WHEN IS $\mu \ast L_1$ CLOSED?

BY

I. GLICKSBERG(1)

Abstract. For a finite measure $\mu$ on a locally compact abelian group, we partially
answer the question of when $\mu \ast L_1$ is closed in $L_1$.

1. Let $G^\wedge$ be a locally compact abelian group with dual $G^\wedge = \Gamma$. Clearly a
measure(2) $\mu \in \mathcal{M} = \mathcal{M}(G)$ has $\mu \ast L_1(G)$ closed in $L_1(G)$ if it is invertible in $\mathcal{M}$, or
an idempotent, or if

$$ (1.1) \mu \text{ is the convolve of an idempotent and an invertible in } \mathcal{M}(G). $$

Our question, for which we are indebted to Edwin Hewitt, might well be answered
by the converse, but so far we can only give a complete answer when $\Gamma$ is connected
(so no nontrivial idempotents appear) or under some special hypothesis.

Theorem 1.1. If $\Gamma$ is connected, $\mu \ast L_1(G)$ is closed if and only if $\mu$ is invertible
in $\mathcal{M}(G)$, or $\mu = 0$.

Theorem 1.2. If $\mu \in L_1(G)$, $\mu \ast L_1(G)$ is closed iff (1.1) holds.

Evidently (1.1) implies that $\mu \ast \mu \ast L_1(G)$ is also closed.

Theorem 1.3. $\mu \ast \mu \ast L_1(G)$ is closed iff (1.1) holds, and then the space coincides
with $\mu \ast L_1(G)$. In particular if $\mu \ast L_1(G)$ is closed and $\mu$ has a square root in $\mathcal{M}$ (which
thus must satisfy (1.1)), then (1.1) holds for $\mu$.

So what we shall leave open can be viewed as the question of whether $\mu \ast L_1(G)$
closed implies $\mu \ast \mu \ast L_1(G)$ is closed.

The same argument used to prove Theorem 1.1 can be applied to yield a bit of
information on the analogous question in which $L_1$ is replaced by a closed ideal.

Theorem 1.4. Suppose $I$ is a proper closed ideal in $L_1$. Then $\mu \ast I$ closed implies
$\hat{\mu}^{-1}(0)$ is interior to hull ($\mu \ast I$) and

$$ (1.2) \mu \ast I = \{ f \in I : \hat{f} = 0 \text{ on } \hat{\mu}^{-1}(0) \}. $$

In the special case that $\Gamma$ is connected, $\mu \in L_1$ (or just if $\hat{\mu} \in C_0$) and hull $I$ is
nowhere dense, $\mu \ast I$ is closed only if $\mu = 0$, or if $\Gamma$ is compact and $\mu$ is invertible.

Received by the editors January 8, 1970 and, in revised form, December 28, 1970.

AMS 1969 subject classifications. Primary 4256; Secondary 4255.

Key words and phrases. Locally compact abelian groups, convolve, Fourier transform.

(1) Work supported in part by the National Science Foundation.

(2) We shall use $\mathcal{M}$, rather than $\mathcal{M}(G)$, for the algebra of measures on $G$, and similarly
$L_1$, $L_\infty$, $C$ for the usual spaces on $G$. 

Copyright © 1971, American Mathematical Society
Various questions related to the original one are in fact equivalent. Indeed, we shall see \( \mu \ast L_1 \) is closed iff \( \mu \ast M \) is closed, and since an operator from one Banach space to another has closed range iff its adjoint has [1, p. 488] this implies

**Corollary 1.5.** If any of \( \mu \ast L_1, \mu \ast M, \mu \ast C_0, \mu \ast L_\infty \) are closed (in \( L_1, M, C_0, L_\infty \), resp.), then all are.

A part of our argument yields a simple characterization of inclusions between ranges of convolution operators, which seems new, and is quite independent of our original problem.

**Theorem 1.6.** For \( \mu, \nu \in M, \mu \ast L_1 \subseteq \nu \ast L_1 \) iff \( \mu \ast M \subseteq \nu \ast M \) (i.e., iff \( \mu \ast M \subseteq \nu \ast M \)); more generally \( \mu \ast L_1 \subseteq \sum_1^a \nu_i \ast L_1 \) iff \( \mu \in \sum_1^a \nu_i \ast M \).

