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CONJUGACY SEPARABILITY OF THE GROUPS OF HOSE KNOTS

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Abstract. Let G be a group. An element g of G is c.d. in G if and only if, given any element h of G, either h is conjugate to g or there is a homomorphism ξ from G onto a finite group such that $\xi(g)$ is not conjugate to $\xi(h)$. Following A. Mostowski, a group is conjugacy separable or c.s. if and only if every element of the group is c.d. In this paper we show that the groups of hose knots are c.s.

According to K. Brauner [1], the groups of hose knots are presented as follows: The groups have generators $P_1, Q_1, \ldots, Q_r$ and relations

$$Q_1^{-m_1}P_1^{n_1} = 1, \qquad Q_i^{m_i}P_i^{n_i - n_{i-1}m_i}Q_{i-1}^{m_{i-1}m_i} = 1, \qquad m_ik_i = n_iu_i + 1,$$

where

$$P_i = Q_{i-1}^{-u_i}P_{i-1}^{k_{i-1}-n_{i-2}u_{i-1}}Q_{i-2}^{m_{i-2}u_{i-1}}, \qquad Q_0 = 1.$$

Clearly these groups can be obtained from a free cyclic group by repeated addition of roots. An element u of a group G is conjugacy distinguished or c.d., in G, if for every element v of G not conjugate to u there is a homomorphism ξ from G onto a finite group such that $\xi(u)$ and $\xi(v)$ are not conjugate. Following A. Mostowski [5], a group is c.s., or conjugacy separable, if every element is c.d. Mostowski proves that the conjugacy problem can be solved in c.s. groups. In this paper we will show that the groups of hose knots are c.s.

A general reference for theorems in infinite group theory is the book by W. Magnus, A. Karass and D. Solitar [4].

A group G is Π_c if and only if given any two elements a and b of G either $a=b^z$ for an integer z or there is a normal subgroup N of finite index in G such that $a \not\equiv b^z \bmod N$ for all z. Let G be a Π_c group. Let $H=(x, G; x^m=g)$ for $m>1$ and $g \in G$ be the free product of G and a cyclic group generated by x with a cyclic amalgamated subgroup generated by g. According to [6, Corollary 3.3], H is Π_c. It is also shown in [6] that free groups and groups with a free subgroup of finite index are Π_c. Since the intersection of finitely many subgroups of finite index in a group G is again of finite index in G, it follows that if $t_1, \ldots, t_n$ and b are elements of G such that $t_i \neq b^{z_i}$ for all z_i and all $i=1, \ldots, n$, and if G is Π_c, then there is a

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single normal subgroup N of finite index in G such that no congruence $t_i \equiv b^{z_i} \bmod N$ has an integral solution.

LEMMA 1. *Let F be a free group. Let α_i be automorphisms of F and let $b_i = \alpha_i(c)$ for an element c of F. Suppose the equations $a_1 = b_1^{-n_0}c^{n_1}, \ldots, a_i = b_i^{-n_{i-1}}c^{n_i}, \ldots, a_m = b_m^{-n_{m-1}}c^{n_0}$ have no integral solution $n_0, \ldots, n_{m-1}$. There is a normal subgroup N of finite index in F such that the system of equations has no solution modulo N.*

Proof. If c is the identity element, all b_i are the identity. In this case the existence of N follows from the fact that F is residually finite. Thus we can assume that c is not the identity.

Suppose there is a subscript i such that b_i commutes with c. By the symmetry of the equations we can assume $(b_1, c) = 1$. Since F is free, there is an element u of F such that $b_1 = u^r$, $c = u^s$. Since $\alpha_1(c) = b_1$, $\alpha_1(u^r) = u^s$. It follows from an easy extension of Theorem 7 of [6] that $r = +s$ or $r = -s$, so that $b_1 = c$ or $b_1 = c^{-1}$. If $b_1 = c$, the given set of equations is equivalent to $a_1 = c^{n_1 - n_0}$, $a_2 = b_2^{-n_1}c^{n_2}, \ldots, a_m a_1 = b_m^{-n_{m-1}}c^{n_1}$. If $b_1 = c^{-1}$, the system of equations is equivalent to $a_1 = c^{n_0 + n_1}$, $a_2 = (b_2^{-1})^{n_1}c^{n_2}, \ldots, a_m a_1^{-1} = b_m^{-n_{m-1}}c^{n_1}$. Since $b_2 = \alpha_2(c)$, $b_2^{-1} = \alpha_2(c^{-1})$ so that $b_2^{-1} = \alpha_2(\alpha_1(c))$. Thus for $b_1 = c$ or c^{-1} the original set of equations is equivalent to a set of equations of the same form with fewer equations and a single equation $a_1 = c^z$. If the equation $a_1 = c^z$ has no solution, the fact that F is Π_c implies that there is a normal subgroup N of finite index in F such that $a_1 \neq c^z \bmod N$ for all z. But the original set of equations will have no solution modulo N. Thus we can ignore the equation $a_1 = c^z$. Repeating this process, it is clear that we can reduce the original set of equations to a set of equations of the same form with $(b_i, c) \neq 1$ for all i. We assume that $(b_i, c) \neq 1$.

