \( M \) with base \( B \);
(ii) \( TL=ι_M L \); and
(iii) \( TL^*=h_2(P[0, \frac{1}{2} f_; E_1] \cup L=L \cup C \cup h(P[0, \frac{1}{2} f_; \text{ed } L]) \) is homeomorphic to \( L \cup C \).

3. **Proof of the main results.** The proof of the main results depends upon some results of piecewise linear manifold theory and the results of the previous section. The terminology that will be used is essentially that used by Zeeman in [10] but is modified to agree with the terminology used by Zeeman and Hudson in [6]. By a *simplicial complex* \( K \) of \( R^p, p \geq 1 \), is meant a locally finite complex consisting of at most a countable number of rectilinear simplexes in a Euclidean \( p \)-space \( R^p \). As usual, \( |K| \) denotes the polyhedron determined by \( K \). An \( n \)-manifold \( M \) is called
combinatorial or piecewise linear, denoted by PL, if there is a homeomorphism \( f \) of \( \vert K \vert \) onto \( M \) where \( K \) is a complex in \( R^n \) such that if \( n \geq 1 \) then \( |lk(v,K)| \) is a combinatorial \((n-1)\)-sphere or \((n-1)\)-ball for each vertex \( v \) of \( K \). The pair \((K,f)\) is called a PL triangulation of \( M \). Let \( X \subset M \), \( X \) a manifold. If there is a subcomplex \( L \) of \( K \) such that \((L,f \mid \vert L \vert)\) is a PL triangulation of \( X \), then we will say that \((K,f)\) restricted to \( X \) gives a PL triangulation of \( X \). An \( m \)-manifold \( L \) which is embedded as a subset of a PL \( n \)-manifold \( M \) is called PL in \( M \) if there is a subdivision \( K' \) of \( K \) and a subcomplex \( P' \) of \( K' \) such that \((P',f \mid \vert P' \vert)\) is a PL triangulation of \( f(|P'|) = L \). Let \( M \) and \( N \) be PL manifolds with PL triangulations \((f,K)\) and \((g,L)\) respectively. A continuous function \( h: M \to N \) is called PL if there are subdivisions \( K' \) of \( K \) and \( L' \) of \( L \) such that \( g^{-1}h(f) \) is a simplicial map. If \( t \) is an \( n \)-simplex, let \( |t| \) denote the polyhedron \( |s(t)| \) where \( s(t) = \{s : s \text{ is a face of } t \} \).

**Lemma 3.1.** Let \( M \) be a PL \( n \)-manifold, \( n \geq 2 \), and let \( L \) be a PL \( n \)-manifold in \( M \), ed \( L \neq \emptyset \). If

(i) \( L \subseteq \text{int } M \) or

(ii) \( L \cap \text{bd } M = Q \) is a PL \((n-1)\)-manifold in \( M \), then \( L \) is collared in \( M \).

**Proof.** The result is easily established and the proof is only outlined. The first step is to use the method employed (with proper modifications in case \( L \cap \text{bd } M \) is a PL \((n-1)\)-manifold in \( M \)) in the proof of Lemma 17 of Chapter 3 of [10] to show that \( M - \text{int } L \) is a PL \( n \)-manifold in \( M \). If (ii) holds, then the above method shows that \( \text{bd } M - \text{int } Q = \emptyset \) or a PL \((n-1)\)-manifold in \( M \) and that \( \text{bd } L - \text{int } Q = \text{ed } L = \text{ed } (M - \text{int } L) \) is a PL \((n-1)\)-manifold in \( M \). Thus both \( L \) and \( M - \text{int } L \) are relative \( M \) manifolds. Since \( L \) is PL in \( M \), \( L \) is closed and it follows from Theorem 2.4 that \( L \) is collared in \( M \).

**Lemma 3.2.** Let \( M \) be a connected PL \( n \)-manifold, \( n \geq 2 \), with PL triangulation \((K,f)\), \( D \) a proper domain of \( M \), and \( C \) a continuum contained in \( D \). Then there exists a connected, PL \( n \)-manifold \( L \) in \( M \) such that

(i) \( C \subseteq i L \subseteq D \), \( L \neq D \);

(ii) \( L = f(|P|) \) where \( P \) is a subcomplex of \( Sd_j K \), the \( j \)th regular barycentric subdivision of \( K \), for some \( j \geq 1 \);

(iii) \( L \subseteq \text{int } M \) if \( C \subseteq \text{int } M \); and

(iv) \( L \cap \text{bd } M = \emptyset \) or \( D \) is a PL \((n-1)\)-manifold if \( C \cap \text{bd } M \neq \emptyset \).

