REPAIRING EMBEDDINGS OF 3-CELLS WITH MONOTONE MAPS OF $E^3$ (1)

BY
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Abstract. If $S_1$ is a 2-sphere topologically embedded in Euclidean 3-space $E^3$ and $S_2$ is the unit sphere about the origin, then there may not be a homeomorphism of $E^3$ onto itself carrying $S_1$ onto $S_2$. We show here how to construct a map $f$ of $E^3$ onto itself such that $f|S_1$ is a homeomorphism of $S_1$ onto $S_2$, $f(E^3 - S_1) = E^3 - S_2$ and $f^{-1}(x)$ is a compact continuum for each point $x$ in $E^3$. Similar theorems are obtained for 3-cells and disks topologically embedded in $E^3$.

1. Introduction. In this paper we show that, for any 2-sphere $S$ wildly embedded in Euclidean 3-space $E^3$, there is a monotone upper semicontinuous decomposition $G$ of $E^3$ whose nondegenerate elements miss $S$ such that $E^3/G$ is $E^3$ and $S$ is taken to a tame 2-sphere in $E^3/G$. If $X$ is a wildly embedded set in a 3-manifold $M^3$, we will say that the embedding of $X$ can be repaired (see [1]) if there exists a monotone upper semicontinuous decomposition $G$ of $M^3$ such that each non-degenerate element of $G$ is disjoint from $X$, $M^3/G = M^3$, and the image of $S$ under the natural projection of $M^3$ onto $M^3/G$ is tamely embedded in $M^3/G$. The main theorem of this paper, Theorem 1, says that any 3-cell in $E^3$ can be repaired. It follows as a corollary of this theorem and a theorem of Hosay [11] and Lininger [14] that any wild embedding of a 2-sphere can be repaired. Another corollary using recent results of Daverman and Eaton [8] is that any 2-cell in $E^3$ and many arcs in $E^3$ can be repaired. In §3, we construct a decomposition of the complement of a 3-cell in $S^3$. It is a kind of triangulation respecting wild embeddings which is difficult to state as a theorem. Therefore, we have been content just with giving a loose description of the decomposition and then proceeding with the construction.

The notation and terminology is largely standard. A cube-with-handles is a space homeomorphic to a regular neighborhood in the 3-sphere $S^3$ of a finite 1-complex and a cube-with-holes is a space homeomorphic to the closure of the complement of a cube-with-handles in $S^3$. The distance between two points $x$ and $y$ in any

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metric space under consideration will be denoted by \( \rho(x, y) \) and \( N(A, r) \) will denote the set of all points \( x \) such that \( \rho(x, A) < r \). If \( \sigma \) is a simplex in a space \( X \) with triangulation \( T \), we will use \( St(\sigma) \) to denote the point set interior in \( X \) of the star of \( \sigma \) in the triangulation \( T \). The \( j \)-skeleton of a triangulation \( T_j \) will be denoted by \( T_j^\bullet \). A Sierpinski curve is the space obtained from a 2-sphere \( S \) by deleting the interiors of a null sequence of mutually disjoint disks in \( S \) whose union is dense in \( S \). If \( X \) is a Sierpinski curve in \( S \) obtained by removing the interiors of the disks \( \{D_i\} \), then the accessible part of \( X \) is the set \( \bigcup \text{Bd} D_i \), and the inaccessible part of \( X \), here denoted by \( \text{Inacc}(X) \), is the set of all points of \( X \) which do not lie in the accessible part of \( X \). We have frequently abbreviated piecewise linear to pwl.

2. Some preliminary lemmas. Lemma 3 below is needed in the construction in §3. Lemma 1 can be proved as in Theorem 4.1 of [3].

**Lemma 1.** Let \( D \) be a disk, \( X \) a Sierpinski curve lying in a 2-sphere \( S \), \( D \cap X = (\text{Bd} \ D) \cap X = \emptyset \), \( A \) an arc lying in the inaccessible part of \( X \). Then there is a null sequence of mutually disjoint disks \( E_1, E_2, E_3, \ldots \) on \( D - A \) such that \( D \cap S \subset A \cup (\bigcup E_i) \) such that, for any \( \epsilon > 0 \) and any point \( p \in A \), there is a neighborhood \( N \) of \( p \) in \( D \) so that only disks \( E_i \) of diameter \( < \epsilon \) intersect \( N \).