Suppose there is a subscript i such that $a_i = b_i^x c^y$ has no solution for integers x and y. According to Lemma 3 of [8], there is a normal subgroup N of finite index in F such that $a_i = b_i^x c^y$ has no solution modulo N. Clearly the set of simultaneous equations has no solution modulo N. Now suppose that for each i there are integers h_i and k_i such that $a_i = b_i^{-k_i}c^{h_i}$. Since the set of equations has no solution, at least one difference $h_i - k_{i+1}$, $h_m - k_0$, is not zero. Let p be a prime relatively prime to at least one nonzero difference. Since $(b_i, c) \neq 1$ for all i, it follows from a theorem of M. Hall [2] that there is a normal subgroup N of finite index in F such that F/N is a p-group and $(b_i, c) \notin N$ for all i. Suppose, to obtain a contradiction, that the given set of equations has a solution modulo N. There are integers $v_0, \ldots, v_{m-1}$ such that $a_i \equiv b_i^{-v_{i-1}}c^{v_i} \bmod N$ $(v_m = v_0)$. Since $a_i = b_i^{-k_i}c^{h_i}$ we have $b_i^{v_{i-1} - k_i} \equiv c^{v_i - h_i} \bmod N$. Since $(b_i, c) \notin N$, neither $v_{i-1} - k_i$ nor $v_i - h_i$ can be relatively prime to p. Thus p divides each difference $h_i - k_{i+1}$. But this result contradicts the definition of p. Thus the system of equations has no solution modulo N.

Let us define a property Π of an element of a group G as follows. Let G be a group. An element g of G has property Π if given a set of equations $a_1 =$

$g^{-r_0}b_1g^{r_1}, \ldots, a_i = g^{-r_{i-1}}b_ig^{r_i}, \ldots, a_m = g^{-r_{m-1}}b_mg^{r_0}$ for a_i and b_i elements of G then either there are integers $r_0, \ldots, r_{m-1}$ such that the equations are true or there is a normal subgroup N of finite index in G such that the system of equations has no solution modulo N.

LEMMA 2. *If G is a group with a free normal subgroup of finite index, every element of G has property* Π.

Proof. Let F be a free normal subgroup of finite index in G. Suppose the equations $a_i = c^{-r_{i-1}}b_ic^{r_i}$, $r_m = r_0$, have no solution. If these equations have no solution modulo F, then F is the subgroup required by the lemma. Assume that there is a finite set of solutions $r_{0,k}, \ldots, r_{m-1,k}$ to the equations modulo F, and that for distinct k the solutions are incongruent modulo the order of c modulo F. Let c have order h modulo F. Now the equations $a_i = c^{-r_{i-1,k}-t_{i-1}h}b_ic^{r_{i,k}+t_ih}$ have no solution $t_0, \ldots, t_{m-1}$ with $t_m = t_0$ and these latter equations are equivalent to the system of equations $b_i^{-1}c^{r_{i-1,k}}a_ic^{-r_{i,k}} = (b_i^{-1}c^hb_i)^{-t_{i-1}}(c^h)^{t_i}$. Since these equations are in F, it follows from Lemma 1 that there is a single normal subgroup N of finite index in F such that none of these equations has a solution modulo N. Let M be the intersection of all the conjugates of N in G. Since F is of finite index in G, M is of finite index in G. Clearly, the original system of equations has no solution modulo M.

LEMMA 3. *Let A_1 be a free cyclic group. If every element of A_2 has property* Π, *then every element of $A_1 \times A_2$ has property* Π.

Proof. Let G be the direct product of A_1 and A_2. Let a_i, b_i for $i = 1, \ldots, m$ and c be elements of G. Suppose that the equations $a_i = c^{-r_{i-1}}b_ic^{r_i}$, $r_0 = r_m$, have no solution. Suppose the equations have no solution modulo A_2. Since every element of A_1 has property Π, there is a normal subgroup N of finite index in A_1 such that the system of equations has no solution modulo $A_2 \times N$. Hence we assume that the system of equations has a solution modulo A_2. Let $a_i = a_{i,1}a_{i,2}$, $b_i = b_{i,1}b_{i,2}$ and $c = c_1c_2$ where $a_{i,j}$, $b_{i,j}$ and c_j are elements of A_j. By our assumption, there are integers $s_0, \ldots, s_{m-1}$ such that $a_{i,1} = c_1^{-s_{i-1}}b_{i,1}c_1^{s_i}$ and $s_m = s_0$. If the equations $a_{i,2} = c_2^{-u}c_2^{-s_{i-1}}b_{i,2}c_2^{s_i}c_2^{u}$ have a solution u, it follows from the fact that A_1 is abelian that the original set of equations has a solution $s_i + u$. Thus the latter equations have no solution u. It will be shown below that there exists a normal subgroup N_2 of A_2 such that the latter set of equations has no solution modulo N_2. Since A_2 is residually finite, it is clear that N_2 can be chosen so that c_2 is not an element of N_2. Let h be the order of c_2 modulo N_2. Let N_1 be a normal subgroup of finite index in A_1 such that c_1 has order h modulo N_1. Since A_1 is free cyclic, N_1 is the subgroup generated by c^h. Let $K = N_1 \times N_2$.