**Proof.** Since \( K \) is locally finite, it may be assumed that \( M \) is embedded in some \( R^n \) as a closed subset and that \( M \) has PL triangulation \((K, \text{id})\), where \( \text{id} \) denotes the identity map. Let \( d \) be the Euclidean metric on \( R^n \) restricted to \( M \) and choose \( \rho > 0 \) such that \( 0 < \rho \leq d(C, M - D) \) and set \( Q = \{x \in M : d(x, C) < \rho \} \). Note that \( Q \subseteq D \) and that if \( C \cap \text{bd } M = \emptyset \) then it may be assumed that \( \rho \) is chosen so that \( Q \subseteq \text{int } M \). The existence of \( L \) is now established using the results and terminology of [6].
If \( J \) is a simplicial complex, \( X \in |J| \), let \( N(X, J) = \{ s \in J \mid s \text{ is a face of } t, t \cap X \neq \varnothing \} \). Let \( L_1 = N(C, K) \). Since \( C \) is compact, \( L_1 \) is finite and \( C \subset |L_1| \). There exists an integer \( q \geq 1 \) such that if \( L_2 = N(C, S_d \cup L_1) \), then, for all \( j \geq q, N(|L_2|, S_d \cup L_1) = N(|L_2|, S_d, K) \) and \( |N(|L_2|, S_d \cup L_1)| \leq Q \). If \( C \cap \partial M \neq \varnothing \), then there is an \( n \)-simplex \( t_n \in L_2 \) such that \( t_n \) has an \( (n-1) \)-face \( t_{n-1} \) contained in \( \partial M \). Let \( b \) denote the barycenter of \( t_{n-1} \). Then there exists an \( n \)-simplex \( t'_n \in S_d \cup L_2 \) such that \( b \in |t'_n|, \{t'_n \cap \partial M = |t'_n-1|, \) where \( t'_{n-1} \) is an \( (n-1) \)-face of \( t'_n \), and \( |t'_n| \subset i_M |t_n| \).

If \( C \cap \partial M \neq \varnothing \), let \( R = S_d \cup L_2 - \{t'_n, t'_{n-1}\} \) and \( S = \{u \mid u \text{ is a proper face of } t'_{n-1}\} \); if \( C \cap \partial M \neq \varnothing \), let \( R = S_d \cup L_2 \) and \( S = \varnothing \). In either case, \( |R| \) is link collapsible on \( |S| \). Furthermore, if \( C \cap \partial M \neq \varnothing \), then \( |R| \cap \partial M \) is link collapsible on \( |S| \cap \partial M \). Let \( J = N(|R| - |S|, S_d \cup L_1 + K) \). Then \( J \) is a subcomplex of \( N(|L_2|, S_d \cup L_1) \) and so \( |J| \subset Q \). Since \( |R| - |S| \) is connected, it follows from Theorem 1 of [6] that \( |J| \) is a compact, connected PL \( n \)-manifold in \( M \) such that \( |J| \cap \partial M = \varnothing \) if \( C \cap \partial M = \varnothing \), and \( |J| \cap \partial M \) is a PL \((n-1)\)-manifold in \( M \) if \( C \cap \partial M \neq \varnothing \). If \( C \cap \partial M = \varnothing \), let \( L = |J| \); otherwise, let \( P = J \cup S_d \cup S(t'_n) \) and let \( L = |P| \). It is easily seen that \( L \) is the required PL \( n \)-manifold in \( M \).

For \( n \geq 1 \), let \( E^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \|x\| \leq 1\} \), \( S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \) and consider \( R^{n-1}, E^{n-1}, \) and \( S^{n-2} \) as embedded in \( \mathbb{R}^n, \mathbb{E}^n, \) and \( S^{n-1} \) respectively in the usual manner. For \( n \geq 2 \), define \( f: E^{n-1} \to R \) by \( f(x) = (1 - \sum_{i=1}^{n-1} x_i^2)^{1/2} \). Then \( f \) is continuous and \( f(x) = 0 \) if and only if \( x \in S^{n-2} \). For each \( t \in [0, 1] \), let \( B^n(t) = \{(x, s) \mid x \in E^{n-1} \text{ and } 0 \leq s \leq tf(x)\}, A^n(t) = \{(x, s) \in B^n(t) \mid x_1 \geq 0\} \) and \( F^{n-1}(t) = \{(x, s) \in A^n(t) \mid x_1 = 0\} \). The following result follows easily from [8] and the proof is omitted.

**Lemma 3.3.** Let \( 0 < s < t < 1 \). There is a homeomorphism \( h[s, t] \) of \( B^n(1) \) onto itself such that

(i) \( h[s, t](B^n(1)) = B^n(s) \);

(ii) \( h[s, t] \mid \partial B^n(1) = \text{identity} \);

(iii) \( g[s, t] = h[s, t] \mid A^n(1) \) is a homeomorphism of \( A^n(1) \) onto itself carrying \( A^n(t) \) onto \( A^n(s) \) and \( g[s, t] \) restricted to \( (\partial B^n(1) \cap A^n(1)) = \text{identity} \);

(iv) \( h[s, t] \) is isotopic to the identity relative to \( \partial B^n(1) \) and \( g[s, t] \) is isotopic to the identity relative to \( \partial B^n(1) \cap A^n(1) \).

If \( X \) is a topological space and \( Y \subset X \), let \( I(X) \) rel \( Y = \{h \mid h \text{ is a homeomorphism of } X \text{ which is isotopic to the identity relative to } Y\} \) (if \( Y = \varnothing \), we will write \( I(X) \)) and let \( CI(X) = \{h \mid h \text{ is a homeomorphism of } X \text{ onto itself which is isotopic to the identity under an isotopy which is the identity outside some proper compact subset of } X\} \).

**Definition 3.4.** Let \( k_j, 1 \leq j \leq n \), be integers. A homeomorphism \( h \) of \( \mathbb{R}^n \) onto itself is called an **integral translation** if \( h \) is defined by

\[
h(x_1, \ldots, x_n) = (x_1 + k_1, \ldots, x_n + k_n).
\]
Definition 3.5. Let $I^n$ denote the unit cube and consider $I^n$ as embedded in $R^n$. A PL triangulation $(K, id)$ of $R^n$ is called an integral PL triangulation if

(i) $(K, id)$ restricted to $I^n$ is a PL triangulation of $I^n$ and

(ii) any integral translation of $R^n$ induces a PL isomorphism of $K$ onto itself.

