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RELATIVE TYPES OF POINTS IN $\beta N-N$

BY

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Abstract. Using the concepts of type and relative type for points in $\beta N-N$, as introduced by W. Rudin, M. E. Rudin, and Z. Frolík, an inductive method is presented for constructing types. The relative types are described for points having these constructed types and a point in $\beta N-N$ is found which has exactly c relative types.

Introduction. Two points in $N^*=\beta N-N$ will be said to have the same type (N^*-type) if there is an autohomeomorphism on βN (N^*) which takes one to the other. The decomposition elements provided by these relations will be called types and N^*-types respectively.

Using the continuum hypothesis, W. Rudin [6] showed the existence of at least two N^*-types, the set of P-points being one of them. M. E. Rudin [5], also using the continuum hypothesis, showed the existence of 2^c N^*-types. Later Z. Frolík [1] proved this without the continuum hypothesis by introducing the notion of relative type. He showed that no point in N^* may have more than c relative types.

The purpose of this paper is to present an inductive method for constructing types and N^*-types. We will also describe the relative types of points having the types constructed. Finally, we find a point in N^* which has c relative types, thus showing that Frolík gave the best upper bound.

1. **Preliminaries.** Let T be the set of types in N^* and τ the mapping on N^* to T which assigns each point its type.

Let X be a countable discrete subset of βN. It is known that $\operatorname{cl} X$ is homeomorphic to βN. If f is such a homeomorphism, then f induces a homeomorphism between $X^*=\operatorname{cl} X-X$ and N^*. Just as a point in N^* may be viewed as an ultrafilter on N, $x\in X^*$ may be viewed as an ultrafilter on X. The type of x relative to X is defined to be $\tau(f(x))$ and will be denoted by $\tau(x, X)$. It is not hard to see that $\tau(x, X)$ is independent of the choice of f, since a permutation on N can be extended to an autohomeomorphism on βN which takes points in N^* to points of the same type. Clearly, $\tau(x, N)=\tau(x)$.

Throughout, we will write X c.d.$\subset Y$ to mean that X is a countable discrete subset of Y.

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It follows immediately from the definition that subsets and homeomorphisms preserve relative types. We state this precisely as

1.1. LEMMA. (a) *If* $Y \subset X$ *c.d.* $\subset \beta N$ *and* $x \in X^* \cap Y^*$, *then* $\tau(x, Y) = \tau(x, X)$.

(b) *If* X, Y *c.d.* $\subset \beta N$, $x \in X^*$ *and* $y \in Y^*$ *then* $\tau(x, X) = \tau(y, Y)$ *if and only if there is a homeomorphism* f *of* cl X *onto* cl Y *such that* $f(x) = y$.

(c) *If* h *is an autohomeomorphism on* βN (N^*), X *c.d.* $\subset \beta N$ (N^*) *and* $x \in X^*$, *then* $\tau(h(x), h[X]) = \tau(x, X)$.

If $x \in N^*$ and $S \subset \beta N$, let $\tau[x, S] = \{\tau(x, X) \mid X \text{ c.d.} \subset S, x \in X^*\}$. We will call $\tau[x, \beta N]$ the set of relative types of x. It follows from the lemma that $\tau[x, \beta N] = \tau[x, N^*] \cup \{\tau(x)\}$.

We now state Frolík's theorem [1] and give his short argument.

1.2. THEOREM (FROLÍK). card $\tau[x, N^*] \leqq c$ *for each* $x \in N^*$.

Proof. Let $x \in N^*$. For each countable decomposition $\{M_n\}$ of N choose an $x_n \in M_n^*$ such that $x \in \{x_n\}^*$, if possible, and consider all $X \subset \{x_n\}$ such that $x \in X^*$. The set $\mathscr{X}$ of all such X with $\{M_n\}$ variable has cardinality at most c. $\mathscr{X}$ also has the property that for each Y c.d. $\subset N^*$ such that $x \in Y^*$, there is an $X \in \mathscr{X}$ such that $X \subset Y$. The proof is concluded by using 1.1(a) above.

Since for each X c.d. $\subset N^*$, $\{\tau(x, X) \mid x \in X^*\} = T$, and card $T = 2^c$, it follows by 1.1(c) and the theorem that there are 2^c N^*-types. M. E. Rudin [5] obtained the same result. The following relation between relative types and N^*-types can be obtained from her corollary [5, p. 151].