Suppose $a_i = c^{-r_{i-1}}b_ic^{r_i}$, $r_0 = r_n$, has a solution $r_0, \ldots, r_{m-1}$ modulo K. We have $b_{i,1}^{-1}a_{i,1} \equiv c_1^{r_i - r_{i-1}} \bmod K$. Since $b_{i,1}^{-1}a_{i,1} = c^{s_i - s_{i-1}}$, $r_i - r_{i-1} \equiv s_i - s_{i-1} \bmod h$. Here subscripts on r and s are taken modulo m. Summing these congruences we obtain $r_i - r_0 \equiv s_i - s_0 \bmod h$ so that $r_i = s_i + (r_0 - s_0) + q_ih$. Since h is the order of c_2 modulo

N_2 we have $a_{i,2} \equiv c_2^{-u} c_2^{-s_i-1} b_{i,2} c_2^{s_i} c_2^u \bmod N_2$ for $u = r_0 - s_0$. Since this contradicts the definition of N_2, the lemma is proven.

Now we will show that if a set of equations $d_i = g^{-u} e_i g^u$ for $i = 1, \ldots, n$ has no solution u for g, e_i, and d_i in A_2 then there is a normal subgroup N_2 of finite index in A_2 such that these equations have no solution modulo N_2. Note that since every element of A_2 has property Π, given elements a and b of A_2 either the equations $a = b^{-r_0} b b^{r_1}$, $a^{-1} = b^{-r_1} b^{-1} b^{r_0}$ have a solution r_0, r_1 or there is a normal subgroup N of finite index in A_2 such that these equations have no solution modulo N. Thus A_2 is Π_c.

Suppose that there is a subscript i such that $e_i^{-1} g^r e_i = g^s$ implies $g^r = g^s = 1$. By the symmetry of the equations we can assume that this subscript is 1. Consider the n systems of equations:

$$\begin{aligned} d_1 &= g^{-r_0} e_1 g^{r_1}, & g &= g^{-r_1} g g^{r_0}, \\ d_1 &= g^{-r_{0,i}} e_1 g^{r_{1,i}}, & d_1^{-1} &= g^{-r_{1,i}} e_1^{-1} g^{r_{2,i}}, \\ d_i &= g^{-r_{2,i}} e_i g^{r_{0,i}}, & i &= 2, \ldots, n. \end{aligned}$$

Suppose each of these systems has a solution. The first set of equations implies that $d_1 = g^{-r_0} e_1 g^{r_0}$. If n is greater than one, the other sets of equations imply that $d_i = g^{-r_{0,i}} e_i g^{r_{0,i}}$ and $d_1 = g^{-r_{0,i}} e_1 g^{r_{1,i}}$. Thus $g^{-r_0} e_1 g^{r_0} = g^{-r_{0,i}} e_1 g^{r_{1,i}}$. It follows from the assumption that $g^{r_{0,i}} = g^{r_0}$. Thus if the above systems of equations have solutions, there is a solution to $d_i = g^{-u} e_i g^u$, $i = 1, \ldots, n$. Thus at least one of these systems has no solution. Since g has property Π, there is a normal subgroup N of finite index in A_2 such that at least one of these equations has no solution modulo N. If the system of equations $d_i = g^{-u} e_i g^u$, $i = 1, \ldots, n$, has a solution modulo N, clearly every one of the other systems of equations has a solution modulo N.

Now suppose that for each i there are integers r_i and s_i such that $e_i^{-1} g^{r_i} e_i = g^{s_i}$. Let r be the least common multiple of the r_i, $i = 1, \ldots, n$. For each i there is an integer v_i such that $e_i^{-1} g^r e_i = g^{v_i}$. Since A_2 is Π_c, it follows from Theorem 7 of [6], that either g has finite order or each v_i is r or $-r$. We now consider several cases.

Suppose that g has finite order or, for each i, $v_i = r$. In this case there are but finitely many distinct n-tuples of the form $(d_1^{-1} g^{-u} e_1 g^u, \ldots, d_n^{-1} g^{-u} e_n g^u)$. Since at least one entry in each n-tuple is not the identity and A_2 is residually finite, there is a single normal subgroup N_2 of finite index in A_2 such that at least one entry in each of the n-tuples is not an element of N_2. Thus the equations $d_i = g^{-u} e_i g^u$ have no solution modulo N_2.

Suppose that g has infinite order and that, for at least one subscript i, $v_i = -r$. We can rearrange the equations so that $v_1 = \cdots = v_k = -r$, $v_{k+1} = \cdots = v_n = r$. Let $\bar{d}_1 = d_1$, $\bar{d}_i = d_1 d_i$ for $1 < i \leqq k$, and $\bar{d}_i = d_i$ for $i > k$. The system of equations $d_i = g^{-u} e_i g^u$ has a solution u if and only if u solves the system $\bar{d}_i = g^{-u} \bar{e}_i g^u$ where $\bar{e}_1 = e_1$, $\bar{e}_i = e_1 e_i$ for $1 < i \leqq k$ and $\bar{e}_i = e_i$ for $i > k$. For all $i > 1$ we have that $\bar{e}_i^{-1} g^r \bar{e}_i = g^r$. Also $\bar{e}_1^{-1} g^r \bar{e}_1 = g^{-r}$. If there is an integer t such that $0 \leqq t < r$ and the equations $g^{-t}\bar{e}_1^{-1}g^t\bar{d}_1=(g^{2r})^z$, $\bar{d}_2=g^{-t}\bar{e}_2g^t,\ldots,\bar{d}_n=g^{-t}\bar{e}_ng^t$ have an integral solution z, then for $u=t+rz$ we have that $\bar{d}_i=g^{-u}\bar{e}_ig^u$. Thus for all t between zero and r the latter system has no solution. Since A_2 is Π_c, there is a single normal subgroup N_2 of finite index in A_2 such that for all t between zero and r, the latter system of equations has no solution. Clearly, $d_i=g^{-u}e_ig^u$ has no solution modulo N_2.

