BOUNDED HOLOMORPHIC FUNCTIONS OF
SEVERAL COMPLEX VARIABLES. I

BY
DONG SIE KIM

Abstract. A domain of bounded holomorphy in a complex analytic manifold is a maximal domain for which every bounded holomorphic function has a bounded analytic continuation. In this paper, we show that this is a local property: if, for each boundary point of a domain, there exists a bounded holomorphic function which cannot be continued to any neighborhood of the point, then there exists a single bounded holomorphic function which cannot be continued to any neighborhood of the boundary points.

Introduction. Let $X$ be a topological space. A subset $D$ of $X$ is said to be a region if it is open and if it is said to be a domain if it is open and connected. We denote by $N(p)$ a fundamental system of open neighborhoods of $p$, where $p \in X$.

1. Definition. Let $X$ be a topological space and $U$ be an open subset of $X$. Let $C(U)$ be the family of all continuous complex-valued functions on $U$, then $C(U)$ is an algebra with 1, and it is equipped with the topology of uniform convergence on compact subsets of $U$. For a pair of open subsets $U$ and $V$ in $X$ such that $V \subseteq U$ we define $\pi_{UV}: C(U) \rightarrow C(V)$ by $\pi_{UV}f=f|V$. Let $A(U)$ be a subalgebra of $C(U)$ with 1 and we assume that $\pi_{UV}A(U) \subseteq A(V)$; then we call $A=\{A(U), \pi_{UV}\}$ a presheaf of algebras of functions. A presheaf $A$ has the local belonging property if, for all open sets $U$ of $X$ and $f$ in $C(U)$, for every $p \in U$ there is $V \in N(p)$, $V \subseteq U$, such that $f|V \in A(V)$; then $f \in A(U)$.

A sheaf $A$ of algebras of functions is a presheaf of algebras of functions with the local belonging property. $A$ is said to be a ringed structure on $X$ and the pair $(X, A)$ is said to be a ringed space. The functions in $A(U)$ are $A$-holomorphic functions. We note that $A(U)$ has the relative topology induced by the topology on $C(U)$.

A ringed structure $A$ on $X$ is an $n$-dimensional complex analytic structure on $X$ if for all $x \in X$ there are $U \in N(x)$ and $f_1, \ldots, f_n \in A(U)$ such that

$$F = (f_1, \ldots, f_n): U \rightarrow \mathbb{C}^n$$
is a homeomorphism of $U$ onto $F(U)$ with the properties: $F(U)$ is open in $\mathbb{C}^n$ and for all $W$ open $\subseteq U$, $(f \circ (F|W)^{-1} : f \in A(W)) = \mathcal{O}(F(W))$, where $\mathcal{O}$ is a complex analytic structure on $\mathbb{C}^n$. If $X$ is a hausdorff space we call this pair $(X, A)$ a complex analytic manifold.

For a subset $U$ of $X$, $A(U)$ is quasi-analytic if for all nonempty open subsets $V$ of $U$ and for $f, g$ in $A(U)$ such that $f=g$ on $V$ then $f=g$ on $U$.

We give a characterization of quasi-analyticity in terms of the hausdorffness of the topology on $A$ in the following proposition. The proof may be found in (3).

2. Proposition. Let $(X, A)$ be a ringed space with $X$ a locally connected hausdorff space. Then $A$ is hausdorff if and only if $A(U)$ is quasi-analytic for all connected subsets $U$ of $X$.

Regions of bounded holomorphy.

3. Definition. Let $(X, A)$ be a ringed space and $D$ be a region. We define $B(D) = \{f \in A(D) : f$ is bounded on $D\}$. For a point $p \in \overline{D} - D$ (boundary of $D$) and $U \in N(p)$, a function $f \in B(D)$ is said to be extendable to $U$ if there is a function $g \in B(U)$ such that $f=g$ on $D \cap U$. $D$ is said to be a weak region of bounded holomorphy if there exists a function $f \in B(D)$ which cannot be extendable beyond the boundary of $D$.

$A$ is said to be montel if for an open set $U$ in $X$ and $F \subseteq A(U)$ there is $M_K > 0$ such that $\|f\|_K < M_K$ for all $f \in F$ and for all compact subsets $K$ of $U$; then $F$ is relatively compact in $A(U)$.

$A$ is c.o. complete if for all open subsets $U$ in $X$, $A(U)$ is complete in the topology of uniform convergence on compact subsets of $U$.