1.3. THEOREM (M. E. RUDIN). *Let* X *and* Y *be countable discrete subsets of the same* N^**-type and* $x \in X^*$, $y \in Y^*$. *If* $\tau(x, X) = \tau(y, Y)$ *then* x *and* y *have the same* N^**-type.*

The next lemma is a consequence of a theorem of Frolík [2] which states that no homeomorphism of βN into N^* has a fixed point.

1.4. LEMMA. *If* Y *c.d.* $\subset X^*$ *and* $x \in Y^*$, *then* $\tau(x, Y) \neq \tau(x, X)$.

Proof. If $\tau(x, X) = \tau(x, Y)$, then there is a homeomorphism f of cl X onto cl Y such that $f(x) = x$. Let h be a homeomorphism of βN onto cl X. The mapping $h^{-1} \circ f \circ h$ is then a homeomorphism of βN into N^* which has $h^{-1}(x)$ as a fixed point.

2. **An order on T.** Following Frolík [3], a partial order can be put on the set T of types.

If $t_1, t_2 \in \tau[x, \beta N]$, then $t_1 >_x t_2$ if there are countable discrete subsets X_1, X_2 in βN such that $\tau(x, X_1) = t_1$, $\tau(x, X_2) = t_2$ and $X_2 \subset X_1^*$.

2.1. LEMMA (FROLÍK). *For each* $x \in N^*$, *the relation* $>_x$ *is a linear order on* $\tau[x, \beta N]$.

2.2. LEMMA. *The relation* $>_x$ *coincides with the relation* $>_y$ *on*

$$\tau[x, \beta N] \cap \tau[y, \beta N].$$

Proof. Suppose $t_1, t_2 \in \tau[x, \beta N] \cap \tau[y, \beta N]$ and that $t_1 >_x t_2$. Let X_1 and X_2 be countable discrete subsets of βN such that $\tau(x, X_1)=t_1$, $\tau(x, X_2)=t_2$ and $X_2 \subset X_1^*$. Since $t_1 \in \tau[y, \beta N]$, there is a Y_1 c.d.$\subset \beta N$ such that $\tau(y, Y_1)=t_1$. By Lemma 1.1(b) there is a homeomorphism f of cl X_1 onto cl Y_1 such that $f(x)=y$. If $Y_2=f[X_2]$, then Y_2 c.d.$\subset Y_1^*$ and $\tau(y, Y_2)=t_2$. Thus $t_1 >_y t_2$.

We can now define an order on T by letting $t_1 > t_2$ if and only if there is an $x \in N^*$ such that $t_1, t_2 \in \tau[x, \beta N]$ and $t_1 >_x t_2$.

2.3. LEMMA. *The relation* $>$ *is a partial order on* T.

Proof. Let $t_1, t_2, t_3 \in T$ such that $t_1 > t_2$ and $t_2 > t_3$. There are sets X_1, X_2, Y_2, Y_3 c.d.$\subset \beta N$ and points x and y in N^* such that $\tau(x, X_i)=t_i$, $i=1, 2$, $\tau(y, Y_j)=t_j$, $j=2, 3$, and $X_2 \subset X_1^*$ and $Y_3 \subset Y_2^*$. Since $\tau(x, X_2)=\tau(y, Y_2)$, there is a homeomorphism f of cl Y_2 onto cl X_2 with $f(y)=x$. By Lemma 1.1(b), $\tau(x, f[Y_3]) =\tau(y, Y_3)=t_3$ and thus $t_3 \in \tau[x, \beta N]$. Since $f[Y_3]$ c.d.$\subset X_1^*$, $t_1 >_x t_3$ and so $t_1 > t_3$.

If $x \in N^*$ is a P-point, then x is not a limit point of any countable subset in N^* and thus $\tau(x)$ is minimal in the partial order on T. There are 2^c P-points in N^* and since card $t=c$ for any $t \in T$, there are 2^c minimal types. Recently, K. Kunen [4] has shown that there exist at least two other types in T which are minimal. Assuming the continuum hypothesis, he has shown that there exists a non-P-point $q \in N^*$ which is not a limit point of any countable set in N^*. He has also shown the existence of a countable set $X \subset N^*$ such that each point of X is a limit point of X and such that no point in X is a limit point of any countable discrete subset of N^*. Since these properties are preserved under autohomeomorphisms on βN, there are at least two minimal types distinct from types of P-points.