LEMMA 4. *Let G be the free product of groups A and B with a cyclic amalgamated subgroup generated by an element h of G. Let G/G' be free cyclic and let $h\notin G'$. If u is a cyclically reduced element of length greater than one in G, and u^2bu^{-n} for $n>0$ begins with a syllable in a different factor from the first syllable of u, then either b commutes with u or u^{-1} is an initial segment of b.*

This lemma was suggested by Lemma 1 in S. Lipschutz [3].

Proof. Suppose u^{-1} is not an initial segment of u, i.e., if $b=b_1\cdots b_j$ in reduced form, we do not have $b_1\cdots b_k=u^{-1}h^e$ where h^e is in the amalgamated subgroup. Let $u=a_1\cdots a_m$ be u written in reduced form. Since u is cyclically reduced, we have not only adjacent a_i in different factors of G but also a_1 and a_m are not in the same factor of G. Thus $u^2bu=a_1\cdots \boldsymbol{a_m}a_1\cdots a_mb_1\cdots b_ju^{-n}$. If the bold syllable $\boldsymbol{a_m}$ cancels, it cannot be by an element of b, since u^{-1} is not an initial segment of b. Thus we have $u^2bu^{-n}=a_1\cdots \boldsymbol{a_m}h^ka_i^{-1}\cdots a_1^{-1}u^{-p}$, so that $h^{k_1}a_i^{-1}$ cancels with the bold $\boldsymbol{a_m}$, and h^{k_1} is in the amalgamated subgroup. If cancellation reaches the initial syllable a_1, we have $a_1\cdots a_mh^{k_1}a_i^{-1}\cdots a_1^{-1}a_m^{-1}\cdots a_{i+1}^{-1}=h^{k_2}$. Since $u^2bu^{-n}=a_1\cdots a_mh^{k_1}a_i^{-1}\cdots a_m^{-1}u^{-p}$ we have $b=u^{-1}h^{k_1}a_{i+1}\cdots a_mu^r$. Since G/G' is free cyclic and $h\notin G'$, $k_1=k_2$ and $uh^{k_1}a_i^{-1}\cdots a_1^{-1}=h^{k_1}a_{i+1}\cdots a_m$ so $uh^{k_1}a_{i+1}\cdots a_mu^{-1}=h^{k_1}a_{i+1}\cdots a_m$. This implies that u commutes with b.

LEMMA 5. *Let G and u satisfy the hypotheses of Lemma 4. If b does not commute with u, there is an integer n_0 such that if $n=n_0$ or $-n_0$ and $m=n/|n|$, then the first syllable of $u^{-n}bu^n$ is in the same factor of G as the first syllable of u^{-m}, and the last syllable of $u^{-n}bu^n$ is in the same factor of G as the last syllable of u^m.*

Proof. If u or u^{-1} is an initial segment of b, it follows from the fact that u has finite syllable length that there is an integer r such that $b=u^rb'$ where u and u^{-1} are not initial segments of b'. If u or u^{-1} is a terminal segment of b', there is an integer s such that $b'=\boldsymbol{b}u^s$ and neither u nor u^{-1} is a terminal segment of $\boldsymbol{b}$. Now if r or s is not zero, the syllable length of b is greater than the syllable length of $\boldsymbol{b}$. If the above process is repeated we must eventually obtain integers x and y and a group element g such that $b=u^xgu^y$, where neither u nor u^{-1} is an initial or terminal segment of g. Since b does not commute with u, g does not commute with u. Let $n_0=2+\max(|x|,|y|)$. We have $u^{-n_0}bu^{n_0}=u^{x-n_0}gu^{y+n_0}$ and $u^{n_0}bu^{-n_0}=u^{x+n_0}gu^{y-n_0}$. In each expression the exponents on u have opposite signs and each exponent in an expression involving g is greater than or equal to 2 in absolute value. The lemma now follows from Lemma 4.

Let G be a group, n an integer greater than 1, and let g be an element of g. Let H be the free product of G and a cyclic group generated by x with the subgroups generated by x^n and g amalgamated. We write $H=(x, G; x^n=g)$.

LEMMA 6. *Let G be a Π_c group. Let $H=(x, G; x^m=g)$ for $g \in G$, $g \notin G'$ and $m>1$. If G/G' and H/H' are free cyclic, u is a cyclically reduced element of length greater than one in H and $u \notin H'$, then u has property Π in H.*

Proof. Suppose the equations $a_i=u^{-r_{i-1}}b_iu^{r_i}$, $i=1,\ldots,n$, $r_0=r_n$, have no solution, where a_i and b_i are elements of H. We consider several cases.