Finally, I am indebted to M. M. Hackman and I. Wik for several helpful conversations, and to Frank Pollard for pointing out a grievous error in an earlier attempt on the problem.

2. One easy special case, which we shall use later on, is that in which \( \mu \ast L_1 \) is closed and \( \hat{\mu} \) never vanishes on \( \Gamma \); then \( \mu \ast L_1 \) is a closed hull-less ideal and so all of \( L_1 \) by Wiener's Tauberian theorem, while \( f \mapsto \mu \ast f \) is 1-1. So the open mapping theorem says this map is topological and \( \mu \ast f \rightarrow f \) is a bounded operator on \( \mu \ast L_1 = L_1 \) which evidently commutes with translations. As is well known [3] it is convolution with a \( \lambda \) in \( M \), whence \( \lambda \ast \mu \ast f = f, f \in L_1 \) and \( \lambda \ast \mu = \delta_0 \) (the point mass at the identity of \( G \)), so \( \mu \) is invertible.

Both Theorems 1.1 and 1.2 follow from the following fact, which we put in a more general form for later use.

**Lemma 2.1.** If \( I \) is an ideal in \( L_1 \) and \( \mu \ast I \) is closed then \( \hat{\mu} \) is bounded away from zero on \( \Gamma \setminus (\hat{\mu}^{-1}(0) \cup \text{hull } I) = \Gamma \setminus \text{hull } (\mu \ast I) \).

Indeed, suppose not, so we can find distinct \( \gamma_n \notin \text{hull } I \) with \( 0 < |\hat{\mu}(\gamma_n)| < 1/n! \). We can find symmetric compact Baire neighborhoods \( V_n \) of the identity in \( \Gamma \) for which \( \{\gamma_n + 2 V_n\} \) is a pairwise disjoint sequence of subsets of \( \Gamma \setminus \text{hull } (\mu \ast I) \), so the functions

\[
\psi_n = (mV_n)^{-1} R_{\gamma_n}(\chi_{V_n} \ast \chi_{V_n})
\]

(where \( m \) is Haar measure in \( \Gamma \) and \( R_{\gamma} \) translation by \( \gamma \)) have pairwise disjoint supports contained in \( \Gamma \setminus \text{hull } (\mu \ast I) \). Now \( \psi_n \) is the Fourier transform of an \( f_n \) in \( L_1 \) with \( \hat{\gamma}_n f_n = (mV_n)^{-1} \hat{\gamma}_n \hat{\psi}_n \geq 0 \) (since \( V_n \) is symmetric) so \( \|f_n\|_1 = \|\hat{\gamma}_n f_n\|_1 = (mV_n)^{-1} \chi_{V_n} \ast \chi_{V_n}(0) = 1 \) by Plancherel say. Hence

\[
f = \sum_{n=1}^{\infty} \frac{2^n}{n!} f_n
\]

is in \( L_1 \) and \( \hat{f}((\gamma_n)) = (2^n/n!) \hat{\psi}_n(\gamma_n) = 2^n/n! \). Since \( 0 < |\hat{\mu}(\gamma_n)| < 1/n! \) we have \( |\hat{f}((\gamma_n))| \hat{\mu}(\gamma_n)| > 2^n \) and \( f \hat{\mu} \) is not bounded on \( \Gamma \setminus \hat{\mu}^{-1}(0) \). On the other hand, since \( f_n \)
has compact support disjoint from hull \((\mu \ast I)\), \(f_n\) lies in the smallest ideal with that hull [3] and so in \(\mu \ast I\). So \(f = \lim_{n \to \infty} \sum_{n=1}^\infty 2^n f_n / n!\) is in the closed ideal \(\mu \ast I\), whence \(f = \mu \ast g\), \(g \in L_1\), and, on \(\Gamma \backslash \hat{\mu}^{-1}(0)\), \(f/\hat{\mu} = \hat{g}\) must be bounded, a contradiction completing our proof(3).