It is easily established that integral PL triangulations exist for $R^n$. Let $H^n = \{ x \in R^n \mid x_n \geq 0 \}$. If $(K, id)$ (id denotes the identity homeomorphism) is an integral PL triangulation of $R^n$, then $(K, id)$ restricted to $H^n$ is a PL triangulation of $H^n$ which will be denoted by $(L, id)$. If $(K, id)$ is an integral PL triangulation of $R^n$, then clearly $(Sd_j, K, id)$ is also for all $j \geq 1$.

Henceforth, let $(K, id)$ denote a fixed integral PL triangulation of $R^n$ and $(L, id)$ the restriction of $(K, id)$ to $H^n$. Let

$$M(K) = \{ P \mid P \text{ is a subcomplex of some } Sd_j K, j \geq 1, \text{ and } |P| \text{ is a compact, connected, PL n-manifold in } R^n \}$$

and

$$M(L) = \{ P \mid P \text{ is a subcomplex of some } Sd_j K, j \geq 1, \text{ and } |P| \text{ is a compact, connected PL n-manifold in } H^n \text{ which is a relative } H^n \text{ manifold, and } |P| \cap bd H^n \text{ is a PL (n-1)-manifold in } H^n \}.$$ 

Lemma 3.6. Let $P \in M(K)$. There exists $h \in CI(R^n)$ such that $h(|P|) = |P'|$ for some $P' \in M(L)$.

Proof. Let $q$ be an integer such that $|P| \subset \text{int } X, X = \{ x \in R^n \mid x_n \geq q \}$. Since $(K, id)$ is an integral PL triangulation of $R^n$, $(K, id)$ restricted to $X$ is a PL triangulation $(Q, id)$ of $X$. Since $|P| \subset \text{int } X$, there exists a PL n-cell $B$ in $Q$ such that $B = |J|\, , \, J$ a subcomplex of some $Sd_j Q, j \geq 1$, and $B \cap |P|$ is a PL $(n-1)$-cell in $X$ contained in $bd |P|$ and $B \cap bd X$ is a PL $(n-1)$-cell in $X$. Let $P^* = Sd_j P \cup J$. Then $P^*$ is a subcomplex of $Sd_j Q$. It follows from Theorem 7, Chapter 3 of [10] that there exists $f \in CI(R^n)$ such that $f(|P|) = |P^*|$. Since $|P^*| \cap bd X$ is a PL $(n-1)$-manifold in $X$, it follows that the integral translation $g: R^n \to R^n$ defined by $g(x_1, \ldots, x_n) = (x_1, \ldots, x_n-q)$ carries $|P^*|$ onto $|P'|$, $P' \in M(L)$. Therefore there exists $g' \in CI(R^n)$ such that $g'(|P^*|) = |P'|$. Then $h = g'f \in CI(R^n)$ is the required homeomorphism.

Lemma 3.7. There exists a continuum $C \subset H^n \subset R^n, \, n \geq 2$, with the following properties:

(i) $C$ is homeomorphic to the 1-point compactification of a connected n-manifold with boundary;

(ii) there exists $C_1$ and $D_1, \, C_1 \equiv C, \, D_1$ a connected n-manifold without boundary, $D_1 = I^n; C_1 \equiv I^n \subset C$ such that if $F_1$ is a continuum in $R^n$ and $U$ is open in $R^n, \, F_1 \subset U$, then there exists $g \in CI(R^n)$ such that $F_1 \subset g(D_1) \subset g(C_1) \subset U$; and

(iii) there exists $C_2$ and $D_2, \, C_2 \equiv C, \, D_2$ a connected n-manifold with boundary,
Proof. If $F_1$ is a continuum in $R^n$ and $F_1 \subset U$, $U$ open in $R^n$, then it follows from Lemma 3.2 that there exists $P \in M(K)$ such that $F_1 \subset |P| \subset U$. Similarly if $F_2$ is a continuum in $H^n$, $F_2 \cap \partial H^n \neq \emptyset$, and $F_2 \subset V$, $V$ open in $H^n$, there exists $Q \in M(L)$ such that $F_2 \subset |Q| \subset V$. Therefore it suffices to establish the result for continua of the form $F_1 = |P|$, $P \in M(K)$ and $F_2 = |Q|$, $Q \in M(L)$.