We note that an analytic structure $A$ in a complex analytic manifold $(X, A)$ has the montel property, and it is hausdorff and c.o. complete.

We show that the weak bounded holomorphy is a local property in the following theorem.

4. Lemma. Let $(X, A)$ be a ringed space. We assume that $X$ is a locally compact and locally connected hausdorff space, and $A$ is hausdorff, c.o. complete and montel. Let $D$ be a region in $X$ and $p \in \overline{D} - D$ such that $X$ is first countable at $p$. Let $B$ be a closed (relative to the topology of uniform convergence on $D$) subalgebra of $B(D)$. Then these are equivalent:

1° For every $U_n \in N(p)$ there is a function $f_n \in B$ which cannot be extended to $U$.

2° There is a function $f \in B$ which cannot be extended to any neighborhood of $P$.

Proof. It is sufficient to show that (1°) implies (2°). Let $\{U_m : m \in \mathbb{Z}_+\}$ be a countable nested basis of open neighborhood of $p$. Let $B_1(U_m, n) = \{f \in B : f=g|D$ where $g \in B(D \cup U_m)$ and $\|g\|_{U_m} \leq n, n \in \mathbb{Z}_+\}$. We claim that $B_2(U_m, n)$ is a closed nowhere dense subset of $B$. For closedness, let $\{f_k\}$ be any net in $B_1(U_m, n)$ converging uniformly on $D$ to $f$. We note that $\{f_k\}$ is c.o. convergent to $f$. Let $\{g_k\}$ $\subseteq B(D \cup U_m)$ such that $g_k|D=f_k$, $\|g_k\|_{U_m} \leq n, k \in \mathbb{Z}_+$. $\{g_k\}$ is uniformly bounded on
compact subsets of \(D \cup U_m\). Since \(A\) is montel \(\{g_k\}\) is relatively compact in \(A(D \cup U_m)\). Thus there is a subnet \(\{g_k\} \subseteq \{g_k\}\) which converges to \(g \in A(D \cup U_m)\). Now \(\lim_{c.o.} g_j|D = \lim_{c.o.} f_j = f\), so \(g_j|D = f\) and since \(\|g_j\|_V \leq n\) for \(j \in \mathbb{Z}^+\), \(\|g\|_V \leq n\), which concludes that \(f \in B_1(U_m, n)\). For nowhere denseness, let \(B_1(U_m, n) = \bigcup_n B_1(U_m, n)\). Take \(f \in B - B_1(U_m)\) and define \(g_j = j^{-1}f + h\) for \(h \in B_1(U_m, n)\), \(j \in \mathbb{Z}^+\). Then \(g_j \notin B_1(U_m)\) \(\Rightarrow B_1(U_m, n)\) and \(\lim_j g_j = h\). Since \(h\) is an arbitrary element \(B_1(U_m, n)\), \(\text{int} B_1(U_m, n) = \emptyset\).

Let \(B_1 = \bigcup \{B_1(U_m) : m \in \mathbb{Z}^+\}\) and \(B_2 = \{f \in B : f\) can be extended to some neighborhood of \(p\}\). Then \(B_1 = B_2\). Now since \(B\) has the baire property, \(B_1 \subseteq B\). Hence there is \(f \in B - B_1\), so \(f \notin B_2\), \(f\) cannot be extended to any neighborhood of \(p\).

5. THEOREM. Let \((X, A)\) be a ringed space. We assume that \(X\) is a locally compact locally connected hausdorff space, and \(A\) is hausdorff, c.o. complete and montel. Let \(D\) be a region in \(X\) such that \(\overline{D} - D\) is separable and \(X\) is first countable on \(\overline{D} - D\). Let \(B\) be a closed subalgebra of \(B(D)\) as in the lemma. Then these are equivalent:

(i) For every \(p \in \overline{D} - D\) there is a function \(f_p \in B\) which cannot be extended to any \(U \in N(p)\).

(ii) There is a function \(f \in B\) which cannot be extended beyond the boundary of \(D\).