Again suppose \(\mu \ast L_1\) is closed. By Lemma 2.1 with \(I = L_1\), \(\hat{\mu}\) is bounded away from zero on \(\Gamma \backslash \hat{\mu}^{-1}(0)\), so the boundary of \(\hat{\mu}^{-1}(0)\) must be void, and \(\hat{\mu}^{-1}(0)\) is open. This, of course, yields Theorem 1.1, since there if \(\mu \neq 0\) we must have \(\hat{\mu}^{-1}(0)\) empty, whence \(\mu\) is invertible by our easy special case.

Now to obtain Theorem 1.2, note that there \(\hat{\mu}\) is in \(C_0\) and is bounded away from zero on \(\Gamma \backslash \hat{\mu}^{-1}(0)\), so this closed set is compact. By regularity there is an \(\eta\) in \(L_1\) with \(\hat{\eta} = 1\) on \(\Gamma \backslash \hat{\mu}^{-1}(0)\) and 0 on \(\hat{\mu}^{-1}(0)\). This \(\eta\) is the desired idempotent, and to complete our proof we note that \(\mu = \eta \ast \mu = \eta \ast (\mu + \delta_0 - \eta)\), while \(\mu + \delta_0 - \eta\) is an invertible in the subalgebra \(C\delta_0 + L_1\) of \(\mathcal{M}\) since its Gelfand representative (on the one point compactification of \(\Gamma\)) has the value 1 where \(\hat{\mu}\) (and hence \(\hat{\eta}\)) vanish, and at infinity, while on \(\Gamma \backslash \hat{\mu}^{-1}(0)\) where \(\hat{\eta} = 1\) the values coincide with those of \(\hat{\mu}\), hence are bounded away from 0.

We obtain Theorem 1.3 by first noting that since \((\mu \ast \mu) \backslash \hat{\mu}^{-1}(0) = \hat{\mu}^{-1}(0)\) is open, it is a set of synthesis so that

\[
\mu \ast L_1 = \mu \ast \mu \ast L_1 = \{ f \in L_1 : f = 0 \text{ on } \hat{\mu}^{-1}(0) \}
\]

since the latter two are closed ideals with hull \(\hat{\mu}^{-1}(0)\), while the first lies between these two. Consequently \(f \to \mu \ast f\) is a map of \(\mu \ast L_1\) onto itself by the first equality, evidently 1-1 by the second and thus an invertible operator on the Banach space \(\mu \ast L_1\). So nearby operators on \(\mu \ast L_1\) are invertible, in particular \(f \to (z\delta_0 - \mu) \ast f\), for \(|z| < \delta\), some \(\delta > 0\). Thus for \(0 \neq |z| < \delta\) we have \((z\delta_0 - \mu) \ast L_1 = \mu \ast L_1\), and so \(f \in L_1\) implies \(\mu \ast f = (z\delta_0 - \mu) \ast \mu \ast g\), \(g \in L_1\), whence \(f = (z\delta_0 - \mu) \ast g\) in \(N_\mu\), the nullity of \(f \to \mu \ast f\) in \(L_1\). But \(h \in N_\mu\) implies \((z\delta_0 - \mu) \ast h = zh\) so that \(N_\mu \subset (z\delta_0 - \mu) \ast L_1\), and we conclude that any \(f\) in \(L_1\) lies in \((z\delta_0 - \mu) \ast L_1\) if \(0 \neq |z| < \delta\), and \(L_1 = (z\delta_0 - \mu) \ast L_1\). By our easy special case \(z\delta_0 - \mu\) is an invertible in \(\mathcal{M}\), and thus 0 is an isolated point in the spectrum of the element \(\mu\) of \(\mathcal{M}\). So the characteristic function \(\chi\) of the complement of a small disc about 0 \(\in \mathbb{C}\) is analytic near the spectrum of \(\mu\), and (via the Cauchy integral) [2] there is an \(\eta\) in \(\mathcal{M}\), necessarily idempotent, with \(\hat{\eta} = \chi \ast \hat{\mu}\) the characteristic function of the spectrum of \(\mathcal{M}\) less