Let $(K, id)$ be the given fixed integral PL triangulation of $R^n$, $(L, id)$ the restriction of $(K, id)$ to $H^n$, $(S, id)$ a fixed PL triangulation of $U^n = \{x \in R^n \mid x_1 > 0\}$, and $(S, id)$ a fixed PL triangulation of $V^n = \{x \in H^n \mid x_1 > 0\}$. For each integer $j \geq 1$, let $c(j) = \{x \in R^n \mid x_1 = 1/j\}$, and for $i, j \geq 1$, $i < j$, let $s(i, j) = \{x \in R^n \mid 1/j \leq x_1 \leq 1/i\}$. Set $q = (1, 0, \ldots, 0)$, $p = (0, 0, \ldots, 0)$, $A = [-1, 1] \times \cdots \times [-1, 1]$, where $[-1, 1]$ appears $(n - 1)$ times, and $B = \{x \in A \mid x_n \geq 0\}$. Let $E$ be the convex hull of $p \cup A$ and $F$ the convex hull of $p \cup B$. Then $E$ is a PL $n$-cell in $R^n$, $F$ is a PL $n$-cell in $H^n$, and $F \cap \partial H^n = G$ is a PL $(n-1)$-cell in $H^n$. Let $X^* = \{R^n, H^n, U^n, V^n\}$. Note that if $N \subset R^n$ is a compact $m$-manifold, and $N$ is PL in $X$, $X \subset X^*$, then $N$ is PL in $Y$ for any $Y \subset X^*$ such that $X \subset Y$. For each $j \geq 1$, let $E_j = E \cap s(j + 1, j)$ and $F_j = F \cap s(j + 1, j)$. Then $E_j$ is a PL $n$-cell in $U^n$ and $F_j$ is a PL $n$-cell in $V^n$ and $F_j \cap \partial H^n$ is a PL $(n-1)$-cell in $V^n$. We now begin the construction of $C$. Let $\{Q_j\}_{j=1}^{\infty}$ be an enumeration of $M(L)$. For each $j \geq 1$, there exists a PL homeomorphism $l_j \in CI(H^n)$, which extends to a PL homeomorphism $k_j \in CI(R^n)$ such that $k_j(Q_j) = M_j$, $M_j \subset i_{H^n} F_j \subset i_{R^n} E_j$. Then $M_j$ is PL in $V^n$. Let $W$ denote the unit $n$-cube, $W = [0, 1] \times \cdots \times [0, 1]$, $X = \{w \in W \mid w_n = 0\}$, $Y = \{x \in W \mid w_n = 1\}$, and $Z = \{w \in W \mid w_{n-1} = 1\}$. By recursive construction a sequence $\{f_j\}_{j=1}^{\infty}$ of embeddings of $W$ in $V^n$ can be constructed such that for $j \geq 1$

(i) $\{f_j(W)\}_{j=1}^{\infty}$ is a disjoint collection of PL $n$-cells in $V^n$;
(ii) $f_j(W) \cap \partial H^n = f_j(Z) = Z_j$ is a PL $(n-1)$-cell in $V^n$;
(iii) $f_j(W) \cap (i_{H^n} (F_j \cup F_{j+1})) \cap (i_{R^n} (E_j \cup E_{j+1}))$;
(iv) $f_j(W) \cap F_j = G_j$ is a PL $n$-cell in $V^n$ and $f_j(W) \cap F_{j+1} = H_{j+1}$ is a PL $n$-cell in $V^n$;
(v) $G_j$ is a PL $(n-1)$-cell in $V^n$ and $H_{j+1} \cap \partial H^n = H_j$ is a PL $(n-1)$-cell in $V^n$;
(vi) $f_j(W) \cap c(j+1) = A_j$ is a PL $(n-1)$-cell in $V^n$ and $A_j \cap H^n = B_j$ is a PL $(n-2)$-cell in $V^n$;
(vii) $f_j(W) \cap (i_{H^n} M_j) = X_j$ is a PL $(n-1)$-cell in $V^n$, $X_j \subset \text{ed}_{H^n} M_j$, and $X_j \subset \partial H^n = X_j$ is a PL $(n-2)$-cell in $V^n$;
(viii) $f_j(W) \cap (i_{R^n} M_{j+1}) = Y_{j+1}$ is a PL $(n-1)$-cell in $V^n$, $Y_{j+1} \subset \text{ed}_{R^n} M_{j+1}$, and $Y_{j+1} \cap H^n = Y_{j+1}$ is a PL $(n-2)$-cell in $V^n$.

Let $M = (\bigcup_{j=1}^{\infty} M_j) \cup (\bigcup_{j=1}^{\infty} f_j(W))$. Then it is easily seen that $M$ is a PL $n$-manifold in both $U^n$ and $V^n$, and $M \cap \partial V^n = M \cap \partial H^n$ is a PL $(n-1)$-manifold in $V^n$. Therefore it follows from Lemma 3.1 that $M$ is collared in $U^n$ and
V^n and hence collared in R^n and H^n. Since cl_{R}^{n} (ed_{R}^{n} M)=ed_{R}^{n} M \cup p and cl_{H}^{n} (ed_{H}^{n} M)=ed_{H}^{n} M \cup p are both compact, it follows from Theorem 2.6 that there exists a tapered collaring TM_1 of M in R^n with base p and a tapered collaring TM_2 of M in H^n with base p. Let D_1 = TM_1, D_2 = TM_2, C_1 = TM_1^n, C_2 = TM_2^n, and C = cl_{R}^{n} M = cl_{H}^{n} M = M \cup p. Then it follows from Theorem 2.6 that C_1 \equiv C \equiv C_2, D_1 \equiv i_{R}^{n} C, and D_2 \equiv i_{H}^{n} C. For k > 1 define

\[ T_k^{1} = f_{k-1}(W) \cup \left( \bigcup_{q=1}^{k-1} F_q \right) = H_k \cup \left( \bigcup_{q=1}^{k-1} F_q \right), \]

\[ T_k^{2} = f_k(W) \cup \left( \bigcup_{q=k+1}^{\infty} F_q \right) \cup p = G_k \cup \left( \bigcup_{q=k+1}^{\infty} F_q \right) \cup p, \]

\[ N_k = R^n - i_{R}^{n} M_k, \] and

\[ P_k = H^n - i_{H}^{n} M_k. \]

Then for k > 1

(i) T_k^{1}, T_k^{2} are PL n-cells in H^n;

(ii) T_k^{1} \cap \text{bd} \ H^n = L_k^{1}, T_k^{2} \cap \text{bd} \ H^n = L_k^{2} are PL (n-1)-cells in H^n;