**Lemma 2.** Let \( \epsilon > 0 \). Let \( C \) be a wild cell in \( E^3 \), \( X \) a tame Sierpinski curve in \( \text{Bd} C \), and \( S \) a 2-sphere. Suppose that \( G: S \times [0, 1] \rightarrow E^3 \) is a homeomorphism which is locally piecewise linear mod \( S \times 0 \), \( G(S \times 0, 1] \) lies in the unbounded complementary domain of \( G(S \times 0) \), \( G(S \times 0) \cap \text{Bd} C = X \), and \( G(S \times 0) \) is tame. Let \( T_1 \) and \( T_2 \) be triangulations of \( S \) such that \( T_2 \) refines \( T_1 \) and \( G(T_2 \times 0) \) lies in the inaccessible part of \( X \). Then, for some integer \( \xi \), there is a homeomorphism \( H \) from \( S \times [0, 1/2^\xi] \) into \( E^3 \), which is locally piecewise linear mod \( S \times 0 \), such that

1. \( \rho(H(x, t), \ G(x, t)) < \epsilon \) for all \( x \in S \) and all \( t \in [0, 1/2^\xi] \),
2. for all \( v \in T_2^\circ \), \( H(v \times (0, 1/2^\xi)) \cap C = \emptyset \),
3. for all \( \sigma \in T_2^\circ \) and \( n = 0, 1, 2, 3, \ldots \), \( G(\sigma \times 1/2^{\xi+n}) \cap C = \emptyset \),
4. for all \( x \in S \), \( H(x, 0) = G(x, 0) \),
5. if \( G \) has properties (2) and (3) with respect to \( T_1 \), then \( H(x, t) = G(x, t) \) for all \( (x, t) \in T_1^\circ \times (0, 1/2^\xi) \) and \( H(T_1^\circ \times (0, 1/2^\xi)) = G(T_1^\circ \times (0, 1/2^\xi)) \).

**Proof.** First we obtain condition (2). Let \( v_1, v_2, v_3, \ldots, v_k, v_{k+1}, \ldots, v_l \) be the vertices of \( T_2 \) with \( v_1, v_2, \ldots, v_k \) being those vertices for which \( G(v_i \times (0, 1]) \cap C = \emptyset \). We will show how to adjust \( G \) so that \( G(v_{k+1} \times (0, 1/2^\xi]) \cap C = \emptyset \) for some nonnegative integer \( \eta \), so that \( G(v_i \times (0, 1]) \cap C = \emptyset \), \( i = 1, 2, \ldots, k \), and so that, if \( T_1^\circ \subset \{v_1, v_2, \ldots, v_k\} \), then \( G|T_1^\circ \times (0, 1] \) is left unaltered.

To do this, let \( v = v_{k+1} \) and suppose that \( \sigma \) and \( \tau \) are 1-simplexes in \( T_2^\circ \) such that \( \sigma \cap \tau = v \). Let \( A = \sigma \cup \tau \) and \( D \) be the disk \( G(A \times [0, 1]) \). By Lemma 1, there is a null sequence \( E_1, E_2, E_3, \ldots \) of mutually disjoint disks in \( D \) such that \( D \cap C \subset G(A \times 0) \cup (\bigcup E_i) \), and, for each \( i = 1, 2, 3, \ldots \), \( G(A \times 0) \cap E_i = \emptyset \). Let \( \alpha \) be a
is a single point \( p \). Let \( \beta \) be the subarc of \( G(v \times [0, 1]) \) joining \( G(v \times 0) \) and \( p \). If \( D' \) is the disk in \( D \) bounded by \( \alpha \cup G(A \times 0) \), then there is a homeomorphism \( f \) of \( D' \) onto itself, which is locally piecewise linear mod \( \alpha \), fixed on \( \text{Bd} \ D' \), so that \( f(\beta) \cap (\bigcup E_i) = \emptyset \). By extending \( f \) piecewise linearly in a sufficiently close neighborhood \( N \) of \( \text{Int} \ D' \) so that \( f \) is fixed on \( \text{Bd} \ N \) and then extending this map to all of \( E^3 \) by the identity, we obtain a map \( f \) such that \( f \circ g \) satisfies requirement (2) for some integer \( \xi \) and the vertex \( v_{k+1} \). Similarly, we alter \( G \) near each of the vertices \( v_{k+2}, v_{k+3}, \ldots, v_1 \) to obtain a homeomorphism \( G_1 \) of \( S \times [0, 1] \), locally piecewise linear mod \( S \times 0 \), such that \( G_1|S \times 0 = G|S \times 0 \) and, for all \( v \in T_2^n \) and some sufficiently large integer \( \eta \), \( G_1(v \times (0, 1/2^n)) \cap C = \emptyset \).

Adjusting \( G_1 \) to obtain a homeomorphism \( G_2 \) satisfying conditions (2) and (3) is similar. Let \( \sigma \in T_2^n \) with \( \text{Int} \sigma \cap T_1^\perp = \emptyset \) if \( G_1 \) satisfies (2) and (3) (replacing \( T_2 \)) with \( T_1 \) and \( H \) with \( G_1 \). Note that we may suppose that \( G_1 \) satisfies (2) and (3) if \( G \) does. Let \( \{v, v'\} = \text{Bd} \sigma \). Lemma 1 can be used to obtain a sequence of “horizontal” arcs in \( G_1(\sigma \times [0, 1/2^n]) \) spanning from \( G_1(\sigma \times [0, 1/2^n]) \) to \( G_1(v' \times [0, 1/2^n]) \) and converging to \( G_1(\sigma \times 0) \) and “vertical” arcs from \( G_1(\sigma \times 0) \) to the interiors of the horizontal spanning arcs. By a suitable choice of these arcs, it is possible to define \( G_2 \) on \( \sigma \times 1/2^r \) for some \( r \geq \eta \) and all \( n = 0, 1, 2, \ldots \) in such a way that it extends \( G_2|S \times 0 = G_1|S \times 0 \). The “vertical” arcs are used to make the “horizontal” arcs converge on \( G_2(S \times 0) = G_1(S \times 0) \) homeomorphically and together they decompose \( G_1(\sigma \times [0, 1/2^n]) \) into disks so that \( G_2 \) can then be extended to take all of \( \sigma \times [0, 1/2^n] \) onto \( G_1(\sigma \times [0, 1/2^n]) \). Doing this for each \( \sigma \in T_2^n \), we then have