Let $T_0 \subset T$ denote the set of all types which are minimal with respect to the partial order on T. Since $\tau[x, \beta N]$ is linearly ordered, it follows that $\tau(x) \in T_0$ if and only if $\tau[x, N^*]=\varnothing$, that is, if and only if x is not a limit point of any countable discrete subset of N^*.

3. **Multiplication in** T. The elements of T can be used as operators on subsets of βN in the following way. If $t \in T$ and $S \subset \beta N$, let $t[S]=\{x \in N^* \mid t \in \tau[x, S]\}$.

In order to introduce multiplication in T we prove

3.1. LEMMA. *If* $t_1, t_2 \in T$, *then* $t_1[t_2] \in T$.

Proof. Let $x, y \in t_1[t_2]$. There are countable discrete subsets X and Y of t_2 such that $\tau(x, X)=t_1=\tau(y, Y)$ and $\tau(w)=t_2=\tau(z)$ for all $w \in X$, $z \in Y$. The first condition implies the existence of a homeomorphism f of cl X onto cl Y such that $f(x)=y$. Since X is discrete, there are disjoint open neighborhoods U_w of w, for $w \in X$, such that $N \subset \bigcup \{U_w \mid w \in X\}$. Similarly, there are disjoint open neighborhoods V_z of $z \in Y$ such that $N \subset \bigcup \{V_z \mid z \in Y\}$. For each $w \in X$, $\tau(w, U_w \cap N)$

$=\tau(w)=\tau(z)=\tau(z, V_z \cap N)$ by Lemma 1.1(a). Let g_w be a homeomorphism of $\text{cl}\,(U_w \cap N)$ onto $\text{cl}\,(V_{f(w)} \cap N)$ such that $g_w(w)=f(w)$. If π is the permutation on N defined by $\pi(n)=g_w(n)$ for $n \in U_w \cap N$, then its extension $\bar{\pi}$ to βN is an autohomeomorphism on βN and $\bar{\pi}(x)=y$. Thus $\tau(x)=\tau(y)$.

Conversely, suppose $\tau(x)=\tau(y)$ and $x \in t_1[t_2]$. Then there is an X c.d.$\subset t_2$ such that $x \in X^*$ and $\tau(x, X)=t_1$. Since $\tau(x)=\tau(y)$, there is an autohomeomorphism h on βN such that $h(x)=y$. Clearly, $y \in (h[X])^*$, $\tau(y, h[X])=t_1$, and $\tau(z)=t_2$ for each $z \in h[X]$. Thus $y \in t_1[t_2]$.

Thus we may define $t_1 \circ t_2$ to be $t_1[t_2]$. From the definitions of relative type and multiplication, $\tau(x, X)=t_1 \circ t_2$ if and only if there is a set X_1 c.d.$\subset X^*$ such that $\tau(x, X_1)=t_1$ and $\tau(z, X)=t_2$ for each $z \in X_1$.

We will use the symbol $X \xrightarrow{t} S$ to mean that X c.d.$\subset \beta N$, $S \subset \beta N$, $s \in X^*$ and $\tau(s, X)=t$ for each $s \in S$. Thus $\tau(x, X)=t_1 \circ t_2$ if and only if we can write $X \xrightarrow{t_2} X_1 \xrightarrow{t_1} \{x\}$ for some X_1 c.d.$\subset \beta N$.

3.2. Lemma. *If $t_1, t_2 \in T$ and $S \subset \beta N$, then $t_1[t_2[S]]=t_1 \circ t_2[S]$.*

Proof. If $x \in t_1[t_2[S]]$ then there is a X c.d.$\subset t_2[S]$ such that $\tau(x, X)=t_1$. For each $y \in X$ there is a set X_y c.d.$\subset S$ such that $\tau(y, X_y)=t_2$. Since X is discrete, there is a family $\{O_y\}$ of disjoint open sets such that $y \in O_y$. If $Y=\bigcup \{O_y \cap X_y \mid y \in X\}$, then Y c.d.$\subset S$ and $X \subset Y^*$. By Lemma 1.1(a), each point in X has type t_2 relative to Y and thus it follows that $\tau(x, Y)=t_1 \circ t_2$ and $x \in t_1 \circ t_2[S]$.

