Some characterizations of n-dimensional F-spaces

by

M. J. Canfell(1)

Abstract. In this paper we obtain characterizations of an n-dimensional F-space in terms of the rings of continuous real-valued and complex-valued functions defined on the space. Motivation for these results is the work of Gillman and Henriksen on U-spaces (F-spaces of dimension 0) and T-spaces (F-spaces of dimension 0 or 1).

1. Introduction. Throughout, X denotes a completely regular (Hausdorff) space, C(X) the ring of continuous real-valued functions on X, and C*(X) the subring of C(X) consisting of the bounded functions in C(X).

By definition, X is an F-space if C(X) has the property that finitely generated ideals in C(X) are principal [5], [6]. Our main concern here is to define a condition on commutative rings with identity in such a way that X is an n-dimensional F-space if and only if C(X) satisfies this condition. The condition we select, called Hn, corresponds to condition T of [4] when n = 1. In Theorem 3, we prove that X is an n-dimensional F-space if and only if C(X) satisfies condition Hn. Characterizations of topological dimension alone in terms of C(X) have been given in [2] and [6, Theorem 16.35].

In Theorems 3 and 4 we give characterizations of F-spaces and n-dimensional F-spaces in terms of the rings of continuous complex-valued functions defined on them. These characterizations are analogous to those in terms of C(X) and are of interest in connection with sup-norm algebras of complex continuous functions [8], and alignable complex Banach lattices [1].

For f ∈ C(X) we define Z(f) = {x ∈ X : f(x) = 0} (the zero-set of f), P(f) = {x ∈ X : f(x) > 0} and N(f) = {x ∈ X : f(x) < 0}. For the elementary properties of zero-sets the reader is referred to [6].

We use the modification of covering dimension involving basic covers given in [6, p. 243]. By a slight modification of Definition 4 of [3], we obtain the following characterization of dimension.
Lemma 1. \( \dim X \leq n \) if and only if given \( n+1 \) disjoint pairs \( C_i, C'_i, i=1, \ldots, n+1 \), of zero-sets of \( X \), there exist functions \( k_i \in C(X) \) such that \( k_i(C_i) = \{1\}, k_i(C'_i) = \{-1\}, -1 \leq k_i \leq 1 \), and \( \bigcap_{i=1}^{n+1} Z(k_i) = \emptyset \).

Proof. Necessity. If \( C_i \) and \( C'_i \) are disjoint zero-sets, we can choose \( f_i \in C(X) \) such that \( f_i(C_i) = \{1\} \), \( f_i(C'_i) = \{-1\} \) and \(-1 \leq f_i \leq 1\). Let \( I^{n+1} = [-1, 1]^{n+1} \) and let \( S^n \) denote the surface of \( I^{n+1} \). Then \( f = (f_1, \ldots, f_{n+1}) \) is a continuous mapping of \( X \) into \( I^{n+1} \). Since \( \dim X \leq n \), we can, by Definition 3 of [3], choose \( k = (k_1, \ldots, k_{n+1}) : X \to S^n \) such that \( k(x) = f(x) \) whenever \( f(x) \in S^n \). Then the functions \( k_i, i=1, \ldots, n+1 \), satisfy the required conditions.

Sufficiency. If functions \( k_i \) exist as stated, then \( C_i \) and \( C'_i \) are separated in \( X-Z(k_i) \) and \( \bigcap_{i=1}^{n+1} Z(k_i) = \emptyset \). By Definition 4 of [3], \( \dim X \leq n \).

We now recall some properties of \( F \)-rings and Hermite rings. In the following \( S \) will denote a commutative ring with identity. The ideal of \( S \) generated by \( n \) elements \( a_1, \ldots, a_n \) will be denoted by \( a_1S + \cdots + a_nS \). A commutative ring \( S \) with identity is called an \( F \)-ring if every finitely generated ideal of \( S \) is principal. Thus \( X \) is an \( F \)-space if and only if \( C(X) \) is an \( F \)-ring.

We take the following characterization of Hermite rings [4, Lemma 4].

Lemma 2. A commutative ring \( S \) with identity is a Hermite ring if and only if it satisfies the conditions:

(i) \( S \) is an \( F \)-ring.

(ii) Whenever \( a_1, a_2, d \in S \) and \( a_1S + a_2S = dS \), there exist \( b_1, b_2 \in S \) such that \( a_1 = b_1d, a_2 = b_2d \) and \( b_1S + b_2S = S \).

A completely regular space \( X \) such that \( C(X) \) is a Hermite ring is called a \( T \)-space. Alternative characterizations of \( T \)-spaces are given in [5, Theorem 3.2]. We will see later that \( X \) is a \( T \)-space if and only if \( X \) is an \( F \)-space and \( \dim X \leq 1 \).