\[
G_2: T_2^n \times [0, 1/2^n] \rightarrow G_1(T_2^n \times [0, 1/2^n])
\]

such that \( G_2|T_2^n \times 0 = G|T_1^n \times 0 \) and, for some \( \nu \geq \eta \), \( G_2(T_2^n \times 1/2^r + \nu) \cap C = \emptyset \) for each \( n = 0, 1, 2, \ldots \). Furthermore, \( G_2 \) can be taken to be locally piecewise linear mod \( T_2^n \times 0 \). Let \( G_2|S \times 0 = G_1|S \times 0 \) and \( G_2|S \times 1/2^n = G_1|S \times 1/2^n \). Then \( G_2 \) is defined on the boundary of each cell \( \tau \times [0, 1/2^n] \), \( \tau \in T_2^n \), and can be extended to take this cell into \( G_1(\tau \times [0, 1/2^n]) \) so that \( G_2 \) satisfies all the conditions of the lemma except possibly (1). Condition (1) is met by using the fact that \( G_2(x, 0) = G(x, 0) \) for all \( x \in S \) and choosing \( \xi \geq \nu \geq \eta \). For this choice of \( \xi \), we set \( H = G_2|S \times [0, 1/2^n] \). This completes the proof of Lemma 2.

The following lemma is a modification of Lemma 2 of a paper by D. R. McMillan, Jr. [15].

**Lemma 3.** Let \( C \) be a 3-cell and \( h: C \rightarrow E^3 \) a homeomorphism. There is a monotone decreasing sequence \( \{\zeta_n\}, 0 < \zeta_n \leq 1/n, \) and for each \( n, \) a pwl homeomorphism

\[
H_n: \text{Bd} \ C \times [-\zeta_n, \zeta_n] \rightarrow E^3
\]
with the following properties:

(i) \( \rho(h(x), H_n(x,t)) < 1/n \), for all \( x \in \text{Bd } C \) and \( t \in [-\zeta_n, \zeta_n] \),

(ii) \( H_n(\text{Bd } C, -\zeta_n) \subset \text{Int } h(C) \),

(iii) \( h(C) \cap H_n(\text{Bd } C, \zeta_n) \) is covered by the interiors of a finite disjoint collection of 2-cells in \( H_n(\text{Bd } C, \zeta_n) \) each of diameter less than \( 1/n \),

(iv) for all \( n \), there exists a finite disjoint collection of topological 3-cells \( C^n_1, C^n_2, \ldots, C^n_k \) in \( h(C) \) such that \( C^n_i \) has diameter less than \( 1/n \) and meets \( h(\text{Bd } C) \) precisely in a 2-cell such that \( h(\text{Bd } C) - H_n(\text{Bd } C \times [-\zeta_n, \zeta_n]) \) is covered by the interiors of these 2-cells and such that

\[ \text{Bd } C^n_i - \text{Int } (C^n_i \cap h(\text{Bd } C)) \subset H_n(\text{Bd } C \times [-\zeta_n, \zeta_n]). \]

Furthermore, there is a sequence of triangulations \( T_1, T_2, \ldots \) of \( \text{Bd } C \) such that mesh \( h(T_n) < 1/n \), \( h(T^n_1) \) is a tame finite graph and \( T_{n+1} \) refines \( T_n \); there is a sequence of homeomorphisms \( G^n : \text{Bd } C \times [-\zeta_n, \zeta_n] \to E^3 \) which are locally piecewise linear mod \( \text{Bd } C \times 0 \) satisfying the following properties:

1. \( \rho(h(x), G^n(x,t)) < 1/n \), for all \( x \in \text{Bd } C \), \( t \in [-\zeta_n, \zeta_n] \),

2. \( G^n(\text{Bd } C \times \zeta_n) = H_n(\text{Bd } C \times \zeta_n) \),

3. \( G^n \) is a tame Sierpinski curve in \( A(\text{Bd } C) \) such that \( h(\text{Bd } C) \cap \text{Int } (C^n_i \cap h(\text{Bd } C)) \subset H_n(\text{Bd } C \times [-\zeta_n, \zeta_n]) \),

4. \( G^n(x, 0) = h(x) \), for any \( x \in T^n_1 \),

5. \( G^n(T^n_i \times \zeta_n) \cap h(C) = \emptyset \), for all \( i \in \mathbb{N} \),