Proof. Let \(\{U_m : m \in \mathbb{Z}^+\}\) be a countable basis of nested open neighborhoods of \(p \in \overline{D} - D\). Let \(B_1(p, U_m, n) = \{f \in B : f = g|D, g \in B(D \cup U_m), \|g\|_V \leq n\}, n, m \in \mathbb{Z}^+\). Then \(B_1(p, U_m, n)\) is a closed nowhere dense subset of \(B\) as in the proof of the lemma. Let \(\{p_i : i \in \mathbb{Z}^+\}\) be a countable dense subset of \(\overline{D} - D\) and \(\{U_m^{(i)}\}\) be a countable basis of nested open neighborhoods of \(p_i\). Let

\[
B_2 = \bigcup \{B_1(p_i, U^{(i)}_m, n) : i, m, n \in \mathbb{Z}^+\}
\]

and

\[
B_3 = \{f \in B : f\) can be extended beyond \(\overline{D} - D\}\).
\]

Then \(B_2 = B_3\). Since \(B\) is baire, \(B_2 \supseteq B\). Hence there is \(f \in B - B_2 = B - B_3\), which asserts (ii).

6. COROLLARY. Let \((X, A)\) be a complex analytic manifold and \(D\) be a region in \(X\) such that \(\overline{D} - D\) is separable and \(X\) is first countable on \(\overline{D} - D\). Let \(B = B(D)\). Then these are equivalent:

(i) For every \(p \in \overline{D} - D\) there is an \(f \in B\) which cannot be extended to any \(U \in N(p)\).

(ii) \(D\) is a weak region of bounded holomorphy.

7. DEFINITION. Let \((X, A)\) be a ringed space and \(D\) be a region in \(X\). Let \(V\) be an open subset of \(X\) such that \(D \cap V \neq \emptyset\) and \(V \supset D\). \(f \in B(D)\) is said to be continued to \(V\) if there is a connected component \(\Omega\) of \(D \cap V\) and \(g \in B(V)\) such that \(f = g\) on \(\Omega\). We say that \(g\) is a continuation of \(f\) to \(V\). A boundary point \(p\) of \(D\) is said to be a boundary singularity for \(f \in B(D)\) if \(f\) cannot be continued to any open
neighborhood of \( p \). A region is called a region of bounded holomorphy if there is an \( f \in B(D) \) for which every boundary point of \( D \) is a boundary singularity.

We give a characterization of a region of bounded holomorphy by a local property in the next theorem.

8. **Lemma.** Let \( (X, A) \) be a ringed space. We assume that \( X \) is a locally compact and locally connected Hausdorff space and \( A \) is Hausdorff, c.o. complete, and Montel. Let \( D \) be a region in \( X \) and \( p \in \overline{D} - D \) such that \( X \) is first countable at \( p \). Let \( B \) be a closed (relative to the topology of uniform convergence on \( D \)) subalgebra of \( B(D) \). Then these are equivalent:

1. For every \( U_a \in N(p) \) and every connected component \( \Omega_{a\beta} \) of \( U_a \cap D \) there is \( f_{a\beta} \in B \) such that \( f_{a\beta} \) has no continuation to \( U_a \).

2. There is \( f \in B \) such that for all \( U \in N(p) \) and for all connected components \( \Omega \) of \( U \cap D \), \( f \) has no continuation to \( U \), i.e. \( p \) is a boundary singularity for \( f \).

**Proof.** It suffices to show that (1°) implies (2°). Let \( \{U_a : a \in \mathbb{Z}^+\} \) be a countable nested basis of open neighborhoods of \( p \) and let \( \{\Omega_{a\beta} : \beta \in \mathbb{Z}^+\} \) be a countable family of connected components of \( U_a \cap D \). Let \( B_1(\Omega_{a\beta}, n) = \{f \in B : \text{there is } g \in B(U_a) \text{ such that } f = g \text{ on } \Omega_{a\beta} \text{ and } \|g\|_{U_a} \leq n\}, n \in \mathbb{Z}^+ \). Then as in the proof of Lemma 4, \( B_1(\Omega_{a\beta}, n) \) is a closed nowhere dense subset of \( B \). Let \( B_1 = \bigcup_{a, \beta, n} B_1(\Omega_{a\beta}, n) \) and let \( B_2 = \{f \in B : f \text{ can be continued to some neighborhood of } p\} \). Then \( B_1 = B_2 \), and since \( B_1 \subseteq B \) there is an \( f \in B - B_2 \).

9. **Theorem.** Let \( (X, A) \) be a ringed space. We assume that \( X \) is a locally compact, locally connected Hausdorff space and \( A \) is Hausdorff, c.o. complete and Montel. Let \( D \) be a region in \( X \) such that \( \overline{D} - D \) is separable and \( X \) is first countable on \( \overline{D} - D \). Let \( B \) be a closed subalgebra of \( B(D) \). Then these are equivalent:

1. For every \( p \in \overline{D} - D \) there is a function \( f_p \in B \) for which \( p \) is a boundary singularity.