Conversely, if $x \in t_1 \circ t_2[S]$, then there is a Y c.d.$\subset S$ such that $\tau(x, Y)=t_1 \circ t_2$. We can thus write $Y \xrightarrow{t_2} X \xrightarrow{t_1} \{x\}$ for some X c.d.$\subset \beta N$. But $Y \xrightarrow{t_2} X$ implies that $X \subset t_2[S]$, and thus $x \in t_1[t_2[S]]$.

From 3.2 it follows that $(t_1 \circ t_2) \circ t_3[S]=t_1 \circ (t_2 \circ t_3)[S]$ for any $S \subset \beta N$. Since $t[N]=t$ for any $t \in T$, we have $(t_1 \circ t_2) \circ t_3=t_1 \circ (t_2 \circ t_3)$. We do not intend to study T as a semigroup here, but will point out that it has no identity, and that multiplication is neither commutative nor compatible with the partial order.

If an element of T can be factored into a product of elements of T_0, then it can be done in only one way. In order to show this we first state the following lemmas.

3.3. Lemma (Frolík [3]). *Let X and Y be countable discrete subsets of βN. Then the set $Z=(X \cap Y) \cup (X^* \cap Y) \cup (X \cap Y^*)$ is discrete,* $\text{cl}\,Z=\text{cl}\,X \cap \text{cl}\,Y$, *and* $Z^*=X^* \cap Y^*$.

3.4. Lemma. *If X and Y are countable discrete subsets of βN having a common limit point x such that $\tau(x, X)=p$ and $\tau(x, Y)=q$ where $p, q \in T_0$, then $p=q$, $X \cap Y$ is infinite, and $x \in (X \cap Y)^*$.*

Proof. Let $Z=(X \cap Y) \cup (X^* \cap Y) \cup (X \cap Y^*)$. Since $x \in X^* \cap Y^*=Z^*$, either $x \in (X \cap Y)^*$, $x \in (X^* \cap Y)^*$ or $x \in (X \cap Y^*)^*$. But $p, q \in T_0$, so x is not a limit point of any countable discrete subset of X^* or Y^*. Therefore, $x \in (X \cap Y)^*$ and by Lemma 1.1(a), $p=\tau(x, X \cap Y)=q$. Clearly, $X \cap Y$ must be infinite.

3.5. THEOREM (UNIQUE DECOMPOSITION). *If* $p_1 \circ p_2 \circ \cdots \circ p_n = q_1 \circ q_2 \circ \cdots \circ q_m$ *where* $p_i, q_j \in T_0$ *for* $1 \leqq i \leqq n$ *and* $1 \leqq j \leqq m$, *then* $n=m$ *and* $p_i = q_i$ *for* $1 \leqq i \leqq n$. *Thus each finite sequence of elements from* T_0 *represents a distinct type in* T.

Proof. The proof will be by induction on n. If $p_1 = q_1 \circ q_2 \circ \cdots \circ q_m$ then $p_1 = \tau(x)$ for some $x \in N^*$ and x is not a limit point of any countable discrete subset of N^*. Thus $\tau(x) = q_1 \circ q_2 \circ \cdots \circ q_m$ implies that $m=1$, and by Lemma 3.4, $p_1 = q_1$.

Assume that for all $k \leqq n$, if $p_1 \circ p_2 \circ \cdots \circ p_k = q_1 \circ q_2 \circ \cdots \circ q_m$ and $p_i, q_j \in T_0$, then $k=m$ and $p_i = q_i$ for $1 \leqq i \leqq k$.

Now suppose $p_1 \circ p_2 \circ \cdots \circ p_{n+1} = q_1 \circ q_2 \circ \cdots \circ q_m$. Then there is an $x \in N^*$ such that $\tau(x) = p_1 \circ p_2 \circ \cdots \circ p_{n+1} = q_1 \circ q_2 \circ \cdots \circ q_m$, so there are sets X c.d.$\subset p_2 \circ \cdots \circ p_{n+1}$ and Y c.d.$\subset q_2 \circ \cdots \circ q_m$ such that $\tau(x, X) = p_1$ and $\tau(x, Y) = q_1$. From Lemma 3.4 it follows that $p_1 = q_1$ and that $X \cap Y$ is infinite. Since $p_2 \circ \cdots \circ p_{n+1}$ and $q_2 \circ \cdots \circ q_m$ are types and are not disjoint, $p_2 \circ \cdots \circ p_{n+1} = q_2 \circ \cdots \circ q_m$. By the induction hypothesis, $n+1=m$ and $p_i = q_i$ for $2 \leqq i \leqq n+1$. This together with $p_1 = q_1$ completes the proof.