6. \( G^n(T^n_1 \times \zeta_n) \cap h(C) = \emptyset \), for all \( i \geq n \),

7. for each \( n \) and each \( n' \), \( n' = 1, 2, \ldots, n-1 \),

\[ G^n(T^n_1 \times [0, \zeta_n]) = G^n(T^n_1 \times [0, \zeta_n]), \]

8. for each \( n \) and each \( n' \), \( n = 1, 2, \ldots, n-1 \), each component of

\[ G^n(\text{Bd } C \times [-\zeta_n, \zeta_n]) \cap G^n(\text{T}^{n'}_1 \times (\zeta_n, \zeta_n')) \]

is a closed set missing

\[ G^n(\text{T}^{n'}_1 \times \zeta_n) \cup G^n(\text{T}^{n'}_n \times \zeta_n) \cup G^n(\text{T}^{n'}_n \times (\zeta_n, \zeta_n')). \]

**Proof.** Step 1. Construction of \( G_1, H_1, T_1 \). Let \( \varepsilon = 1 \) and let \( \delta \) be a positive number such that for any homeomorphism \( g : \text{Bd } C \to E^3 \) differing from \( h|\text{Bd } C \) by less than \( \delta \) and for any compact set \( Y \) in \( g(\text{Bd } C) \) whose components have diameter less than \( \delta \), then there is a finite collection of \( \varepsilon \)-disks in \( g(\text{Bd } C) \) such that \( Y \) lies in the union of the interiors of these disks. Let \( T^n_1 \) be a triangulation of \( \text{Bd } C \) such that \( h(T^n_1) \) has mesh less than \( \varepsilon \) and \( h(T^n_1) \) is tame [2]. Let \( X_1 \) be a tame Sierpinski curve in \( h(\text{Bd } C) \) such that \( h(T^n_1) \cap \text{Inacc } (X_1) \) and the diameter of each component of \( h(\text{Bd } C) - X_1 \) is less than \( \delta \) [6, Theorem 9.1]. Let \( g_1 : \text{Bd } C \to E^3 \) be a homeomorphism obtained by pushing \( h(\text{Bd } C) - X_1 \) slightly into \( h(C) \) so that \( g_1 \) is locally pwl mod \( h^{-1}(X_1) \), differs from \( h \) by less than \( \delta \), \( g_1|h^{-1}(X_1) = h|h^{-1}(X_1) \), and the closures of components of \( h(C) - g_1(\text{Bd } C) \) form a null sequence of 3-cells \( C^1_1, C^2_1, C^3_1, \ldots \).
Since \( g_1(Bd C) \) is locally tame mod a tame Sierpinski curve, it is tame [6, Theorem 8.2]. It follows from the tameness of \( g_1(Bd C) \) and Theorem 2 of [5] that there is a homeomorphism \( G_1: Bd C \times [-1, 1] \rightarrow E^3 \) which is locally pwl mod \( Bd C \times 0 \) satisfying \( G_1(x, 0) = g_1(x) \) for all \( x \in Bd C \), \( G_1(Bd C \times -1) \subset \text{Int } h(C) \), and condition (1). By Lemma 2, we may suppose that \( G_1 \) satisfies conditions (5) and (6). Take \( H_1 \) to be a sufficiently close pwl approximation to \( G_1 \) using Theorem 3 of [5] in order to obtain conditions (2) and (3). There is a \( k \) such that \( C_1^1, C_2^1, \ldots, C_k^1 \) are the only cells of the null sequence not lying in \( H_1(Bd C \times (-1, 1)) \) and these cells are the ones of condition (iv). By our choice of \( \delta \), \( H_1 \) satisfies condition (iii).

**Step 2. Construction of \( G_n, H_n, T_n \).** Choose \( \delta \) as in Step 1, but with \( \epsilon = 1/n \). Choose a Sierpinski curve \( X_n \) by adding on to \( X_{n-1} \) in the following way. Let \( D_1, D_2, \ldots, D_n \) be those component disks of \( h(Bd C) \setminus \text{Inacc} (X_{n-1}) \) such that the diameter \( D_i \geq \delta \) or \( \rho(x, G_{n-1}(x, 0)) \geq \delta \) for some \( x \in D_i \). We add these disks back on to \( X_{n-1} \) and remove a null sequence of disks from their interiors to obtain \( X_n \) such that components of \( h(C) - X_n \) have diameter \(<\delta \). Let \( T_n \) be a triangulation of \( Bd C \) such that \( h(T_n^2) \) is a finite graph in the inaccessible part of \( X_n, T_n \) refines \( T_{n-1} \), and mesh \( h(T_n^2) < 1/n \) [2], [6].