Suppose every b_i commutes with u. In this case the given system of equations is equivalent to the system $a_ib_i^{-1}=u^{r_i-r_{i-1}}$ for $i=1,\ldots,n-1$ and $a_1\cdots a_n=b_1\cdots b_n$. Since the r_i-r_{i-1} are independent, the first $n-1$ of these equations can be replaced by $a_i=u^{z_i}$ for z_i an arbitrary integer. Thus at least one of the equations in the latter set has no solution. Since Corollary 3.3 [6] implies that H is Π_c and hence residually finite, there is a normal subgroup N of finite index in H such that at least one of the equations of the latter set has no solution modulo N. But then the original system of equations has no solution modulo N.

Now suppose that at least one of the b_i does not commute with u. By the symmetry of the set of equations we can assume that b_1 does not commute with u. Let η be the natural homomorphism from H to H/H' and let v be a generator of the free cyclic group H/H'. Let $\eta(a_1)=v^{e_1}$, $\eta(b_1)=v^{e_2}$ and $\eta(u)=v^s$. Now $\eta(a_1)=\eta(u)^{-r_0}\eta(b_1)\eta(u)^{r_1}$ implies that $e_1=e_2+s(r_1-r_0)$. We consider several cases.

Suppose that s does not divide e_1-e_2. Then the equations

$$\eta(a_i) = \eta(u)^{-r_{i-1}}\eta(b_i)\eta(u)^{r_i}, \qquad i = 1,\ldots,n, \qquad r_0 = r_n,$$

have no solution. Since H/H' is free cyclic, $\eta(u)$ has property Π in H/H' and there is a homomorphism ξ from H/H' onto a finite group so that

$$\xi\eta(a_i) = \xi\eta(u)^{-u_{r-1}}\xi\eta(b_i)\xi\eta(u)^{r_i}, \qquad i = 1,\ldots,n, \qquad r_0 = r_n,$$

has no solution. If K is the kernel of $\xi\eta$, the given set of equations has no solution modulo K and K is a normal subgroup of finite index in H.

If s divides e_1-e_2, set $k=(e_1-e_2)/s$. Consider the equation $a_1=u^{-r_0}(b_1u^k)u^{r_0}$. Since b_1 does not commute with u, b_1u^k does not commute with u. By Lemma 5, there is a positive integer n_0 such that the first syllable of $u^{-n_0}(b_1u^k)u^{n_0}$ is in the same factor of H as the first syllable of u^{-1}, the last syllable of $u^{-n_0}(b_1u^k)u^{n_0}$ is in the same factor of H as the last syllable of u, the first syllable of $u^{n_0}(b_1u^k)u^{-n_0}$ is in the same factor of H as the first syllable of u, and the last syllable of $u^{n_0}(b_1u^k)u^{-n_0}$ is in the same factor of H as the last syllable of u^{-1}. Let N_1 be a normal subgroup of finite index in G such that $h\not\equiv g^z \bmod N_1$ for all integers z and all group elements h ranging over the syllables of u, a_1, $u^{-y}(b_1u^k)u^y$ for $y>n_0$, and b_1 contained in the factor G of H. N_1 exists since G is Π_c. If M is a normal subgroup of G and a subgroup of N_1 and K is the kernel of the naturally defined homomorphism from H

to $H^*=(x, G/M; x^m=g^*)$, where g^* is the image of g in G/M, then

$$a_1 = u^{-r_0}(b_1u^k)u^{r_0} \bmod K$$

is possible for only finitely many values of r_0. This is true because the homomorphism from H to H^* preserves the syllable lengths of u, a_1, b_1, and $u^{-y}(b_1u^k)u^y$ for $y>n_0$, and maps the syllables of these elements onto the syllables of their images in H^*. Thus the image of u is cyclically reduced in H^* and has syllable length greater than one. Thus for only finitely many integral r_0 can the syllable length of the image of a_1 equal the syllable length of the image of $u^{-r_0}(b_1u^k)u^{r_0}$, and these values of r_0 are independent of the way M is chosen in N_1. Let $t_1, \dots, t_w$ be the values of r_0 such that the syllable length of the image of a_1 in H^* equals the syllable length of the image of $u^{-r_0}(b_1u^k)u^{r_0}$ in H^*.

Given a subscript j between 1 and w, we consider the system of equations (I_j) as follows:

$$(\mathrm{I}_j)\qquad \begin{aligned} b_1^{-1}u^{t_j}a_1 &= u^{r_1}\\ b_2^{-1}b_1^{-1}u^{t_j}a_1a_2 &= u^{r_2}\\ &\vdots\\ b_{n-1}^{-1}\cdots b_1^{-1}u^{t_j}a_1\cdots a_{n-1} &= u^{r_{n-1}}\\ b_n^{-1}\cdots b_1^{-1}u^{t_j}a_1\cdots a_n &= u^{t_j}.\end{aligned}$$