(iii) T_k^{1}, T_k^{2} are relative N_k and P_k manifolds, T_k^{1} \cap M_k = T_k^{2} \cap \text{bd} \ M_k = f_{k-1}(Y) = Y_k and T_k^{1} \cap M_k = T_k^{2} \cap \text{bd} \ M_k = f_k(X) = X_k are PL (n-1)-cells contained in \text{bd} N_k and \text{bd} P_k; and

(iv) L_k^{1} \cup Y_k = T_k^{1} \cap \text{bd} P_k, L_k^{2} \cup X_k = T_k^{2} \cap \text{bd} P_k, and L_k^{1} \cap Y_k, L_k^{2} \cap X_k are PL (n-2)-cells in P_k.
Therefore it follows from Theorem 2.6 that there exist embeddings
\( \alpha_1, \alpha_2 : B^n(1) \to N_k \) such that
\( \text{(i)} \quad \alpha_1(B^n(1)) \cap \alpha_2(B^n(1)) = \emptyset, \)
\( \text{(ii)} \quad \alpha_q(B^n(0)) = T^2_q \cap \text{bd } N_k = \alpha_q(B^n(1)) \cap \text{bd } N_k, \quad q = 1, 2, \)
\( \text{(iii)} \quad \alpha_q(B^n(1)) \cap M_k = \alpha_q(B^n(0)), \quad q = 1, 2, \text{ and} \)
\( \text{(iv)} \quad \alpha_q(B^n(1)) = T^2_q, \quad q = 1, 2; \)
and embeddings \( \beta_1, \beta_2 : (A^n(1), F^{n-1}(1)) \to (P_k, P_k \cap \text{bd } H^n) \) such that
\( \text{(v)} \quad \beta_1(A^n(1)) \cap \beta_2(A^n(1)) = \emptyset, \)
\( \text{(vi)} \quad \beta_q(A^n(0)) = T^2_q \cap M_k, \quad q = 1, 2, \)
\( \text{(vii)} \quad \beta_q(A^n(1)) \cap M_k = \beta_q(A^n(0)), \quad q = 1, 2, \)
\( \text{(viii)} \quad \beta_q(A^n(1)) = T^2_q, \quad q = 1, 2, \text{ and} \)
\( \text{(ix)} \quad \beta_q(A^n(1)) \cap \text{bd } H^n = \beta_q(F^{n-1}(1)), \quad q = 1, 2. \)

Note that \( C \subset M_k \cup T^1_k \cup T^2_k. \)

Suppose that \( P \in M(K) \) and that \( U \) is open in \( R^n, \) \( |P| \subset U. \) It follows from Lemma 3.6 that there exists \( h \in CI(R^n) \) such that \( h(|P|) = |Q_k|, \) \( Q_k \in M(L). \) Since \( Sd, Q_k \in M(L), \) \( j \geq 1, \) we may assume that \( k > 1. \) Let \( k_\ast \in CI(R^n) \) be the previously chosen PL homeomorphism with the property that \( k_\ast(|Q_\ast|) = M_k. \) Then
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Let \( h_k = k(h \in C(R^n)) \), \( h_k(\{P\}) = M_k \), and \( M_k \subset U_1 = h_k(U) \). Since \( U_1 \) is open in \( R^n \) and \( \alpha_q(B^n(0)) \subset U_q, q = 1, 2, \) there exists \( t \in (0, \frac{1}{2}) \) such that \( \alpha_q(B^n(t)) \subset U_1, q = 1, 2 \). Define \( g_1 : R^n \to R^n \) by

\[
g_1(x) = x, \quad x \notin i_{R^n} (\alpha_1(B^n(1)) \cup \alpha_2(B^n(1))),
\]

\[
g_1(x) = \alpha_q h[t, \frac{1}{2}] \alpha_q^{-1}(x), \quad x \in \alpha_q(B^n(1)), \quad q = 1, 2.
\]

(See Lemma 3.3 for the definition of \( h[t, \frac{1}{2}] \).) Then \( g_1 \in C(R^n) \), \( g_1(C) \subset U_1 \), and \( g_1|M_k = \text{identity} \). Let \( U_2 = g_1^{-1}(U_1) \); then \( U_2 \) is open in \( R^n \), \( C \subseteq U_2 \). Since \( D_1 \) is a tapered collar of \( M \) in \( R^n \) with base \( p \) and compact support, it follows from Definition 2.5 that there exists \( g_2 \in C(R^n) \) such that \( g_2|C = \text{identity} \), \( C - \{p\} \subseteq g_2(D_1) \subseteq g_2(C) \subseteq U_2 \). Let \( g = h_1 g_1 g_2 \); then \( g \in C(R^n) \) and \( |P| \subset g(D_1) \subseteq g(C) \subseteq U \) and thus (ii) of the lemma has been established.

Now suppose that \( Q_k \in M(L) \) and that \( V \) is open in \( H^n \), \( |Q_k| \subseteq V \). Again we may assume that \( k > 1 \). Let \( l_k \in C(H^n) \) be the previously chosen PL homeomorphism with the property that \( M_k = l_k(|Q_k|) \). Then \( M_k \subset V_1 = l_k(V), V_1 \text{ open in } H^n \).