We obtain \( g_n \) as we did \( g_1 \), but in a more careful way to get \( g_n: Bd C \rightarrow h(C) \) such that \( g_n((Bd C - h^{-1}((-1, 1) \cup \text{Int } D_i))) = G_{n-1}((Bd C - h^{-1}((-1, 1) \cup \text{Int } D_i))) \times 0 \) by pushing the little disks in \( \bigcup D_i \) into \( \text{Int } h(C) \) but not so far as \( D_i \) was pushed by \( G_{n-1} \) on \( Bd C \times 0 \) nor as far as \( \delta \). Thus \( \rho(g_n(x), h(x)) < \delta \) and \( g_n(h^{-1}(D_i)) \cup G_{n-1}(h^{-1}(D_i)) \times 0 \) bounds a little cell \( C_i^1 \) containing \( g_n(h^{-1}(D_i)) \). Let \( N \) be a neighborhood of \( h(T_{n-1}^2) \) in \( E^3 \) such that \( (C\text{l } N) \cap (\bigcup C_i^1) = \emptyset \). Let \( N_1 \) be a neighborhood of \( \bigcup C_i^1 \) missing \( \text{Cl } N \). We take a space homeomorphism \( f \) fixed outside \( N \) which moves

\[
G_{n-1}(Bd C \times 0)
\]

onto \( g_n(Bd C) \) as follows: The \( C_i^1 \)'s are tame [6, Theorem 8.2], so fatten the \( C_i^1 \)'s in \( N_1 \) except at \( Bd D_i \)'s to form cells and move \( G_{n-1}(h^{-1}(D_i)) \times 0 \) onto \( g_n(h^{-1}(D_i)) \). We do this inside the fattened \( C_i^1 \)'s in such a way that \( f \) is fixed on \( h(Bd C) \setminus \bigcup \text{Int } D_i \) and on \( G_{n-1}((-1, 1) \cup \text{Int } D_i)) \times 0 \), and so that \( f \circ G_{n-1}(x, 0) = g_n(x) \) for all \( x \in Bd C \). Extend \( f \) to a homeomorphism of \( E^3 \) onto itself which is fixed outside of the fattened \( C_i^1 \)'s.

We obtain \( g_n \) from \( f \circ G_{n-1} \). Choose a power \( t_n \) of \( 1/2, 0 < t_n \leq \delta_{n-1} \), so small that \( G_{n-1}(T_{n-1}^1 \times [-t_n, t_n]) \subset N \) and, for all \( x \in Bd C, \rho(h(x), G_{n-1}(x, t)) < 1/n \). In order to get property (8), we also choose \( t_n \) so small that

\[
f \circ G_{n-1}(h^{-1}(\bigcup D_i) \times [-t_n, t_n]) \cap G_{n}(T_{n}^1 \times \xi_{n}/2^j) = \emptyset,
\]

for \( n' = 1, 2, \ldots, n - 1 \) and \( j = 0, 1, 2, \ldots \), and so small that

\[
f \circ G_{n-1}(h^{-1}(\bigcup D_i) \times [-t_n, t_n]) \cap G_{n}(v \times (0, \xi_{n}^j)) = \emptyset,
\]

for \( n' = 1, 2, \ldots, n - 1 \) and \( v \in T_{n}^0 \). These last two conditions can be met, because \( h(C) \) misses \( G_{n}(T_{n}^1 \times \xi_{n}/2^j) \) and \( G_{n}(v \times (0, \xi_{n}^j)) \), while \( f \circ G_{n-1}(h^{-1}(\bigcup D_i) \times 0) \)
= g_n(h^{-1}({\cup D_i})) \text{ lies in } h(C). \text{ Set } G'_n = f \circ G_{n-1} | \text{Bd} C \times [-t_n, t_n] \text{ and make it locally pwl mod Bd } C \times 0 \text{ without changing its values on }

\text{(Bd } C \times 0) \cup (T^1_{n-1} \times [-t_n, t_n])

by using Theorem 3 of [5] in such a manner as preserve property (8).

We use Lemma 2 to get \( G_n : \text{Bd } C \times [-t_n, t_n] \rightarrow E^3 \), for some power \( \zeta_n \) of \( \frac{1}{2} \), from \( G'_n \). In applying Lemma 2 we choose \( \epsilon \) sufficiently small to preserve properties (1) and (8). Lemma 2 gives us properties (5) and (6) without destroying (4) or (7).

We obtain \( H_n \) from \( G_n \) as we obtained \( H_1 \) from \( G_1 \) using Theorem 3 of [6]. This also gives properties (2) and (3).

3. A decomposition. Cube-with-holes decompositions. For convenience, we make the following definition: A cube-with-holes decomposition of a space \( X \) is a “triangulation” of \( X \) with cubes-with-holes replacing 3-simplexes and disks-with-handles replacing their 2-faces. In a cube-with-holes decomposition we will allow each cube-with-holes to have any finite number of faces, not just four as in a simplicial 3-complex. In this section we construct a cube-with-holes decomposition of the complement in \( S^3 \) of a wild 3-cell. In this case all but one cube-with-holes has five faces; the one has many more faces. Some of the disks-with-handles have four 1-faces, each a 1-simplex, whereas, others have three 1-faces.