2. There is a function \( f \in B \) for which every boundary point is a boundary singularity.

**Proof.** Follows by the lemma and in a similar way as the proof of Theorem 5.

10. **Corollary.** Let \( (X, A) \) be a complex analytic manifold and \( D \) be a region in \( X \) such that \( \overline{D} - D \) is separable. Let \( B = B(D) \). Then these are equivalent:

1. For every \( p \in \overline{D} - D \) there is \( f_p \in B \) for which \( p \) is a boundary singularity.

2. \( D \) is a region of bounded holomorphy.

In the following, we show that a weak region of bounded holomorphy is a region of bounded holomorphy when the region is locally connected on the boundary.

11. **Definition.** Let \( X \) be a topological space and \( D \) be a region in \( X \). We say that \( D \) is locally connected at \( p \in \overline{D} - D \) if \( p \) has a base of open neighborhoods whose intersections with \( D \) are connected. \( D \) is locally connected on the boundary of \( D \) if \( D \) is locally connected at every point of the boundary.

The following lemma will give the proof of Theorem 13.
12. Lemma. Let $X$ be a locally connected Hausdorff space and let $D$ be a region in $X$ which is locally connected on the boundary. Let $V \in N(p)$, $p \in \overline{D} - D$ and $U$ be an open subset of $V \cap D$. Then there is an open set $V_1 \subset U$ such that $V_1 \cap (\overline{D} - D) \neq \emptyset$, $V_1 \cap D$ is connected and $V_1 \cap U \neq \emptyset$.

Proof. We assume that $V$ is a connected neighborhood of $p$.

(i) We show that for every connected component $\Omega$ of $V \cap D$, $V \cap (\overline{\Omega} - \Omega) \subset \overline{D} - D$. Note that $V \cap (\overline{\Omega} - \Omega) \neq \emptyset$, for otherwise we have $V = (V - \overline{\Omega}) \cup \Omega$ which contradicts its connectedness. Now $\Omega \cap D$ so that $V \cap (\overline{\Omega} - \Omega) \subset V \cap \overline{D}$. If $V \cap (\overline{\Omega} - \Omega) \subset D \neq \emptyset$, take $p \in V \cap (\overline{\Omega} - \Omega) \cap D$ then there is a connected open set $U' \in N(p)$ such that $U' \subset V \cap D$ and $U' \cap \Omega \neq \emptyset$. Thus $U' \cup \Omega \subset V \cap D$ is connected. But then $U' \cup \Omega = \Omega$ and $p \in \Omega$, which is a contradiction. It follows that $V \cap (\overline{\Omega} - \Omega) \subset D = \emptyset$ so that $V \cap (\overline{\Omega} - \Omega) \subset \overline{D} - D$.

(ii) Choose a connected component $\Omega$ of $V \cap D$ such that $\Omega \cap U \neq \emptyset$. Take $q \in V \cap (\overline{\Omega} - \Omega) \subset \overline{D} - D$ and choose a neighborhood $V'$ of $q$ such that $V' \subset V$ and $V' \cap D$ is connected. Let $V_1 = \Omega \cup V'$. Since $\Omega \cap V' \neq \emptyset$, $V_1$ has the required property.

13. Theorem. Let $(X, A)$ be a ringed space. We assume that $X$ is a locally compact, locally connected Hausdorff space, and $A$ is Hausdorff, $c._{o.}$ complete and Montel. Let $D$ be a region in $X$ which is locally connected on the boundary. Let $B$ be a closed subalgebra of $B(D)$. Then these are equivalent:

(1) There is a function $f \in B$ which cannot be extended beyond $D$.

(2) There is a function $f \in B$ which cannot be continued beyond $D$.

Proof. It is immediate from the lemma.

14. Corollary. Let $(X, A)$ be a complex analytic manifold. Let $D$ be a region which is locally connected on the boundary. Let $B$ be a closed subalgebra of $B(D)$. Then these are equivalent:

(1) $D$ is a weak region of bounded holomorphy.

(2) $D$ is a region of bounded holomorphy.

We investigate regions of bounded holomorphy in $(\mathbb{C}^n, \emptyset)$. First, we have a useful lemma for searching domains of bounded holomorphy.