Therefore there exists \( t \in (0, \frac{1}{2}) \) such that \( \beta_q(A^n(t)) \subset V_1, q = 1, 2 \). Define \( h_1 : H^n \to H^n \) by

\[
h_1(x) = x, \quad x \notin i_{H^n} (\beta_1(A^n(1)) \cup \beta_2(A^n(1))),
\]

\[
h_1(x) = \beta_q g[t, \frac{1}{2}] \beta_q^{-1}(x), \quad x \in \beta_q(A^n(1)), \quad q = 1, 2.
\]

Then \( h_1 \in C(H^n) \), \( h_1(C) \subset V_1 \), and \( h_1|M_k = \text{identity} \). Let \( V_2 = h_1^{-1}(V_1) \); then \( V_2 \) is open in \( H^n \) and \( C \subseteq V_2 \). Since \( D_2 \) is a tapered collar of \( M \) in \( H^n \) with base \( p \) and compact support, it follows from Definition 2.5 that there exists \( h_2 \in C(H^n) \) such that \( h_2|C = \text{identity} \), \( C - \{p\} \subset h_2(D_2) \subset h_2(C) \subset V_2 \). Let \( h = l_k h_1 h_2 \). Then \( h \in C(H^n) \) and \( |Q_k| \subset h(D_2) \subset h(C) \subset V \) and (iii) of the lemma is established and the proof is complete.

**Lemma 3.8.** Let \( E \) be an \( n \)-cell, \( n \geq 2 \). There exists \( C_3 \subset E \) such that

(i) \( C_3 = C_1 \) (see Lemma 3.7 for definition of \( C_1 \));

(ii) \( D_3 = i_{E} C_3 \text{ is a connected } n \text{-manifold and } \partial D_3 = \partial E; \text{ and}
\]

(iii) if \( F \) is a proper continuum, \( \partial E \subset F, \text{ and } U \text{ is open in } E, F \subset U, \text{ then there exists } g \in I(E) \text{ rel } \partial E \text{ such that } F \subset g(D_3) \subset g(C_3) \subset U \).

**Proof.** It follows from the construction of \( M \) in Lemma 3.7 that there exists a sequence \( \{B_j\}_{j=2}^{\infty} \) of PL \( n \)-cells in \( R^n \) such that for \( j \geq 2 \)

(i) \( B_j \cap M = B_j \cap \partial M \) is a PL \((n-1)\)-sphere in \( R^n \);

(ii) \( B_j = i_{R^n} F_{M,j}, k(j) \geq 2; \text{ and}
\]

(iii) \( k(j+1) > k(j) \).

Let \( B_1 \) be a PL \( n \)-cell in \( R^n \), \( B_1 \subset i_{R^n} M_1 \). Then it is easily seen that \( C - \text{int } B_1 \equiv C \) where \( C \) is the continuum defined in Lemma 3.7. Let \( f : C \to C_1 \) be a homeomorphism and let \( B = f(B_1) \) and \( C_3 = f(C - \text{int } B_1) = C_1 - \text{int } B \). Then \( C_3 \equiv C \) and \( \partial B \) is a bicollared \((n-1)\)-sphere in \( R^n \). Let \( S \) denote the 1-point compactification of \( R^n \).
with compactification point \( q \) and consider \( R^n \) and all its subsets as subsets of \( S \). Let \( E = S - \text{int } B \), then \( E \) is an \( n \)-cell and \( i_E C_3 = D_3 \) is a connected \( n \)-manifold with \( \text{bd } D_3 = \text{bd } E \). Note that \( C_1 = C_3 \cup B \) and \( D_1 = D_3 \cup B \).

Suppose that \( F \) is a proper continuum in \( E \), \( \text{bd } E \subset F \), and \( F \subset U \), \( U \) open in \( E \). Let \( F_1 = F \cup B \) and \( U_1 = U \cup B \); then \( F_1 \) is a continuum in \( S \) and \( U_1 \) is open in \( S \). There exists \( f \in I(S) \) such that \( f|B = \text{identity} \), \( f|E \in I(E) \), and \( f(F_1) = F \subset R^n \). Let \( U' = f(U_1) \cap R^n \); then \( U' \) is open in \( R^n \) and \( F \subset U' \). Suppose that there exists \( h' \in I(S) \) such that \( h'(E) = E \), \( h'(q) = q \), \( h'|B = \text{identity} \), and \( F \subset h'(D_1) \subset h'(C_1) \subset U' \). Let \( g' = f^{-1} h' \); then \( g'(E) = E \), \( g'|\text{bd } E = \text{identity} \), and \( g'|\text{int } B = \text{int } B \). Since \( h'(\text{int } B) = \text{int } B \) and \( F \subset h'(D_1) \subset \text{int } B \subset h'(C_1) \subset \text{int } B \), it follows that \( F \subset g'(D_3) \subset g'(C_3) \subset U \). Therefore \( g = g'|E \) satisfies (iii) (the fact that \( g \in I(E) \) rel \( \text{bd } E \) follows from [1]). Therefore it suffices to establish the existence of \( h' \).