Let \( C \) be a 3-cell and let \( h : C \rightarrow S^3 \) be a topological embedding of \( C \). We construct a sequence of triangulations \( T_1, T_2, \ldots \) of \( \text{Bd } C \) with mesh \( h(T_i) \rightarrow 0 \) as \( i \rightarrow \infty \) and, with \( T_{i+1} \) refining \( T_i \). Our decomposition of \( S^3 - h(C) \) is into small cubes-with-holes \( \Gamma_{\sigma,m} \) with \( \sigma \) being a 2-simplex of \( T_{m-1} \). For a fixed \( m, m \geq 2 \), the \( \Gamma_{\sigma,m} \) may be thought of as lying in a shell, \( S_m \), about \( h(C) \) and this shell, which is a 3-manifold with two boundary components (actually a cell-with-handles with a cell-with-handles containing \( h(C) \) removed from its interior), consists of

\[ \cup \{ \Gamma_{\sigma,m} : \sigma \in T^2_{m-1} \} \]

The shell \( S_{m+1} \) formed by \( \cup \{ \Gamma_{\sigma,m+1} : \sigma \in T^2_m \} \) is the next shell in toward \( h(C) \) from the one formed by \( \cup \{ \Gamma_{\sigma,m} : \sigma \in T^2_{m-1} \} \), and \( S_{m+1} \cap S_m \) is the outer boundary of \( S_{m+1} \) and the inner boundary of \( S_m \). \( \Gamma_{\sigma,m} \), for \( m = 1 \), is a single cube-with-holes \( \Gamma_1 \), with \( \Gamma_1 \) being the closure of the complementary domain of \( S_2 = \cup \Gamma_{\sigma,2} \) not containing \( h(C) \). Schematically the situation is shown in Figure 1.

Each shell \( S_m \) is a thickened sphere or hollow ball with holes and handles (see Figure 2). The shell’s two surfaces are divided into disks-with-handles in the same pattern into which \( h(T^2_{m-1}) \) divides \( \text{Bd } h(C) \); the shell itself is like a Cartesian product of a sphere and an interval with handles added to the “outer” boundary and removed from the “inner” boundary so that the shell lies in \( S^3 - h(C) \). Figure 3 shows what a cross-section of such a shell might look like. It is a union of cubes-with-holes, each having five faces, with each face being a disk-with-handles. Any two of the cubes-with-holes intersect along a disk-with-handles face or a
1-simplex in the boundary of such a face or not at all. If $\sigma$ is a 2-simplex of $T_{m-1}$, then $\Gamma_{e,m}$ is the cube-with-holes “above” $h(\sigma)$ in the $m$th shell $S_m$.

First we construct a sequence of triangulations $T_1, T_2, \ldots$ of $\text{Bd } C$ and a sequence of cubes-with-handles $M_1, M_2, \ldots$ converging to $h(C)$ such that $M_m = L_m \cup (\bigcup_{i=1}^{k_m} H^n_i)$, $L_m$ is a tame 3-cell and each $H^n_i$, $i = 1, 2, \ldots, k_m$, is a small cube-with-handles such that $L_m \cap H^n_i = F^n_i$ is a disk in $\text{Bd } L_m \cap \text{Bd } H^n_i$. On each $L_m$ we will
put a copy of $T^4_{m-1}$ which will have a collar running down to a copy of $T^4_{m+1}$ on $L_{m+1}$ dividing up the space between $Bd M_m$ and $Bd L_{m+1}$. By adjusting this collar it will miss $Bd M_{m+1}$ except in the copy of $T^4_{m-1}$ on $Bd L_{m+1}$ and divide up the space between $Bd M_m$ and $Bd M_{m+1}$ into cubes-with-holes in such a fashion that any two will intersect along a common disk-with-handles in their boundaries or along an arc in the boundary of such a disk-with-handles, or not at all. In this fashion we get a cube-with-holes decomposition of $S^3 - h(C)$. Each cube-with-holes, $\Gamma_{\sigma,m}$, will be named by the triangulation $T_m - 1$ and the 2-simplex $\sigma \in T_m - 1$ associated with it by a map $G_{\sigma_m}$, given by Lemma 3. The size of $\Gamma_{\sigma,m} \to 0$ as $m \to \infty$ and if $\sigma_1, \sigma_2, \ldots$ is a sequence of 2-simplexes with $\sigma_i \in T^2$ and $\sigma_{i+1}$ lying in $\sigma_i$, then $\Gamma_{\sigma_{m+1}} \to \Gamma_{\sigma_m}$ as $m \to \infty$.

**Construction of $M_1$.** Consider $G_1: Bd C \times [-\zeta_1, \zeta_1] \to E^3$ and $T^1$ from Lemma 3. Choose $e_1$ as follows:

(a) $e_1 < \eta_1$, where $\eta_1$ is less than $\frac{1}{3}$ the distance from $G_1(v \times [0, \zeta_1])$ to

$$G_1((T^1 - \text{St}(v)) \times [0, \zeta_1])$$

for each $v \in T^0_1$.