2. \( n \)-dimensional \( F \)-spaces.

Definition. Let \( n \) be a nonnegative integer. A commutative ring \( S \) with identity is said to be an \( H_n \)-ring, or to satisfy the condition \( H_n \), if

(i) \( S \) is an \( F \)-ring.

(ii) Whenever \( a_1, \ldots, a_{n+1}, d \in S \) and \( a_1S + \cdots + a_{n+1}S = dS \), there exist \( b_1, \ldots, b_{n+1} \in S \) such that \( a_1 = b_1d, \ldots, a_{n+1} = b_{n+1}d \) and \( b_1S + \cdots + b_{n+1}S = S \).

Thus \( S \) is an \( H_1 \)-ring if and only if it is a Hermite ring, and \( S \) is an \( H_0 \)-ring if and only if it is an \( F \)-ring in which generators of principal ideals are unique (up to associates).

Theorem 3. For every completely regular space \( X \), the following statements are equivalent:

(a) \( X \) is an \( F \)-space and \( \dim X \leq n \).

(b) \( C(X) \) is an \( H_n \)-ring.

(c) \( C^*(X) \) is an \( H_n \)-ring.

(d) For all \( f_1, \ldots, f_{n+1} \in C(X) \), there exist \( k_1, \ldots, k_{n+1} \in C(X) \) such that \( f_1 = k_1|f_1|, \ldots, f_{n+1} = k_{n+1}|f_{n+1}| \) and \( k_1C(X) + \cdots + k_{n+1}C(X) = C(X) \).
Proof. (a) $\Rightarrow$ (d). Suppose $f_1, \ldots, f_{n+1} \in C(X)$. Since $X$ is an $F$-space, $P(f_i)$ and $N(f_i)$ are contained in disjoint zero-sets. By Lemma 1, there exist functions $k_i$ such that $k_i(P(f_i))=\{1\}$, $k_i(N(f_i))=\{-1\}$, and $\bigcap_{i=1}^{n+1} Z(k_i) = \emptyset$. Hence $f_i = k_i(f_i)$, $i=1, \ldots, n+1$, and $k_1C(X) + \cdots + k_{n+1}C(X) = C(X)$.

(d) $\Rightarrow$ (b). The hypothesis implies that $X$ is an $F$-space and hence that $C(X)$ is an $F$-ring. Suppose that $f_1C(X) + \cdots + f_{n+1}C(X) = hC(X)$. There exist $g_1, \ldots, g_{n+1} \in C(X)$ and $s_1, \ldots, s_{n+1} \in C(X)$ such that $f_i = g_i h$, $f_{n+1} = s_{n+1} h$ and $h = s_1 f_1 + \cdots + s_{n+1} f_{n+1} = s_1 g_1 h + \cdots + s_{n+1} g_{n+1} h$. Put $q = 1 - s_1 g_1 - \cdots - s_{n+1} g_{n+1}$.

Then $hq = 0$ and for any elements $t_i \in C(X)$ we have $(g_i + t_i q) h = f_i$. We will choose the $t_i$ so that the elements $g_i + t_i q$ generate $C(X)$. Since $X$ is an $F$-space, there exists $p \in C(X)$ such that $pq = |q|$. By hypothesis, there exist $m_i \in C(X)$ such that $g_i = m_i |g_i|$, $i=1, \ldots, n+1$, and $\bigcap_{i=1}^{n+1} Z(m_i) = \emptyset$. Let $t_i = pm_i$ and let $g_i = g_i + t_i q$. Then for each $x \in X$, we have $g_i(x) \neq 0$ for some $i$. To see this, suppose first that $g_i(x) \neq 0$ for some $i$. Now $(t_i q)(x) = p(x) m_i(x) q(x) = m_i(x) |q(x)|$ has the same sign (or argument) as $g_i(x)$ so that $g_i(x) \neq 0$. On the other hand, if $g_i(x) = 0$ for all $i$, then $q(x) = 1$, $p(x) = 1$, and $g_i(x) = t_i(x) = m_i(x)$. Since $\bigcap_{i=1}^{n+1} Z(m_i) = \emptyset$, then $g_i(x) \neq 0$ for some $i$. Hence $g_1C(X) + \cdots + g_{n+1}C(X) = C(X)$.