(3) I am indebted to M. Hackman for suggesting this type of construction, and to I. Wik for pointing out how simple the final argument could be made. A shorter but slightly less elementary alternate proof of the lemma can be obtained from the open mapping theorem and the fact that for \(y \in \Gamma F\), \(F\) closed, we have \(|y + (kF)^i| = \inf \{|y + h|^i : h \in (kF)^i\} = \|y\|\) (which also follows from spectral synthesis for open sets). Indeed since the dual to \(I \to \mu \ast I \subset L_1\) induces a topological map of \(L_{\infty} / (\mu \ast I)^1\) into \(L_{\infty} / I^1\), \(|\varphi + (\mu \ast I)^1| \leq k \|\mu \ast \varphi + I^1\| \leq k \|\mu \ast \varphi\|\) for \(\varphi \in L_{\infty}\). With \(\varphi = y \in \Gamma \backslash \text{hull} (\mu \ast I)\) then \(1 = \|y\| = \|y + (\mu \ast I)^1\| \leq k \|\mu \ast y\| = k (\hat{\mu})(y)\).
\( \hat{\mu}^{-1}(0) \), where the hat now denotes the Gelfand representative. So \( \mu = \eta \ast \mu = \eta \ast (\mu + \delta_0 - \hat{\eta}) \), and we see \( \mu + \delta_0 - \hat{\eta} \) is invertible in \( M \) exactly as before(4).

To obtain Theorem 1.4, suppose \( I \) is a closed ideal and \( 0 \neq \mu \ast I \) is closed. By Lemma 2.1 no element of \( \hat{\mu}^{-1}(0) \) can lie in the boundary of hull \( (\mu \ast I) \), and thus the boundary of \( \hat{\mu}^{-1}(0) \) is interior to hull \( I \) as asserted.

Now \( \mu \ast I \subseteq \{ f \in I : f = 0 \text{ on } \hat{\mu}^{-1}(0) \} \), while if \( f \) is in the larger set then \( f = 0 \) near \( \hat{\mu}^{-1}(0) \) so that \( f \in \mu \ast L_1 \), and we have an \( h \in L_1 \) with \( f = \mu \ast h \in L_1 \). Near any point of \( \Gamma \setminus \hat{\mu}^{-1}(0) \), \( 1/\mu \) coincides with a Fourier transform, so \( h \) belongs locally to \( I \) at all \( \gamma \) in \( \Gamma \setminus \hat{\mu}^{-1}(0) \). On the other hand \( f \) vanishes on an open set \( U \supset \hat{\mu}^{-1}(0) \), so that \( f/\mu = h \) vanishes on \( U \setminus \hat{\mu}^{-1}(0) \) and if we set \( \varphi = h \) off \( U \) and \( \varphi = 0 \) on \( U \) then \( \varphi \) belongs locally to \( I \) at all points of \( \Gamma \), and even at \( \infty \) if \( f \) has compact support, so \( \varphi = g, g \in I \). For such an \( f \) we thus have \( f \in \mu \ast I \), as desired; and for any \( f \in I \) with \( f = 0 \) on \( \hat{\mu}^{-1}(0) \) we have a net \( \{ f_\alpha \} \) in \( L_1 \) with \( f_\alpha \ast f \to f, f_\alpha \equiv 0 \) near \( \infty \), so that \( f_\alpha \ast f \in \mu \ast I \), and therefore \( f \in \mu \ast I \) since \( \mu \ast I \) is closed, proving (1.2).

Suppose hull \( I \) is nowhere dense. Then since \( \hat{\mu}^{-1}(0) \) is the interior of \( \hat{\mu}^{-1}(0) \) \( \cup \) hull \( I \), one easily concludes \( \hat{\mu}^{-1}(0) \) is open (if \( U \) is the interior, \( \Gamma \setminus \hat{\mu}^{-1}(0) \) is an open set contained in hull \( I \)). So if \( \mu \neq 0 \), \( \hat{\mu}^{-1}(0) = \varnothing \) if we assume \( \Gamma \) to be connected, and by Lemma 2.1 \( \mu \) is bounded away from zero on \( \Gamma \setminus \text{hull}(\mu \ast I) = \Gamma \setminus \text{hull} I \). Since \( \mu \) lies in \( C_0 \) by hypothesis, we conclude then that \( \Gamma \setminus \text{hull} I \) must have compact closure, which easily implies that \( \Gamma \) is compact since hull \( I \) is nowhere dense. But now \( \hat{\mu}^{-1}(0) = \varnothing \) means \( \mu \) is invertible in \( M = L_1 \) and Theorem 1.4 is proved.