(b) $e_1$ less than $\frac{1}{3}$ the distance from $G_1(\sigma \times [0, \zeta_1]) - N(G_1(Bd \sigma \times [0, \zeta_1]), \eta_1)$ to $G_1((T^1 - \text{Int} \sigma) \times [0, \zeta_1])$ for every $\sigma \in T^1_1$.

Condition (a) says any $e_1$-set intersecting $N(G_1(v \times [0, \zeta_1]), \eta_1)$ cannot intersect $G_1(T^1_1 \times [0, \zeta_1])$ outside $G_1(\text{St}(v) \times [0, \zeta_1])$. Condition (b) says any $e_1$-set intersecting $G_1(\sigma \times [0, \zeta_1])$ but not $N(G_1(Bd \sigma \times [0, \zeta_1]), \eta_1)$ cannot intersect $G_1(T^1_1 \times [0, \zeta_1])$ except in $G_1(\text{Int}(\sigma) \times [0, \zeta_1])$. Together, these conditions imply that any $e_1$-set intersecting $G_1(T^1_1 \times [0, \zeta_1])$ lies in $G_1((\text{St}(\sigma^0) \cap T^1_1) \times [0, \zeta_1])$ for some $\sigma^0 \in T^0_1$—that is, any $e_1$-set intersecting $G_1(T^1_1 \times [0, \zeta_1])$ intersects it only in fins radiating from one post $G_1(\sigma^0 \times [0, \zeta_1])$. 

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Following McMillan’s Theorem 1 of [15] and using Lemma 3, we find an integer \( n_1 \) such that \( 1/n_1 < \delta_1/3 \), where \( \delta_1 \) is a positive number chosen as McMillan does \( \delta \) in his Theorem 1 for \( \varepsilon = \varepsilon_1 \). We use \( H_{n_1} \), as given by Lemma 3 above, for his \( H \) in his Theorem 1. This gives a cube-with-handles \( M_1 = L_1 \sqcup (\bigcup_{i=1}^{k_1} H^1_i) \), where \( L_1 \) is a cube with \( \text{Bd} \ L_1 \) \( \varepsilon_1 \)-homeomorphic to \( h(\text{Bd} \ C) \) and each \( H^1_i \) is an \( \varepsilon_1 \)-cube-with-handles for each \( i = 1, 2, \ldots, k_1 \); \( h(C) \) lies in \( \text{Int} \ M_1 \). By Lemma 3, condition (7),
\[
G_{n_1}(T^1_1 \times [0, \zeta_{n_1}]) = G_1(T^1_1 \times [0, \zeta_{n_1}]) \subset G_1(T^1_1 \times [0, \zeta_1])
\]
so that the \( \varepsilon_1 \)-set intersection properties prescribed by conditions (a) and (b) above for \( G_1 \) and \( T_1 \) hold also for \( G_{n_1} \) and \( T_1 \).

**Construction of \( M_m \).** We assume \( M_{m-1} \) is already constructed. We construct \( M_m \) just as \( M_1 \) but with additional restrictions on the closeness of \( M_m \) to \( h(C) \). Choose \( \varepsilon_m \) as follows:

(a) \( \varepsilon_m < \eta_m \), where \( \eta_m \) is less than \( \frac{1}{2} \) the distance from \( G_m(v \times [0, \zeta_m]) \) to
\[
G_m(\{T^1_m - \text{St}(v)\} \times [0, \zeta_m])
\]
for each \( v \in T^*_m \).

(b) \( \varepsilon_m \) less than \( \frac{1}{2} \) the distance from \( G_m(\sigma \times [0, \zeta_m]) - N(G_m(\text{Bd} \ \sigma \times [0, \zeta_m], \eta_m)) \) to
\[
G_m(\{T^1_m - \text{Int} \ \sigma\} \times [0, \zeta_m])
\]
for every \( \sigma \in T^*_m \).

Conditions (a) and (b) insure that any \( \varepsilon_m \)-set intersecting \( G_m(T^1_m \times [0, \zeta_m]) \) lies in \( G_m(\text{St}(\sigma^0) \times [0, \zeta_m]) \) for some \( \sigma^0 \in T^0_m \).

(c) \( \varepsilon_m < \rho(h(C), \text{Bd} \ M_{m-1}) \).

(d) \( \varepsilon_m < 1/m \).

(e) \( \varepsilon_m < \varepsilon_{m-1} \).