(b) $\Rightarrow$ (a). By hypothesis, $C(X)$ is an $F$-ring and hence $X$ is an $F$-space. Suppose that $C_1, C'_i$, $i=1, \ldots, n+1$, are $n+1$ disjoint pairs of zero-sets. Choose $f_i \in C(X)$ such that $f_i(C_1) = \{1\}$, $f_i(C'_i) = \{-1\}$, for $i=1, \ldots, n+1$, and let $h = |f_1| + \cdots + |f_{n+1}|$. Since $X$ is an $F$-space, $f_1C(X) + \cdots + f_{n+1}C(X) = hC(X)$. By hypothesis, there exist $g_i \in C(X)$ such that $f_i = g_i h$ and $g_1C(X) + \cdots + g_{n+1}C(X) = C(X)$. Thus $\bigcap_{i=1}^{n+1} Z(g_i) = \emptyset$. Now $P(f_i) \subseteq P(g_i)$, $N(f_i) \subseteq N(g_i)$ for $i=1, \ldots, n+1$. Also $|g_i(x)| \leq 1$ for $f_i(x) \neq 0$ and we can arrange that $|g_i(x)| \leq 1$ everywhere (take $g_i(x) = g_i(x)$ if $|g_i(x)| \leq 1$ and $g_i(x) = g_i(x)/|g_i(x)|$ if $|g_i(x)| \geq 1$). Since $P(g_i)$ and $N(g_i)$ are completely separated, we can choose $s_i$ so that $s_i P(g_i) = \{1\}$ and $s_i N(g_i) = \{0\}$.

Let $m_i \in C(X)$ satisfy $f_i = m_i |f_i|$, $-1 \leq m_i \leq 1$. Define $k_i = s_i \max \{m_i, g_i\} + (1-s_i) \min \{m_i, g_i\}$. Then $f_i = k_i |f_i|$ and $Z(k_i) \subseteq Z(g_i)$. Hence $\bigcap_{i=1}^{n+1} Z(k_i) = \emptyset$. Since $k_1(C_1) = \{1\}$ and $k_{n+1}(C'_i) = \{-1\}$ we have dim $X \leq n$ by Lemma 1.

(b) $\Leftrightarrow$ (c). $C^\ast(X)$ is isomorphic to $C(\beta X)$ where $\beta X$ is the Stone-$\check{C}$ech compactification of $X$. Since dim $X = \dim \beta X$ [6, p. 245] and $X$ is an $F$-space if and only if $\beta X$ is an $F$-space, the result follows from (a) $\Leftrightarrow$ (b) above.

Example. $\beta R^n - R^n$ is an $n$-dimensional $F$-space.

That $\beta R^n - R^n$ is an $F$-space follows from Theorem 14.27 of [6], and it is shown in [7] that dim $(\beta R^n - R^n) = n$.

As a simple consequence, we have an example of an $F$-ring which is not a Hermite ring (the first example of this was given in [5]). $\beta R^2 - R^2$ is an $F$-space which is not a $T$-space, hence $C(\beta R^2 - R^2)$ is an $F$-ring which is not a Hermite ring.

3. Continuous complex functions on $F$-spaces. We turn now to the problem of characterizing $F$-spaces in terms of the ring $C_c(X)$ of all continuous complex-
valued functions on $X$. We also consider $C^*_c(X)$, the subring of $C_c(X)$ consisting of the bounded functions in $C_c(X)$.

Since $Z(f) = Z(|f|)$, the family of zero-sets of $C_c(X)$ is the same as the family of zero-sets of $C(X)$.

An ideal $I$ of $C_c(X)$ is said to be selfadjoint if and only if $f \in I \Rightarrow \overline{f} \in I$, where $\overline{f}$ is the complex conjugate of $f$.

**Theorem 4.** The following conditions are equivalent:

(a) $X$ is an $F$-space.

(b) $C_c(X)$ is an $F$-ring.

(c) $C^*_c(X)$ is an $F$-ring.

(d) Each ideal $I$ of $C_c(X)$ is selfadjoint.

(e) For all $f, g \in C_c(X)$, $fC_c(X) + gC_c(X) = (|f| + |g|)C_c(X)$.

(f) Given a zero-set $Z$ of $X$, every function $\theta \in C^*_c(X - Z)$ has a continuous extension $h \in C^*_c(X)$.

(g) Given $f \in C_c(X)$, there exist $k_1, k_2 \in C_c(X)$ such that $f = k_1|f|$ and $|f| = k_2f$.

**Proof.** (g) $\Rightarrow$ (d). Let $f \in I$. There exist $k_1, k_2 \in C_c(X)$ such that $f = k_1|f| = k_1|f|$ and $|f| = k_2f$. Hence $f = k_1k_2f$ so that $f \in I$.