3. Proof of Corollary 1.5 and Theorem 1.6. We shall first obtain 1.5 as a corollary to 1.6. If \( \mu \ast L_1 \) is closed then, as we have noted, since \( \hat{\mu}^{-1}(0) \) is open

\[
\mu \ast L_1 = \{ f \in L_1 : f = 0 \text{ on } \hat{\mu}^{-1}(0) \},
\]

and this implies any \( v \in M \) with \( \hat{v} = 0 \) on \( \hat{\mu}^{-1}(0) \) has \( v \ast L_1 \subseteq \mu \ast L_1 \), so \( v \in \mu \ast M \) by 1.6. Conversely \( v \in \mu \ast M \) implies \( \hat{v} = 0 \) on \( \hat{\mu}^{-1}(0) \), so

\[
\mu \ast M = \{ v \in M : \hat{v} = 0 \text{ on } \hat{\mu}^{-1}(0) \}
\]

which shows \( \mu \ast M \) is closed.

On the other hand that \( \mu \ast M \) is closed is equivalent to continuity of the inverse of the induced map of \( M/N_\mu^0 \) into the subspace \( \mu \ast M \) of \( M \) (where \( N_\mu^0 \) is the

(4) At this point it should be apparent that the gap between Theorem 1.1 and the result in full generality lies in showing that \( 0 \) isolated in the range of \( \mu \) means it must be isolated in the spectrum of \( \mu \) in \( M \). Actually, any root of \( \mu \) will do in the final assertion, and more generally, if \( \mu = \lambda \ast \nu \), where \( \hat{\lambda}^{-1}(0) = \hat{\nu}^{-1}(0) = \hat{\mu}^{-1}(0) \), the conclusion follows, by just the same argument: \( \lambda \) is invertible on \( \nu \ast L_1 = \lambda \ast \nu \ast L_1 \), whence \( 0 \) is isolated in the spectrum of \( \lambda \) in \( M \), so the Gelfand representative \( \hat{\lambda} \) of \( \lambda \) is bounded away from \( 0 \) where it is not \( 0 \). By symmetry the same is true of \( \hat{\nu} \), hence \( \hat{\lambda} \hat{\nu} = \hat{\mu} \) is bounded away from \( 0 \) where it is not \( 0 \).
nullity of $\nu \rightarrow \mu \ast \nu$), i.e. to the existence of a $k$ with

$$
\|\nu + N^0_\mu\| \leq k\|\mu \ast \nu\|, \quad \nu \in M.
$$

With $N_\mu$ the nullity in $L_1$, for $\nu \in L_1$ one easily sees $\|\nu + N_\mu\| \leq \|\nu + N^0_\mu\|$ using an approximate identity in $L_1$ and the fact that $L_1$ is an ideal in $M$, so (3.1) implies our map of $L_1/N_\mu$ into the subspace $\mu \ast L_1$ of $L_\infty$ is topological, whence $\mu \ast L_1$ is closed. Because of the theorem on adjoints cited earlier the proof of 1.5 is complete, and we turn to 1.6.

First we need the simple

**Lemma 3.1.** For $\lambda, \mu$ in $M$ with

$$
\|\lambda \ast \varphi\| \leq k\|\mu \ast \varphi\|, \quad \varphi \in C_0,
$$

we have $\lambda \in \mu \ast M$.

We have only to note that $\lambda \ast \varphi \in C_0$ so $\mu \ast \varphi \rightarrow \lambda \ast \varphi(0)$ is a bounded linear functional on a subspace of $C_0$. By Hahn-Banach it is given by a $\nu \in M$, $\lambda \ast \varphi(0) = \nu \ast \mu \ast \varphi(0)$, and replacing $\varphi$ by $R^{-1}\varphi$, $\lambda \ast \varphi(x) = \nu \ast \mu \ast \varphi(x)$, hence $\lambda = \mu \ast \nu$ as desired.