If the system (I_j) has a solution $s_1, \dots, s_{n-1}$, then the system of equations $a_i = u^{-r_{i-1}}b_iu^{r_i}$, $r_0=r_n$, has the solution $r_0=t_j$, $r_1=s_1, \dots, r_{n-1}=s_{n-1}$. Thus for each j between 1 and w the equations (I_j) have no solution. It follows from the remarks preceding Lemma 1 that H is Π_c and that there is a single normal subgroup N_2 of finite index in H such that all the systems (I_j) for $j=1, \dots, w$ have no solution modulo N_2. Let M be a normal subgroup of G contained in $G \cap N_2$, let ψ_1 be the natural homomorphism of G onto G/M, and let φ_1 be the extension of ψ_1 to a homomorphism of H onto $H^*=(x, G; x^m=\psi_1(g))$. Since $G \cap N_2 \supset M$, there is a natural homomorphism ψ_2 of G/M onto $G/(G \cap N_2)$, and ψ_2 can be extended to a homomorphism φ_2 of H^* onto $H^{**}=(x, G/(G\cap N_2); x^m=\psi_2\psi_1(g))$. Since $\psi_2\psi_1$ is the restriction of the natural homomorphism of H onto H/N_2 to G, there is a homomorphism φ_3 of H^{**} onto H/N_2. Since N_2 is the kernel of $\varphi_3\varphi_2\varphi_1$, N_2 contains the kernel of φ_1. Thus for $j=1, \dots, w$, the equations (I_j) have no solution modulo the kernel of φ_1. Let $N_3=(G \cap N_2) \cap N_3$, ψ be the natural homomorphism of G onto G/N_3 and let φ be the extension of ψ to a homomorphism of H onto $L=(x, G/N_3; x^m=\psi(g))$. It follows from the results of this paragraph and the one preceding that the equations $\varphi(a_i)=\varphi(u)^{-r_{i-1}}\varphi(b_i)\varphi(u)^{r_i}$, $r_0=r_n$, have no solution with $r_1=r_0+k$. Note also that since N_1 and $G \cap N_2$ have finite index in G, N_3 has finite index in G.

Let η be the natural homomorphism of H onto H/H'. Let δ be the homomorphism of H onto $L \times H/H'$ defined by $\delta(h)=\eta(h)\varphi(h)$ for all $h \in H$. Since $\eta(a_i)$

$=\eta(u)^{-r_{i-1}}\eta(b_i)\eta(u)^{r_i}$, $r_0=r_n$, implies $r_1=r_0+k$, the equations

$$\delta(a_i) = \delta(u)^{-r_{i-1}}\delta(b_1)\delta(u)^{r_i}, \qquad r_0 = r_n,$$

have no solution. Since G/N_3 is finite, it follows from theorems of H. Neumann and B. H. Neumann (cf. [8, Lemma 16]) that L has a free subgroup of finite index. Since H/H' is free cyclic, it follows from Lemmas 2 and 3 that every element of $L\times H/H'$ has property Π. Thus $\delta(u)$ has property Π and there is a homomorphism ξ of $L\times H/H'$ onto a finite group such that the equations

$$\xi\delta(a_i) = \xi\delta(u)^{-r_{i-1}}\xi\delta(b_i)\xi\delta(u)^{r_i}, \qquad r_0 = r_n,$$

have no solution. If K is the kernel of $\xi\delta$, K is of finite index in H. Thus the lemma is proven.

LEMMA 7. *Let G be a Π_c group and let g be an element of G with property Π in G. Let m be an integer greater than one and let $H=(x, G; x^m=g)$. If h is a cyclically reduced element of syllable length greater than one in H, h is c.d. in H.*

Proof. Let t be an element of H not conjugate to h. We can assume that t is cyclically reduced. Let $h=h_1\cdots h_r$ and $t=t_1\cdots t_k$, where the elements h_i and t_i are the syllables of h and t respectively. Since G is Π_c, if s ranges over the h_i and t_i in G, then there is a single normal subgroup N_1 of finite index in G such that $s\not\equiv g^z \bmod N_1$ for all integers z and all values of s. If s is a syllable of h in some representation of h as a product of syllables, and s is in G, there are integers a, b, and c such that $s=g^a h_b g^c$ and h_b is an element of G. Since h_b is not congruent modulo N_1 to a power of g, s is not congruent modulo N_1 to a power of g. Similarly, no syllable of t in G is congruent modulo N_1 to a power of g.

Suppose now that the syllable length of t is not equal to the syllable length of h, i.e. $r\neq k$. Let ψ be the natural homomorphism of G onto G/N_1, and let φ be the extension of ψ to a homomorphism of H onto $H^*=(x, G/N_1; x^m=\psi(g))$. The syllable length of $\varphi(h)$ is r, the syllable length of $\varphi(t)$ is k, and both $\varphi(h)$ and $\varphi(t)$ are cyclically reduced. Since $r\neq k$, $\varphi(h)$ is not conjugate to $\varphi(t)$. Since $\varphi(h)$ is cyclically reduced and has syllable length greater than one, it has infinite order. Since G/N_1 is finite, it follows as in the proof of Lemma 6 that H^* has a free subgroup of finite index. According to [7, Theorem 2], $\varphi(h)$ is c.d. in H^*. Hence there is a homomorphism ξ from H^* onto a finite group such that $\xi\varphi(h)$ is not conjugate to $\xi\varphi(t)$.