(d) $\Rightarrow$ (a). Let $f \in C(X)$. Then $f - i|f| \in C_c(X)$ and by hypothesis its complex conjugate $f + i|f|$ is in the principal ideal generated by $f - i|f|$. There exists $h \in C_c(X)$ such that $f + i|f| = h(f - i|f|)$. On multiplying both sides by $f - i|f|$ we have

$$f^2 + |f|^2 = h(f^2 - 2i|f||f - |f|^2),$$

and on simplifying and equating real parts, we get

$$|f|^2 = f^2 = I(h)f|f|.$$

It follows that $f = I(h)|f|$ so that $X$ is an $F$-space.

The rest of the proof is a routine modification of the proofs in Theorem 14.25 of [6]. For example, (a) $\Rightarrow$ (f) since the real and imaginary parts of $\theta$ can be extended over $X$.

Although $C_c(X)$ is an $F$-ring if and only if $C(X)$ is an $F$-ring, the situation is slightly different for $H_n$-rings.

**Theorem 5.** The following conditions are equivalent:

(a) $C_c(X)$ is an $H_n$-ring.

(b) $C(X)$ is an $H_{2n+1}$-ring.

(c) $X$ is an $F$-space and $\dim X \leq 2n + 1$.

(d) For all $f_1, \ldots, f_{n+1} \in C_c(X)$, there exist $k_1, \ldots, k_{n+1} \in C_c(X)$ such that $f_1 = k_1|f_1|, \ldots, f_{n+1} = k_{n+1}|f_{n+1}|$ and $k_1C_c(X) + \cdots + k_{n+1}C_c(X) = C_c(X)$.

**Proof.** (a) $\Rightarrow$ (b). First we observe that if $X$ is an $F$-space and $f_1, f_2 \in C(X)$, then $f_1C(X) + f_2C(X) = (f_1^2 + f_2^2)^{1/2}C(X)$. In fact, since $(f_1^2 + f_2^2)^{1/2} \leq |f_1| + |f_2|$.
\[ \leq 2(f_1^2 + f_2^2)^{1/2}, \] then it follows from Theorem 14.25(6) of [6], that \(|f_1| + |f_2|\) and \((f_1^2 + f_2^2)^{1/2}\) are multiples of each other. Similarly if \(f_1, f_2 \in C(X)\), then \(f_1C(X) + f_2C(X) = (f_1^2 + f_2^2)^{1/2}C(X)\).

Now suppose that \(f_1, \ldots, f_{2n+2}, d \in C(X)\) and \(f_1C(X) + \cdots + f_{2n+2}C(X) = dC(X)\). Let \(h = (f_1^2 + \cdots + f_{2n+2}^2)^{1/2}\). By hypothesis and Theorem 4, \(X\) is an \(F\)-space and, by the preceding remarks, \(dC(X) = hC(X)\). Let \(g_i = f_{2i-1} + if_{2i}, i = 1, \ldots, n+1\). Again by the preceding remarks, \(g_1C(X) + \cdots + g_{n+1}C(X) = (|g_1|^2 + \cdots + |g_{n+1}|^2)^{1/2}C(X) = hC(X)\). Therefore \(g_1C(X) + \cdots + g_{n+1}C(X) = dC(X)\). By hypothesis, there exist elements \(s_{2i-1} + is_{2i} \in C(X)\) which generate \(C(X)\) and which satisfy \(g_i = s_{2i-1} + is_{2i}\). Thus \(f_i = s_1d, i = 1, \ldots, 2n+2\) and \(\bigcap_{i=1}^{2n+2} Z(s_i) = \emptyset\), i.e., \(g_1C(X) + \cdots + g_{n+1}C(X) = C(X)\).

(b) \(\Rightarrow\) (c). This has been shown in Theorem 2.

(c) \(\Rightarrow\) (d). Let \(f_1, \ldots, f_{n+1} \in C_c(X)\). By Theorem 4, there exist \(k_1', \ldots, k_{n+1}' \in C_c(X)\) such that \(f_i = k_i'|f_i|, i = 1, \ldots, n+1\). If \(f_i(x) \neq 0\), then \(|k_i(x)| = 1\), and we may assume that \(|k_i(x)| \leq 1\) for \(x \in X, i = 1, \ldots, n+1\).

Let \(D\) be the closed unit disc in the complex plane and \(D_1\) its surface; that is, \(D = \{z \in C : |z| \leq 1\}\) and \(D_1 = \{z \in C : |z| = 1\}\). Then \(k' = (k_1', \ldots, k_{n+1}')\) is a continuous mapping of \(X\) into \(D^{n+1} \subset R^{2n+2}\). Since \(dim X \leq 2n+1\), we may, as in Definition 3 of [3], choose \(k = (k_1, \ldots, k_{n+1}) : X \rightarrow D_{1}^{n+1}\) such that \(k(x) = k'(x)\) whenever \(k'(x) \in D_{n+1}^{n+1}\). Thus \(f_i = k_i|f_i|, i = 1, \ldots, n+1\), and \(\bigcap_{i=1}^{2n+2} Z(k_i) = \emptyset\).