15. Lemma. Let $(X, A)$ be a complex analytic manifold and $D$ be a region in $X$. Let $U$ be a domain such that $D \cap U \neq \emptyset$ and $U \notin D$. If every function $f \in B(D)$ can be continued to $U$ and $\tilde{f}$ denotes the continuation of $f$ to $U$, then $\tilde{f}(U) \subseteq \text{cl}(f(D))$ for all $f \in B(D)$.

Proof. Let $\alpha \notin \text{cl}(f(D))$, then $g = (f - \alpha)^{-1} \in B(D)$, and so has a continuation $\tilde{g} \in B(U)$. Now $g \cdot (f - \alpha) \equiv 1$ on $D$, and $g \cdot (f - \alpha) = \tilde{g} \cdot (\tilde{f} - \alpha) \equiv 1$ on a connected component $\Omega$ of $D \cap U$. So by analytic continuation, $\tilde{g} \cdot (\tilde{f} - \alpha) \equiv 1$ on $U$. Hence $\alpha \notin \tilde{f}(U)$. So $\tilde{f}(U) \subseteq \text{cl}(f(D))$.

16. Simple examples of domains of bounded holomorphy in $(\mathbb{C}^n, \emptyset)$.

(1) An open polydisc

$$P(w:r) = P(w_1, \ldots, w_n : r_1, \ldots, r_n) = \{s \in \mathbb{C}^n : |s_i - w_i| < r_i, 1 \leq i \leq n\} \subset \mathbb{C}^n$$
is a domain of bounded holomorphy. For, take a boundary point \( s \in \overline{P(w;r)} \); then 
\(|s_j|=r_j\) for some \( j \). Now for any polydisc \( P(s;\varepsilon), \|z_j\|_{P_j}>r_j \). Hence \( z(P) \neq \text{cl}(Z(P)) \). By Lemma 15, \( P \) is a domain of bounded holomorphy. Moreover, an analytic polyhedron and a bounded complete Reinhardt domain are domains of bounded holomorphy.

(2°) A simply connected domain \( D \) in \( C \) which is locally connected on the boundary of \( D \) is a domain of bounded holomorphy.

17. Proposition. Let \( \{D_j : j \in \mathbb{Z}^+\} \) be an indexed set of regions of bounded holomorphy in \( C^n \). Let \( D=\bigcap_{j=1}^\infty D_j \) and assume that \( D \) is open. Then \( D \) is a region of bounded holomorphy in \( C^n \).

**Proof.** For a point \( p \in \overline{D} - D \) there exists \( m \in \mathbb{Z}_+ \) such that \( p \notin D_m \). Then there exists \( f \in B(D_m) \) which is a singular function at \( p \). Thus \( f|D \in B(D) \) is singular at \( p \).

18. Proposition. A finite cartesian product of regions of bounded holomorphy is a region of bounded holomorphy.

**Proof.** We shall prove this for the case of a product of two regions. Let \( D_1 \) and \( D_2 \) be regions of bounded holomorphy in \( C^n \) and let \( f_i \in B(D_i), i=1,2 \), be singular functions. Define \( F_1 \in B(D \times C^n) \) by \( F_1(s,t)=f_1(s) \) and \( F_2 \in B(C^n \times D_2) \) by \( F_2(s,t)=f_2(t) \). Then \( F_1 \) is a singular function at every point of \( (\text{bdry } D_1) \times C^n \) and so is \( F_2 \) for \( C^n \times (\text{bdry } D_2) \). For, if \( F_1 \) is not, then there is \( V \in N(p), p \in (\text{bdry } D_1) \times C^n \) such that \( F_1 \) can be continued to \( V \). Let \( W \) be the image of \( V \) into \( C^n \Rightarrow D_1 \) then \( F_1|W=f_1 \) can be continued to \( W \). But \( W \) is a neighborhood of a boundary point of \( D_1 \). This is absurd (similarly for \( F_2 \)). Now \( \text{bdry } (D_1 \times D_2)=(\text{bdry } D_1) \times D_2 \cup D_1 \times (\text{bdry } D_2) \). Thus if \( p \in \text{bdry } (D_1 \times D_2) \), then \( F_1 \) or \( F_2 \) is a singular function at \( p \). Hence \( D_1 \times D_2 \) is a domain of bounded holomorphy.