We can now obtain 1.6.

Suppose that $\mu \ast L_1 \subset \nu \ast L_1$. Then we have a linear map of $L_1$ into $L_1/N_\nu$ sending $f$ into $g + N_\nu$ if $\mu \ast f = \nu \ast g$, which has a closed graph: if $f_n \rightarrow f$ in $L_1$ and $g_n + N_\nu \rightarrow g + N_\nu$ in $L_1/N_\nu$, $\mu \ast f_n = \nu \ast g_n$, then $\mu \ast f = \lim \mu \ast f_n = \lim \nu \ast g_n = \nu \ast g$. So our map is continuous, and has a continuous adjoint sending $(L_1/N_\nu)^*$ into $L_\infty$. Of course $(L_1/N_\nu)^*$ is the subspace $N^\perp_\nu$ of $L_\infty$, which clearly contains $(5) \nu \ast L_\infty$. With $\varphi \in L_\infty$, $f, g \in L_1$ and $\mu \ast f = \nu \ast g$,

$$
\langle g + N_\nu, \nu \ast \varphi \rangle = \langle \nu \ast g, \varphi \rangle = \langle \mu \ast f, \varphi \rangle = \langle f, \mu \ast \varphi \rangle,
$$

so the adjoint maps $\nu \ast \varphi$ into $\mu \ast \varphi$, and we have $k > 0$ for which

$$
\|\mu \ast \varphi\|_\infty \leq k\|\nu \ast \varphi\|_\infty, \quad \varphi \in L_\infty,
$$

so $\mu \in \nu \ast M$ follows from Lemma 3.1. Conversely $\mu \in \nu \ast M$ implies $\mu \ast L_1 \subset \nu \ast L_1$ trivially.

The more general assertion in Corollary 1.5 follows in the same way, noting that $\mu \ast L_1 \subset \sum_i \nu_i \ast L_1$ yields a continuous linear map of $L_1$ into $(L_1 \oplus \cdots \oplus L_1)/N$, where $N = \{(f_1, \ldots, f_n) : f_i \in L_1, \sum \nu_i \ast f_i = 0\}$, with an adjoint taking $(\nu_1 \ast \varphi, \ldots, \nu_n \ast \varphi)$ in $N_\nu$ into $\mu \ast \varphi$. So for $\varphi \in C_0$, $(\nu_1 \ast \varphi, \ldots, \nu_n \ast \varphi) \rightarrow \mu \ast \varphi(0)$ defines a continuous linear functional on a subspace of $C_0 \oplus \cdots \oplus C_0$, whence we obtain $\lambda_1, \ldots, \lambda_n \in M$ with $\mu \ast \varphi(0) = \sum \lambda_i \ast \nu_i \ast \varphi(0)$, yielding the result.

Finally, the argument applies to a more general setting: suppose $E$ is a space of Radon measures on $G$ which is a Banach space under some norm, closed under

$$(5)$$

We pair $L_1$ and $L_\infty$ via $\langle f, \varphi \rangle = f \ast \varphi(0)$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
convolution with finite measures, with the property

\[(3.2) \quad \mu \in M \text{ and } e_n \to 0 \text{ in } E \text{ imply } |\mu * e_n|(K) \to 0 \quad \text{ for all compact } K \text{ in } G.\]