(d) \(\Rightarrow\) (a). The proof is identical with (d) \(\Rightarrow\) (b) of Theorem 3.

**COROLLARY.** \(X\) is a \(T\)-space if and only if given \(f \in C_c(X)\), there exists \(k \in C_c(X)\) such that \(f = k|f|\) and \(Z(k) = \emptyset\).

**Proof.** This is (d) \(\Rightarrow\) (b) above with \(n = 0\) but we give a simple direct proof. If \(X\) is an \(F\)-space and \(f = f_1 + if_2, k = k_1 + ik_2\), then \(f = k|f|\) and \(Z(k) = \emptyset\) if and only if \(f_i = k_i(f_i^2 + f_2^2)^{1/2}, f_2 = k_2(f_1^2 + f_2^2)^{1/2}\) and \(Z(k_1) \cap Z(k_2) = \emptyset\). Since \(f_1C(X) + f_2C(X) = (f_1^2 + f_2^2)^{1/2}C(X)\), then (b) \(\Rightarrow\) (d) is immediate, while (d) \(\Rightarrow\) (b) follows from Lemma 4 of [4].

As the example \(X = \beta R^2 - R^2\) shows, \(C_c(X)\) may be a Hermite ring while \(C(X)\) is not a Hermite ring.

4. \(U\)-spaces and \(T\)-spaces. An element \(u\) of \(C(X)\) (or \(C_c(X)\)) is said to be **unitary** if \(|u(x)| = 1\) for all \(x \in X\). If \(f = v|f|\) and \(Z(v) = \emptyset\), then \(u = v/|v|\) is unitary, and since \(|f| = |v| |f|\), we have \(f = v|f| = u|v| |f| = u|f|\).

From Theorem 3 and the corollary to Theorem 5, we have the following characterization.

**Lemma 6.** \(X\) is a \(U\)-space (respectively \(T\)-space) if and only if for each \(f \in C(X)\) (respectively \(C_c(X)\)), there exists a unitary element \(u\) of \(C(X)\), (respectively \(C_c(X)\)) such that \(f = u|f|\).

Finally we given an unpublished result of Bonsall in which \(T\)-spaces are characterized in terms of linear operators on the complex vector space \(C_c(X)\).
A rotation on $C_c(X)$ is a linear operator $D$ mapping $C_c(X)$ onto $C_c(X)$ such that $|Df| = |f|$ for all $f \in C_c(X)$. $C_c(X)$ is said to be alignable if and only if given $f_0 \in C_c(X)$ there exists a rotation $D$ on $C_c(X)$ such that $D|f_0| = f_0$.

Alignable spaces were considered in [1].

**Theorem 7.** $X$ is a $T$-space if and only if $C_c(X)$ is alignable.

**Proof.** If $u \in C_c(X)$ is a unitary element for which $f_0 = u|f_0|$, then clearly the operation of multiplication by $u$ is a rotation on $C_c(X)$ with the required property.

Conversely, suppose that $D$ is a rotation on $C_c(X)$ for which $D|f_0| = f_0$. We show that $D1$ is unitary and that $D$ is the operation of multiplication by $D1$. Given $x \in X$, let $\Psi_x$ and $\Phi_x$ denote the linear functionals on $C_c(X)$ defined by $\Psi_x(f) = f(x)$ and $\Phi_x(f) = (Df)(x)$. Then $|\Psi_x(f)| = |\Phi_x(f)|$ for each $f \in C_c(X)$. Hence $\Psi_x$ and $\Phi_x$ have the same null space and therefore differ only by a scalar factor. Thus $\Phi_x = \lambda_x \Psi_x$ for some $\lambda_x \in \mathbb{C}$ with $|\lambda_x| = 1$. Now $(Df)(x) = \Phi_x(f) = \lambda_x \Psi_x(f) = \lambda_x f(x)$. In particular, $(D1)(x) = \lambda_x$ so that $(Df)(x) = ((D1)(x))f(x)$. This holds for all $x \in X$, so that $Df = (D1)f$. Finally, for each $x \in X$, $|(D1)(x)| = |\lambda_x| = 1$ so that $D1$ is unitary and $D|f_0| = D|f_0| = f_0$.

**References**


**university of new england, armidale, australia**