Now suppose that h and t have the same syllable length, i.e. $r=k$. It follows from a theorem of D. Solitar [4, Theorem 4.6] that h is conjugate to t if and only if there is an integer u_0 and a cyclic permutation σ of $i,\ldots,k$ such that $h=g^{-u_0}t_{\sigma(1)}\cdots t_{\sigma(n)}g^{u_0}$. Assume without loss of generality that the first syllable of h is an element of G. Let σ be a permutation of $1,\ldots,k$ such that $t_{\sigma(i)}$ is in the same factor of H as h_i, and for h_i in the factor of H generated by x, $h_i=x^{u_i}$, $t_{\sigma(i)}=x^{v_i}$ with $u_i=v_i \bmod m$. If the above assumptions are true for σ, $t_{\sigma(1)}\cdots t_{\sigma(k)}=s_1h_2\cdots s_{k-1}h_k$, where $s_i=t_{\sigma(i)}g^{(v_{i+1}-u_{i+1})/m}$. Thus $h=g^{-r_0}t_{\sigma(1)}\cdots t_{\sigma(k)}g^{r_0}$ if and only if $h_1\cdots h_k=$

$g^{-r_0}s_1h_2\cdots s_{k-1}h_kg^{r_0}$. The latter equation holds if and only if there are integers $r_0,\ldots,r_{k-1}$ such that

$$\begin{aligned} h_1 &= g^{-r_0}s_1g^{r_1} \\ h_2 &= g^{-r_1}h_2g^{r_2} \\ &\vdots \\ h_{k-1} &= g^{-r_{k-2}}s_{k-1}g^{r_{k-1}} \\ h_k &= g^{-r_{k-1}}h_kg^{r_0}. \end{aligned}$$

The equations for h_{2n} imply that these equations have a solution if and only if there is a solution to the equations

$$\begin{aligned} h_1 &= g^{-r_0}s_1g^{r_2} \\ h_3 &= g^{-r_2}s_3g^{r_4} \\ &\vdots \\ h_{k-1} &= g^{-r_{k-2}}s_{k-1}g^{r_0}. \end{aligned}$$

Since h is not conjugate to t, the latter system of equations has no solution. Since h_{2i+1} and s_{2i+1} are elements of G and g has property in G, there is a normal subgroup N of finite index in G such that the latter equations have no solution modulo N. Let N be the intersection of N_1 and N_σ for σ such that $t_{\sigma(i)}$ is in the same factor of H as h_i, and for h_i in the factor of H generated by x, $h_i\equiv t_{\sigma(i)}$ mod x^m. Let η be the natural homomorphism from G onto G/N and let ψ be the extension of η to a homomorphism from H onto $H^*=(x, G/N; x^m=\eta(g))$. Clearly N is a normal subgroup of finite index in G so it follows as in the proof of Lemma 6 that H^* has a free subgroup of finite index.

Suppose that $\psi(h)$ is conjugate to $\psi(t)$. Now $\psi(h)=\psi(h_1)\cdots\psi(h_k)$ and $\psi(t)=\psi(t_1)\cdots\psi(t_k)$, where $\psi(h_i)$ and $\psi(t_i)$ are the syllables of $\psi(h)$ and $\psi(t)$ respectively. Since the theorem of D. Solitar used in the last paragraph applies to H^*, there is an integer r_0, and a cyclic permutation σ of $1,\ldots,k$ such that

$$\psi(h) = \psi(g)^{-r_0}\psi(t_{\sigma(1)})\cdots\psi(t_{\sigma(k)})\psi(g)^{r_0}.$$

Thus there are integers $r_1,\ldots,r_{k-1}$ so that $\psi(h_i)=\psi(g)^{-r_{i-1}}\psi(t_{\sigma(i)})\psi(g)^{r_i}$, $r_n=r_0$. These equations are soluble only if $\psi(h_i)$ and $\psi(t_{\sigma(i)})$ are in the same factor of H^*, and if $\psi(h_i)$ is in the factor of H^* generated by x, $\psi(t_{\sigma(i)})\equiv(h_i)$ mod $\psi(g)$. Since $N\subset N_1$, h_i and $t_{\sigma(i)}$ must be in the same factor of H, and if h_i and $t_{\sigma(i)}$ are in the factor of H generated by x, $h_i\equiv t_{\sigma(i)}$ mod x^m. Note that $x^m=g$. Thus σ must be a permutation for which N_σ is defined and $N_\sigma\supset N$. But N_σ was chosen so that the relations between elements of G implied by $h=g^{-r_0}t_{\sigma(1)}\cdots t_{\sigma(k)}g^{r_0}$ are not valid modulo N_σ. Thus $\psi(h)$ is not conjugate to $\psi(t)$ in H^*.

Since H^* has a free normal subgroup of finite index, it follows from [7, Theorem 2] that elements of infinite order in H^* are c.d. in H^*. Since $\psi(h)$ is cyclically reduced and has syllable length greater than one, $\psi(h)$ is c.d. in H^*. Since $\psi(t)$ is not conjugate to $\psi(h)$, there is a homomorphism ξ of H^* onto a finite group such that $\xi\psi(h)$ is not conjugate to $\xi\psi(t)$.

It now follows that the given element h is c.d. in H.