3.6. THEOREM. *If* S *is an* N^**-type and* $t \in T$, *then* $t[S]$ *is an* N^**-type.*

Proof. If $x, y \in t[S]$ then there exist subsets X, Y c.d.$\subset S$ such that $\tau(x, X) = t = \tau(y, Y)$. Since elements in X and Y are in the same N^*-type, x and y are in the same N^*-type (M. E. Rudin, Theorem 1.3). Conversely, if $x \in t[S]$ and x and y are in the same N^*-type, then there is an autohomeomorphism h on N^* such that $h(x) = y$. Since $t \in \tau[x, S]$, it follows from Lemma 1.1(c) that $t \in \tau[h(x), h[S]] = \tau[y, S]$ and thus $y \in t[S]$.

We will now prove an analogue of the unique decomposition theorem for N^*-types.

3.7. THEOREM. *If* S *is an* N^**-type such that* $\tau[S] \subset T_0$, *and* $p_1 \circ p_2 \circ \cdots \circ p_n[S] = q_1 \circ q_2 \circ \cdots \circ q_m[S]$ *where* $p_i, q_j \in T_0$, *then* $n=m$ *and* $p_i = q_i$ *for* $1 \leqq i \leqq n$. *Thus each finite sequence from* T_0 *gives rise to a distinct* N^**-type which can be obtained from* S.

Proof. The proof will be by induction on n. If $x \in p_1[S] = q_1 \circ q_2 \circ \cdots \circ q_m[S]$, then there are sets X c.d.$\subset S$ and Y c.d.$\subset q_2 \circ \cdots \circ q_m[S]$ such that $\tau(x, X) = p_1$ and $\tau(x, Y) = q_1$. Since $p_1, q_1 \in T_0$, it follows from Lemma 3.4 that $p_1 = q_1$ and $X \cap Y$ is infinite. If $y \in X \cap Y$, then $y \in S$, $\tau(y) \in T_0$, and thus y is not a limit point of any countable discrete subset of N^*. This implies that $m=1$.

Assume that for $k \leqq n$, $p_1 \circ p_2 \circ \cdots \circ p_k[S] = q_1 \circ q_2 \circ \cdots \circ q_m[S]$ implies that $k=m$ and $p_i = q_i$ for $1 \leqq i \leqq k$.

If $x \in p_1 \circ p_2 \circ \cdots \circ p_{n+1}[S] = q_1 \circ q_2 \circ \cdots \circ q_m[S]$, it again follows from 3.4 that $p_1 = q_1$ and that the N^*-types $p_2 \circ \cdots \circ p_{n+1}[S]$ and $q_2 \circ \cdots \circ q_m[S]$ have points in common and thus are identical. By the induction hypothesis, $n+1=m$ and $p_i = q_i$ for $2 \leqq i \leqq n+1$. This together with $p_1 = q_1$ completes the proof.

4. **Relative types.** Using the previous lemmas, we can now describe the sets of relative types of some points in N^*.

4.1. THEOREM. *If* $\tau(x)=t_1 \circ t_2 \circ \cdots \circ t_n$ *where each* $t_i \in T_0$, *then* $\tau[x, \beta N] =\{t_1, t_1 \circ t_2, \ldots, t_1 \circ t_2 \circ \cdots \circ t_n\}$.

Proof. From Lemma 3.2 it follows that there exist countable discrete sets $X_1, X_2, \ldots, X_{n-1}, X_n=N$ in βN such that

$$N = X_n \xrightarrow{t_n} X_{n-1} \xrightarrow{t_{n-1}} \cdots \longrightarrow X_2 \xrightarrow{t_2} X_1 \xrightarrow{t_1} \{x\}.$$

Then for each i, $\tau(x, X_i)=t_1 \circ t_2 \circ \cdots \circ t_i$ and $\{t_1 \circ \cdots \circ t_i \mid 1 \leqq i \leqq n\} \subset \tau[x, \beta N]$.