Since \( F \subset U' \subset R^n \), it follows from Lemma 3.7 that there exists \( h_1 \in C(I(R^n)) \) such that \( F \subset h_1(D_1) \subset h_1(C_1) \subset U' \). Therefore \( h_1 \) extends to a homeomorphism \( g_1 \in I(S) \) such that \( g_1 \) is the identity in a neighborhood of \( q \). Let \( A = \{ g \in I(S) \mid g \) is the identity in a neighborhood of \( q \) and \( g|(S - g_1(D_1)) = \text{identity} \} \). Note that if \( g \in A \), then \( g(g_1(D_1)) = g_1(D_1) \) and \( F \subset g(g_1(D_1)) = g_1(D_1) \). If \( g \in I(S) \), then \( g(B) \) and \( g(E) \) are \( n \)-cells in \( S \) with bicollared boundaries—this fact will be used without explicit mention. Since \( B \cup g_3(B) \subset g_3(D_1) \), there exists \( g_2 \) such that \( g_2 \in A \) and \( g_2 \subset \text{int } B \). It now follows from Lemma 2 of [3] that there exists \( g_3 \in A \) such that \( B \subset \text{int } g_3 g_2 g_1(B) \). Let \( g_4 = g_3 g_2 \) and \( g_5 = g_4 g_1 \); then \( g_4 \in A \) and \( F \subset g_4(D_1) \subset U' \). Let \( \alpha_1 : \text{bd } E \times [1, 2] \to E \) be such that \( \alpha_1(x, 1) = x, x \in \text{bd } E \), and \( \alpha_1(\text{bd } E \times [1, 2]) \subset D_1 \). Define \( \beta_1 : \text{bd } E \times [1, 2] \to g_3(E) \) by \( \beta_1(y) = g_5 \alpha_1(y) \). Since \( g_5 \) is the identity in a neighborhood of \( q \), \( \rho : \text{bd } E \to S \) defined by \( \rho(x) = \beta_1(x, 1) \) and \( j : \text{bd } E \to S \) defined by \( j(x) = x \) have similar orientations (see [4, p. 3]). Therefore it follows from Theorem 3.5 of [4] that there exists a homeomorphism \( \beta_0 : \text{bd } E \times [0, 1] \to g_3(B) - \text{int } B \) such that \( \beta_0(x, 0) = x \) and \( \beta_0(x, 1) = \beta_1(x, 1), x \in \text{bd } E \). Therefore \( \beta : \text{bd } E \times [0, 2] \to E \) defined by

\[
\beta(x, t) = \begin{cases} 
\beta_0(x, t), & t \in [0, 1], \\
\beta_1(x, t), & t \in [1, 2], 
\end{cases}
\]

is an embedding into \( E \). Furthermore, \( B \subset \text{int } g_5(B) \subset g_5(D_1) \) and thus \( \beta(\text{bd } E \times [0, 2]) \subset g_5(D_1) \). Let \( f \) be a homeomorphism of \( \text{bd } E \times [1, 2] \) onto \( \text{bd } E \times [0, 2] \) such that \( f(x, 2) = (x, 2) \) and \( f(x, 1) = (x, 0) \), \( x \in \text{bd } E \). Define \( h' : S \to S \) by

\[
h'(y) = \begin{cases} 
y, & y \in B, \\
\beta f \alpha_1^{-1}(y), & y \in \alpha_1(\text{bd } E \times [1, 2]), \\
g_5(y), & y \in E - \alpha_1(\text{bd } E \times [1, 2]).
\end{cases}
\]

Then \( h' \) is the required homeomorphism.
4. Proof of the engulfing and approximation theorems. Using the results of §3, these theorems are easily established.

Proof of the Engulfing Theorem. Let \( e \) be an embedding of \( \mathbb{R}^n \) onto \( \text{int} \ E \), \( E \) an \( n \)-cell, and let \( e_1 \) be an embedding of \( H^n \) onto \( \mathbb{E}^n - p \), \( p \in \text{bd} \ E \). Consider \( \mathbb{R}^n \) and \( H^n \) as subsets of \( E \). Let \( X \) be a proper continuum, \( X \subset G \), \( G \) open in \( E \). If \( X \subset \text{int} \ E \), then it follows from Lemma 3.7 that there exists \( g \in CI(\mathbb{R}^n) \) such that \( X \subset g(D_1) \subset G \cap \mathbb{R}^n \). Since \( g \in CI(H^n) \), \( g \) extends to \( h \in I(E) \) rel \( \text{bd} \ E \) such that \( X \subset h(D_1) \subset G \). If \( X \cap \text{bd} \ E \) is a proper subset of \( \text{bd} \ E \), there exists \( f \in I(E) \) such that \( f(X) \subset H^n \). It follows from Lemma 3.7 that there exists \( g \in CI(H^n) \) such that \( f(X) \subset g(D_2) \subset f(G) \cap H^n \). Since \( g \in CI(H^n) \), \( g \) extends to \( h' \in I(E) \). Let \( h = f^{-1}h' \). Then \( h \in I(E) \) and \( X \subset h(D_2) \subset G \). If \( \text{bd} \ E \subset X \), the existence of \( h \in I(E) \) rel \( \text{bd} \ E \) such that \( X \subset h(D_3) \subset G \) follows immediately from Lemma 3.8. Thus the theorem is established.

Proof of the Approximation Theorem. If \( D \) is a proper domain of \( E \), \( E \) an \( n \)-cell, \( n \geq 2 \), then it is easily established that \( D = \bigcup_{k=1}^\infty C_k \), where, for \( k \geq 1 \), \( C_k \) is a continuum and \( C_k \subset \text{int} C_{k+1} \). If \( F \) is any proper continuum of \( E \), it is easily established that \( F = \bigcap_{k=1}^\infty C_k \) where, for \( k \geq 1 \), \( C_k \) is a continuum and \( C_k \subset \text{int} C_{k+1} \). In view of the above remarks, the proof of the Approximation Theorem follows immediately from the results of the preceding section.