LEMMA 8. *Let G be a $\Pi_c \cdots$ and c.s. group. Let g be an element of G with property Π in G. Let m be an integer greater than one. Let $H=(x, G; x^m=g)$. If h is an element of the subgroup G of H, h is c.d. in H.*

Proof. Let t be an element of H not conjugate to h. Without loss of generality, we can assume that t is cyclically reduced. If t has syllable length greater than one, t is c.d. in H and there is a homomorphism ξ of H onto a finite group such that $\xi(t)$ and $\xi(h)$ are not conjugate. If t has syllable length one and is not in G, t is a power of x not in G. Let η be the homomorphism of H onto a cyclic group of order m generated by the image of x, where the kernel of η is the normal closure of G in H. Clearly, $\eta(t)$ is not conjugate to $\eta(h)$. Thus we can assume that t is an an element of G. Since G is c.s. and h is not conjugate to t in G, there is a normal subgroup N of finite index in G such that $u^{-1}tu \not\equiv h \bmod N$ for all u in G. Let η be the natural homomorphism of G onto G/N. Let P be the $m \times m$ permutation matrix corresponding to the cycle $(1 \cdots m)$. Let H^* be the group of $m \times m$ matrices with entries in the integral group ring of G/N with generators $\operatorname{diag}(a, \ldots, a)$ for all $a \in G/N$ and $\operatorname{diag}(1, \ldots, 1, \eta(g))P$. The assignments $u \to \operatorname{diag}(\eta(u), \ldots, \eta(u))$, for $u \in G$ and $x \to \operatorname{diag}(1, \ldots, 1, \eta(g))P$ clearly define a homomorphism ψ of H onto H^*. If D is diagonal, $DP=PD^*$ where D^* is diagonal. Thus it is clear that H^* is finite and that if $\psi(h)$ and $\psi(t)$ are conjugate, then $\eta(h)$ and $\eta(t)$ are conjugate. Thus $\psi(h)$ is not conjugate to $\psi(t)$.

It now follows that h is c.d. in H.

LEMMA 9. *Let G be a Π_c and c.s. group. Let g be an element of G with property Π in G. Let m be an integer greater than one and let $H=(x, G; x^m=g)$. If H/H' is a free cyclic group, if g is not an element of H' and if h is a power of x not in G, then h is c.d. in H.*

Proof. Let t be an element of H not conjugate to h. Without loss of generality, we can assume that t is cyclically reduced. If t is an element of G or has syllable length greater than one, t is c.d. in H, so there is a homomorphism ξ of H onto a finite group such that $\xi(t)$ and $\xi(h)$ are not conjugate. Thus we can assume that t is a power of x. Let $h=x^{n_1}$, $t=h^{n_2}$. Since h is not conjugate to t, $n_1 \neq n_2$. Let η be the natural homomorphism of H onto H/H'. Since $\eta(g)\neq 1$ and H/H' is torsion free, $\eta(x)\neq 1$. Thus $\eta(h)\neq\eta(t)$. Let ψ be a homomorphism of H/H' onto a finite group such that $\psi\eta(h)\neq\psi\eta(t)$. Since $\psi\eta(H)$ is abelian, $\psi\eta(h)$ is not conjugate to $\psi\eta(t)$.

It follows that the given element h is c.d. in H.

LEMMA 10. *Let G be a Π_c and c.s. group. Let g be an element of G with property Π in G. Let m be an integer with absolute value greater than one. Let $H=(x, G; x^m=g)$. If H/H' is a free cyclic group, H is c.s.*

Proof. First we show that g is not an element of H'. Suppose, to obtain a contradiction, that $g \in H'$. Since there is a homomorphism of H onto a cyclic group of order m generated by the image of x, $x \notin H'$. Since $x^m = g$, $g \in H'$ implies that H/H' has torsion, contrary to hypothesis. Now H is isomorphic to $(x, G; x^{-m} = g)$ so that we may assume that m is an integer greater than one. By Lemmas 7, 8 and 9, every cyclically reduced element of H is c.d. in H. Since every element of H is conjugate to a cyclically reduced element, H is c.s.

THEOREM 1. *The groups of hose knots are c.s.*

Proof. Let H be the group of a hose knot. We refer to the presentation given in the Introduction. Since H is a knot group, H/H' is free cyclic. According to §4 of [6], the groups of hose knots are Π_c. According to Theorem 1 of [8], the groups of torus knots, i.e. $(a, b, a^r = b^s)$, are c.s. Thus it will follow that the groups of hose knots are c.s. if at each stage of the construction of the group by adding roots of elements, a root of a Π element is added and the resultant group has free cyclic commutator quotient group. Since at each stage in the construction a knot group is obtained, at each stage we obtain a group with free cyclic commutator quotient group.

Let us examine a stage of the construction to see that a root of a Π element is added. Let H^* be the subgroup of the group of a hose knot generated by $P_1, Q_1, \ldots, Q_{i-1}$. We add Q_i to H^*, where Q_i is a root of the element

$$g_i = P_i^{n_i - n_{i-1} m_i} Q_{i-1}^{m_{i-1} n_i}.$$

It was shown in the proof of Lemma 10 that the resultant group has free cyclic commutator quotient group only if g_i is not an element of the commutator subgroup of H^*. If we regard H^* as the free product of its subgroups generated by Q_{i-1} and $Q_{i-2}, \ldots, Q_1, P_1$ with a cyclic amalgamated subgroup, it is clear from the relations between the exponents in the presentation of H^* and those of g_i that g_i is conjugate to a cyclically reduced element of length greater than one in H^*. Thus g_i is, by Lemma 6, conjugate to a Π element of H^* and hence is itself a Π element of H^*. It thus follows inductively, using Lemmas 6 and 10, that the groups of hose knots are c.s.

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