19. Proposition. Every convex (in the geometric sense) domain \( D \) in \( C^n \) is a domain of bounded holomorphy.

**Proof.** Since such a domain \( D \) is the intersection of the open halfspaces in \( C^n \) (as a real vector space \( R^{2n} \)) which contain it, by Proposition 17 it suffices to show that every open halfspace in \( C^n \) is a domain of bounded holomorphy. Let \( S=\{(z_1, \ldots, z_n) \in C^n : \text{Re } z_i>0, i=1, \ldots, n\} \). Then any open halfspace in \( C^n \) can be identified as \( S \) by a translation and a complex linear transformation. Hence again it suffices to show that \( S \) is a domain of bounded holomorphy. But this is so; for, let \( H=\{z \in C : \text{Re } z>0\} \), then since \( H \) can be mapped onto the open unit disc by a Riemann map, \( H \) is a domain of bounded holomorphy. Now \( S=\prod^n H \), a finite cartesian product. Hence \( S \) is a domain of bounded holomorphy by Proposition 18.

20. Proposition. Let \( D \) be a region in \( C^n, n>1 \), and let \( K \) be a compact subset of \( D \) such that \( D-K \) is connected. Then for every \( f \in B(D-K) \) there exists \( f^* \in B(D) \) such that \( f=f^* \) on \( D-K \).
Proof. Since $B(D-K) \subset \mathcal{O}(D-K)$, for every function $f \in B(D-K)$ there is $f' \in \mathcal{O}(D)$ such that $f = f'$ on $D-K$ by a theorem of Hartog's. So it suffices to show that those extensions are still bounded on $D$. But this is clear from Lemma 15.

21. Let $D$ be a region in $\mathbb{C}^n$ and let $B = B(D)$. Then $B$ is a Banach algebra with the supremum norm on $D$. The spectrum of $B$, denoted by $S(B)$, is the set of nonzero complex homomorphisms of $B$. For $z \in D$ if we define $h_z(f) = f(z)$, $f \in B$, then $h_z \in S(B)$. Hence we obtain a mapping $\rho: D \to S(B)$, $\rho(z) = h_z$. To each $f \in B$ we associate a function $\tilde{f}$ defined on $S(B)$ by defining $\tilde{f}(h) = f(h)$. Since $\tilde{f} \circ \rho = f$, the mapping $f \mapsto \tilde{f}$ is one-to-one. We endow $S(B)$ with the weakest topology which makes $\tilde{f}$ continuous. Then $S(B)$ is compact and the mapping $f \mapsto \tilde{f}$ is an isometry of $B$ onto $\tilde{B} = \{ \tilde{f} : f \in B \}$. Hence we may assume that $B$ is defined on $S(B)$. Let $f_1, \ldots, f_n \in B$. The joint spectrum of $f_1, \ldots, f_n$ is the set; $\sigma(f_1, \ldots, f_n) = \{(f_1(h), \ldots, f_n(h)) : h \in S(B)\}$. For given $f_1, \ldots, f_n \in B$ we define $\pi: S(B) \to C^n$ by $\pi(h) = (f_1(h), \ldots, f_n(h))$, then $\pi$ is a continuous map. If $D$ is relatively compact in $C^n$ then the coordinate functions $z_1, \ldots, z_n$ belong to $B$ and $\pi S(B) \supset D$ since the point evaluation maps are in $S(B)$. Furthermore, since $S(B)$ is compact $\pi S(B) \supset D$.

Now we have the following theorem:

22. Theorem. Let $D$ be a relatively compact region in $\mathbb{C}^n$ with $\text{int } D = D$. If $\pi S(B) = \bar{D}$ then $D$ is a region of bounded holomorphy.

Proof. If we assume that $D$ is not a region of bounded holomorphy, then every function $f \in B$ has an extension $\tilde{f}$ to a neighborhood $V$ of a boundary point $p$ of $D$. By Lemma 15, $\tilde{f}(V) \subset \text{cl } (f(D))$. Hence the extensions $\tilde{f}, f \in B$ are continuous with respect to the supnorm on $D$. Now take a point $z \in V - \bar{D}$, consider the point evaluation map $h_z$, $h_z(\tilde{f}) = \tilde{f}(z)$ for all $f \in B$, then $h_z \in S(B)$ and $\pi(h_z) = z \in V - \bar{D}$, which is absurd.

We note that if $\text{int } D \neq D$ then the theorem is false; consider

$$D = \{ z \in \mathbb{C} : 0 < |z| < 1 \}$$

then $B(D) = B(D \cup \{0\})$ and $S(B) = \bar{D}$. But $D$ is not a domain of bounded holomorphy.

References

3. F. Quigley, Lectures on several complex variables, Tulane University, New Orleans, La., 1964/65, 1965/66.

University of Florida, Gainesville, Florida 32601