(For example \(E\) could be \(C(G)m\), where \(m\) is Haar measure.) Defining \(\mu^*\) on the dual \(E^*\) as the adjoint operator, so \(\langle e, \mu^* e^* \rangle = \langle \mu * e, e^* \rangle, e^* \in E^*\), suppose \(D\) is a closed subspace of \(E^*\) which is invariant and continuously translating (so \(\| \delta_x * d - d \| \to 0 \text{ as } x \to 0 \text{ in } G\)). Then if \(\mu * L_1 \subseteq \nu * E\), where \(\mu, \nu \in M\), (3.2) shows the nullity \(N_{\nu}\) of \(\nu * e\) is closed, and then that the map of \(L_1\) into \(E/N_{\nu}\) sending \(f\) into \(e + N_{\nu}\) if \(\mu * f = \nu * e\) has a closed graph, hence is continuous, as before. The adjoint, mapping \(N_\nu^*\) in \(E^*\) into \(L_\infty\) in particular takes \(\nu * D \subseteq \nu * E^* \subseteq N_\nu^*\) into \(L_\infty\), sending \(\nu * d\) into \(\mu * d\) since \(\langle e + N_{\nu}, \nu * d \rangle = \langle \nu * e, d \rangle = \langle \mu * f, d \rangle = \langle f, \mu * d \rangle\) when \(\mu * f = \nu * e\). Moreover, since \(\delta_x * \mu * d = \mu * \delta_x * d, \mu * d\) is a continuously translating element of \(L_\infty\), hence can be taken as a continuous function, so that \(\nu * d \to \mu * d(0)\) is a well-defined continuous linear functional on \(D\). Thus there is a \(d^* \in D^*\) with

\[(3.3) \quad \mu * d(0) = \langle \nu * d, d^* \rangle = \langle d, \nu * d^* \rangle, \quad d \in D.\]

For example, if \(E = C(G)m\) and we take \(D = L_1(G)\) (with \(\langle e * m, f \rangle = e * f(0)\) of course) then (3.3) says there is an \(h \in L_\infty\) for which

\[\mu * d(0) = d * \nu * h(0) \quad \text{ all } d \in L_1,\]

whence we can identify \(\mu\) with the measure \((\nu * h) * m\) in \(L_\infty * m\). So \(\mu * L_1 \subseteq (\nu * C)m\) (\(= \nu * (C * m)\)) implies \(\mu \in (\nu * L_\infty)m\); conversely that implies \(\mu * L_1 \subseteq (\nu * L_\infty * L_1)m \subseteq (\nu * C)m\) of course.

4. We should note that if \(\sum_{i=1}^n \mu_i * L_1\) is closed the argument of Lemma 2.1 can be used to show \(\sum |\hat{\mu}_i|\) is bounded away from zero on \(\Gamma \setminus \bigcap \hat{\mu}_i^{-1}(0)\), and thus that \(\bigcap \hat{\mu}_i^{-1}(0)\) is open. So if \(\Gamma\) is connected \(L_1\), has no proper nonzero closed subspaces of the form \(\sum_{i=1}^n \mu_i * L_1\).

Finally, we note that the arguments of §2 apply in a more general setting: in place of \(L_1\) we can take any regular commutative Banach algebra \(A\) which is tauberian (so the analogue of Wiener's theorem holds) and in place of \(M(G)\) the (Banach) algebra of multipliers of \(A\) (the algebra \(M\) of all operators \(T\) on \(A\) satisfying \(T(ab) = a \cdot Tb\)). With just the special hypothesis that for each neighborhood \(V\) of any element \(\gamma\) of the spectrum \(\Gamma\) of \(A\) there is an \(a\) in \(A\) with Gelfand representative \(\hat{a}\) supported by \(V, \hat{a}(\gamma) = 1\) and \(|a| \leq c, \) a fixed constant, one has:

\[(4.1) \quad \text{If } \Gamma \text{ is connected } TA \text{ is closed iff } T = 0 \text{ or } T \text{ is invertible in } M.\]

\[(4.2) \quad \text{If } T \text{ is multiplication by an element of } A, TA \text{ is closed iff } T \text{ is the product of an idempotent and an invertible in } M.\]

\[(4.3) \quad T^2A \text{ is closed iff } T \text{ is a product as in (4.2).}\]

As is well known, \(T \in M\) corresponds to a continuous function \(\varphi\) on \(\Gamma\) satisfying \((Ta)^\sim = \varphi \hat{a}\), and the arguments of §2, in particular of Lemma 2.1, apply with \(\varphi\) in place of \(\hat{\mu}\), as the reader can easily verify.
WHEN IS $\mu \ast L_1$ CLOSED?

REFERENCES


                      
                       UNIVERSITY OF WASHINGTON,  
                       SEATTLE, WASHINGTON 98105