5. Applications to domain rank. If \( X \) is a topological space and \( U \) an open subset, a topological space \( g(U) \) is called a generator of \( U \) if \( U \) is an open monotone union of \( g(U) \); that is \( U = \bigcup_{k=1}^\infty U_k \) where, for all \( k \geq 1 \), \( U_k = g(U) \), \( U_k \subset U_{k+1} \), and \( U_k \) is open in \( X \). A set \( G \) of connected nonempty topological spaces is called a set of generating domains for \( X \) if for each proper domain \( D \) of \( X \) there exists \( G_x \in G \) such that \( G_x \) generates \( D \). The domain rank of \( X \), denoted by \( DR(X) \), is defined by \( DR(X) = \text{glb} \{ |A| \mid A \text{ is a set of generating domains for } X \} \), where \( |A| \) denotes the cardinality of the set \( A \). A set \( B \) of generating domains for \( X \) is called basic if \( |B| = DR(X) \). We now give a proof of the result mentioned in [9].

Theorem 5.1. \( DR(\mathbb{R}^n) = 1 \), \( DR(H^n) = 2 \), and \( DR(E^n) = 3 \).

Proof. Clearly \( DR(\mathbb{R}^n) \geq 1 \), \( DR(H^n) \geq 2 \), and \( DR(E^n) \geq 3 \). It follows easily from the results of §3 that \( \{D_1\} \), \( \{D_1, D_2\} \), and \( \{D_1, D_2, D_3\} \) are basic sets of generating domains for \( \mathbb{R}^n \), \( H^n \), and \( E^n \) respectively.

For \( n \geq 2 \), let \( F(n) = \{M \mid M \text{ is a connected } n\text{-manifold, } DR(M) < \infty \} \). We are interested in defining an equivalence relation in the class \( F(n) \) which equates those \( n\)-manifolds which have a common basic set of generating domains. Let \( M \), \( N \in F(n) \). \( M \) and \( N \) are called domain equivalent, denoted by \( M \equiv N \), if \( M \) and \( N \) have a common basic set of generating domains. \( M \) and \( N \) are called compactly equivalent, denoted by \( M \equiv N \), if

(i) for each proper compact set \( C \) of \( M \) there is an embedding \( h \) of \( (C, C \cap \text{bd} M) \) into \( (N, \text{bd} N) \) such that \( h(C) \neq N \), and
(ii) for each proper compact set $C$ of $N$ there is an embedding $h$ of $(C, C \cap \text{bd } N)$ into $(M, \text{bd } M)$ such that $h(C) \neq M$.

Clearly $\equiv$ is an equivalence relation in $F(n)$. The following theorem establishes that $\cong$ is also an equivalence relation in $F(n)$ and the relation between $\cong$ and $\equiv$.

**Theorem 5.2.** Let $M, N \in F(n)$. $M \cong N$ if and only if $M \equiv N$.

**Proof.** Suppose that $M \equiv N$. If $C$ is a proper compact set of $M$, then since $n \geq 2$, there exists a proper domain $D$ of $M$ such that $C \subset D$. Since $M$ and $N$ have a common basic set of generating domains, it follows easily that there is an embedding $h$ of $(C, C \cap \text{bd } M)$ into $(N, \text{bd } N)$ such that $h(C) \neq N$. Similarly if $C$ is a proper compact set of $N$, then there is an embedding $h$ of $(C, C \cap \text{bd } N)$ into $(M, \text{bd } M)$ such that $h(C) \neq M$. Therefore $M \equiv N$.

Now suppose that $M \equiv N$. Let $B = \{B_1, \ldots, B_k\}$ be a basic set of generating domains for $M$ and let $G$ be a proper domain of $N$. Since $n \geq 2$, there exists a sequence $\{G_j\}_{j=1}^\infty$ of domains of $N$ such that

(i) $\text{cl}_G G_j$ is compact, $\text{cl}_G G_j \subset G$, $j \geq 1$;
(ii) $\text{cl}_G G_j \subset G_{j+1}$, $j \geq 1$; and
(iii) $G = \bigcup_{j=1}^\infty \text{cl}_N G_j$.

Since $M \equiv N$, for each $j \geq 2$, there exists an embedding $f_j$ of $(\text{cl } G_j, \text{cl } G_j \cap \text{bd } N)$ into $(M, \text{bd } M)$ such that $f_j(\text{cl } G_j) \neq M$. Then $f_j(\text{cl } G_j)$ is compact, $f_j(G_j)$ is a proper domain of $M$, and $f_j^{-1}$ is an embedding of $(f_j(G_j), f_j(G_j) \cap \text{bd } M)$ into $(N, \text{bd } N)$. Since $B$ is a basic set of generating domains for $M$, there exists an integer $q(j)$, $1 \leq q(j) \leq k$, and an embedding $h_j$ of $(B_{q(j)}, \text{bd } B_{q(j)})$ into $(M, \text{bd } M)$ such that $f_j(\text{cl } G_j) \subset h_j(B_{q(j)}) \subset f_j(G_j)$. Therefore $\text{cl } G_{j-1} \subset f_j^{-1} h_j(B_{q(j)}) \subset G_j$ and $f_j^{-1} h_j(B_{q(j)})$ is a domain of $N$. Since $|B|$ is finite, it follows that $B$ is a set of generating domains for $N$ and thus $\text{DR } (N) \leq \text{DR } (M)$. Similarly it can be shown that $\text{DR } (M) \leq \text{DR } (N)$. Therefore $B$ is a basic set of generating domains for $N$ and $M \cong N$.

We close noting that if $M$ is a $W$-body (see [7]) which is not embeddable in $R^3$, then $M \cong R^3$ so $\text{DR } (M) = \text{DR } (R^3) = 1$. Thus there exists a domain of $R^3$ which generates an open connected 3-manifold which is not embeddable in $R^3$.

**Bibliography**