Now, let W be a countable discrete subset of βN such that $x \in W^*$. Let $W_i = W \cap (\mathrm{cl}\, X_i - \mathrm{cl}\, X_{i-1})$ for $2 \leqq i \leqq n$ and $W_1 = W \cap \mathrm{cl}\, X_1$. Since the W_i's are disjoint and $W - \bigcup_1^n W_i \subset N = X_n$, either $\tau(x, W)=\tau(x)$ (by Lemma 1.1(a)) or $x \in W_i^*$ for exactly one i. If $x \in W_i^*$ then either $x \in (W_i \cap X_i)^*$ or $x \in (W_i \cap X_i^*)^*$. In the first case, $\tau(x, W_i)=\tau(x, X_i)=t_1 \circ t_2 \circ \cdots \circ t_i$. The second case cannot hold. If $i=1$, then $W_1 \cap X_1^*$ c.d.$\subset X_1^*$, and since $\tau(x, X_1)=t_1 \in T_0$, $x \notin (W_1 \cap X_1^*)^*$. If $i \geqq 2$, $(W_i \cap X_i^*)^* \cap X_{i-1}^* = \varnothing$, and $x \in X_{i-1}^*$. Thus $\tau(x, W)=t_1 \circ t_2 \circ \cdots \circ t_i$ for some i, $1 \leqq i \leqq n$.

If $\tau(x)=t_1 \circ t_2 \circ \cdots \circ t_n$, where the types t_i are not necessarily in T_0, it is still the case that $t_1 \circ t_2 \circ \cdots \circ t_k$ is in $\tau[x, \beta N]$ for $1 \leqq k \leqq n$, but x may have other relative types. It follows from the definition of T_0 that if $\tau(x)=t_1 \notin T_0$, then there is a $t_2 \in T$ such that $t_2 < t_1$ and so t_2 is also a relative type of x.

4.2. THEOREM. *If Q is an N^*-type such that $\tau[Q] \subset T_0$ and $t_1, t_2, \ldots, t_n \in T_0$, then for each $x \in t_1 \circ t_2 \circ \cdots \circ t_n[Q]$, $\tau[x, N^*]=\{t_1, t_1 \circ t_2, \ldots, t_1 \circ t_2 \circ \cdots \circ t_n\}$.*

Proof. From Theorem 3.6, $t_1 \circ t_2 \circ \cdots \circ t_n[Q]$ is an N^*-type. If $x \in t_1 \circ t_2 \circ \cdots \circ t_n[Q]$ then $\tau(x, X)=t_1 \circ t_2 \circ \cdots \circ t_n$ for some X c.d.$\subset Q$. Since $\mathrm{cl}\, X$ is homeomorphic to βN, $\tau[x, \mathrm{cl}\, X]=\{t_1 \circ t_2 \circ \cdots \circ t_k \mid 1 \leqq k \leqq n\}$ from Theorem 4.1. A countable discrete subset of $\mathrm{cl}\, X$ is also a countable discrete subset of N^* so $\tau[x, \mathrm{cl}\, X] \subseteq \tau[x, N^*]$.

If W c.d.$\subset N^*$ such that $x \in W^*$, then by Lemma 3.3, either $x \in (X \cap W)^*$, $x \in (W^* \cap X)^*$ or $x \in (W \cap X^*)^*$. In the first case, $\tau(x, W)=\tau(x, X)$, the second case cannot occur since no point in $X \subset Q$ is a limit point of any countable discrete subset of N^*, and in the third case, $x \in (W \cap \mathrm{cl}\, X)^*$. Since $W \cap \mathrm{cl}\, X$ is a countable discrete subset of $\mathrm{cl}\, X$, $\tau(x, W)=\tau(x, W \cap \mathrm{cl}\, X)$ is in $\tau[x, \mathrm{cl}\, X]$. Thus $\tau[x, N^*] =\tau[x, \mathrm{cl}\, X]=\{t_1 \circ t_2 \circ \cdots \circ t_k \mid 1 \leqq k \leqq n\}$.

5. A point with c relative types.

By

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

we will mean that A c.d.$\subset \beta N$, that B, $C \subset A^*$, and $\mathrm{cl}\, B \cap \mathrm{cl}\, C = \varnothing$. When $C = \varnothing$, we will merely write $A \to B$. The diagram

$$\begin{array}{l}
N \to A(1) \to B(0) \to B(1) \to \{x_1\} \\
\qquad \downarrow \\
\qquad A(2) \to B(0,0) \to B(0,1) \to B(1,0) \to B(1,1) \to \{x_2\} \\
\qquad \downarrow \\
\qquad \vdots \\
\qquad \downarrow \\
\qquad A(n) \to B(\overbrace{0,\dots,0}^{n}) \to B(0,\dots,1) \to \cdots \to B(1,\dots,1) \to \{x_n\} \\
\qquad \downarrow \\
\qquad \vdots
\end{array}$$

can be constructed in the following way.

Let $A(1)$ be a countable discrete subset of N^*. $A(1)$ can be written as the union of two disjoint infinite subsets K_1 and K_2. Let $A(2)$ c.d.$\subset K_1^*$, $B(0)$ c.d.$\subset K_2^*$, $B(1)$ c.d.$\subset B(0)^*$ and $x_1 \in B(1)^*$. (This is the construction of the first line.)

If $k > 1$, and the set $A(k)$ has been defined, let K_1 and K_2 be infinite disjoint sets whose union is $A(k)$. Let

$$A(k+1) \text{ c.d.} \subset K_1^* \quad \text{and} \quad B(\overbrace{0,\dots,0}^{k}) \text{ c.d.} \subset K_2^*.$$

For each sequence of k zeros and ones, let $B(\alpha_1, \alpha_2, \dots, \alpha_k)$ c.d.$\subset B(\beta_1, \beta_2, \dots, \beta_k)^*$ if $(\alpha_1, \alpha_2, \dots, \alpha_k) > (\beta_1, \beta_2, \dots, \beta_k)$ and $x_k \in B(1, 1, \dots, 1)^*$. ($(\alpha_1, \alpha_2, \dots, \alpha_k) > (\beta_1, \beta_2, \dots, \beta_k)$ if for the first i where $\alpha_i \neq \beta_i$, $\alpha_i > \beta_i$.)

Let $X = \{x_n \mid n \geqq 1\}$ and for each infinite sequence $\{\alpha_n\}$ of zeros and ones, let $B\{\alpha_n\} = \bigcup \{B(\alpha_1, \dots, \alpha_i) \mid i \geqq 1\}$. It follows that X c.d.$\subset N^*$ and $B\{\alpha_n\}$ c.d.$\subset N^*$ for each sequence $\{\alpha_n\}$. Since $x_n \in B(\alpha_1, \dots, \alpha_n)^*$ for each finite sequence of length n, it follows that $x \in B\{\alpha_i\}^*$ for each $x \in X^*$ and each sequence $\{\alpha_i\}$. Let $x_0 \in X^*$. We will show that x_0 has c relative types by proving $\tau(x_0, B\{\alpha_i\}) \neq \tau(x_0, B\{\beta_i\})$ if $\{\alpha_i\} \neq \{\beta_i\}$.

Let k be the first index such that $\alpha_k \neq \beta_k$ and assume $\alpha_k > \beta_k$. Let

$$Y = \bigcup \{B(\alpha_1, \dots, \alpha_n) \mid n \geqq k\} \subset B\{\alpha_i\}.$$

Since $x_n \in B(\alpha_1, \dots, \alpha_n)^*$ for $n \geqq k$, it follows that $x_0 \in Y^*$ and by 1.1(a), $Y \subset B\{\alpha_i\}$ implies that $\tau(x_0, Y) = \tau(x_0, B\{\alpha_i\})$. From the construction, Y c.d.$\subset B\{\beta_i\}^*$ and thus by Lemma 1.4, $\tau(x_0, Y) \neq \tau(x_0, B\{\beta_i\})$.

Since there are c sequences of zeros and ones, the point x_0 has at least c relative types. By Frolík's Theorem 1.2, it has at most c relative types and the proof is complete.

We might point out that while the continuum hypothesis seems necessary to obtain minimal types and thus points with only finitely many relative types, the continuum hypothesis is not necessary in the construction of this example. We do not know if there are points with exactly $\aleph_0$ relative types.

References

1. Z. Frolík, *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 87–91. MR **34** #3525.

2. ———, *Fixed points of maps of βN*, Bull. Amer. Math. Soc. **74** (1968), 187–191. MR **36** #5897.

3. ———, *Homogeneity problems for extremally disconnected spaces*, Comment. Math. Univ. Carolinae **8** (1967), 757–763.

4. K. Kunen, *On the compactification of the integers*, Notices Amer. Math. Soc. **17** (1970), 299. Abstract #70T-G7.

5. M. E. Rudin, *Types of ultrafilters*, Topology Seminar (Wisconsin, 1965), Ann. of Math. Studies, no. 60, Princeton Univ. Press, Princeton, N. J., 1966, pp. 147–151. MR **35** #7284.

6. W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419. MR **18**, 324.

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