We choose an integer \( n_m > n_{m-1} > \cdots > n_1 \) such that \( 1/n_m < \delta_m/3 \), where \( \delta_m \) is chosen as \( \delta \) in McMillan’s Theorem 1 for \( \varepsilon = \varepsilon_m \). We use \( H_{n_m} \) from Lemma 3 for his \( H \) in his Theorem 1. With this \( H \) his Theorem 1 gives \( M_m = L_m \sqcup (\bigcup_{i=1}^{k_m} H^m_i) \) with \( \text{Bd} \ L_m \) \( \varepsilon_m \)-homeomorphic to \( h(\text{Bd} \ C) \), \( L_m \) a cell in an \( \varepsilon_m \)-neighborhood of \( h(C) \), and each \( H^m_i, i = 1, 2, \ldots, k_m \), an \( \varepsilon_m \)-cube-with-handles. We also have
\[
h(C) \subset \text{Int} \ M_m \subset M_m \subset \text{Int} \ M_{m-1}.
\]
By Lemma 3, \( G_{n_m}(T^1_m \times [0, \zeta_{n_m}]) = G_n(T^1_n \times [0, \zeta_{n_m}]) \) so that conditions (a) and (b) tell us that any \( \varepsilon_m \)-set intersecting \( G_{n_m}(T^1_m \times [0, \zeta_{n_m}]) \) lies in \( G_{n_m}(\text{St}(\sigma^0) \times [0, \zeta_{n_m}]) \) for some \( \sigma^0 \in T^0_m \). According to McMillan’s theorem, \( H^m_i \cap L_m = F^m_i \) is a disk in \( \text{Bd} \ H^m_i \) and in \( \text{Bd} \ L_m \). The rest of the proof will consist mostly of simplifying intersections between the \( M_m \) and the “collars” \( G_{n_m-1}(T^1_{n-1} \times [0, \zeta_{n_m-1}]) \) of \( h(C) \) by altering the “collars”.

Before going on let us make the following simplification in notation. Rename \( G_{n_m} \) and \( \zeta_{n_m} \). We will use \( G_m \) instead of \( G_{n_m} \) and \( \zeta_m \) instead of \( \zeta_{n_m} \).

**Simplifying intersections with \( F^m_i \).** We would like to say that \( G_m(T^1_m \times \zeta_m) \) lies in \( L_m - \bigcup F^m_i \). To achieve this we must look at how McMillan arrives at the \( F^m_i \).
Each $H^m_l$ comes from a $W^m_l$, a polyhedral cube-with-handles such that each component of $W^m_l \cap L_m$ is a 2-cell in the common boundary of $W^m_l$ and of $L_m$. He finds an $\varepsilon_l/2$-cell, $F^m_l$, in $\partial L_m$ such that $W^m_l \cap L_m = F^m_l$. (No two of these $F^m_l$ intersect.) Then $H^m_l$ is obtained by adding to $W^m_l$ a cell obtained by thickening $F^m_l$ (in $L_m$). This pushes $\partial L_m$ into $L_m$ slightly so that $H^m_l \cap L_m = F^m_l$ (or rather the pushed-in $F^m_l$) and $H^m_l \cap L_m$ is a single 2-cell $F^m_l$.

With an $\varepsilon_m$-homeomorphism of $S^3$, we can adjust $G_m(T^1_m \times [0, \varepsilon_m])$ [before the assumption just prior to this section this set would have been written $G_m(T^1_m \times [0, \varepsilon_m])$] in a sufficiently small neighborhood of $F^m_l$ so that $G_m(T^1_m \times \varepsilon_m)$ lies in $L_m - \bigcup_{l=1}^{m-1} F^m_l$. This homeomorphism also adjusts $G_{m-1}(T^1_{m-1} \times [0, \varepsilon_{m-1}])$ so that $G_{m-1}(T^1_{m-1} \times \varepsilon_m) = G_m(T^1_{m-1} \times \varepsilon_m)$ lies in $L_m - \bigcup_l F^m_l$. In constructing this space homeomorphism we just take a homeomorphism of $\partial L_m$ onto itself which is fixed outside a small neighborhood of $F^m_l$ and slips $G_m(T^1_m \times \varepsilon_m)$ off $F^m_l$ and extend to a homeomorphism of $S^3$ onto itself that is also fixed outside a small neighborhood of $F^m_l$. These neighborhoods are to be so small that nothing is moved near any other $F^m_l$ and so small that nothing is moved near any other $\partial M_m$. The foregoing shows that we may assume that $G_m(T^1_m \times \varepsilon_m)$ lies in $L_m - \bigcup F^m_l$ and that $G_m(T^1_m \times \varepsilon_{m+1})$ lies in $L_{m+1} - \bigcup F^m_{m+1}$ and $F^m_l \cap G_{m}(T^1_m \times \varepsilon_{m+1}, \varepsilon_m) = \emptyset$.

If $\sigma^1$ is a 1-simplex of $T_{m-1}$, let us use the notation $\sigma^{1(m-1)}$ to denote the disk $G_{m-1}(\sigma^1 \times [\varepsilon_m, \varepsilon_{m-1}])$ and $T(m-1)$ to denote $G_{m-1}(T^1_{m-1} \times [\varepsilon_m, \varepsilon_{m-1}])$. $G_m$ (as adjusted) imposes a triangulation $Q_m$ on $\partial L_m$ such that $Q_m = \{G_m(\sigma \times \varepsilon_m) : \sigma \in T_{m-1}\}$. Each simplex of $Q_m$ is $(\varepsilon_m + 1/n_m)$-homeomorphic to its corresponding simplex of $T_{m-1}$. The 1-skeleton of $Q^1_m$ is a sub-polyhedron of $\partial L_m$. By construction, if $\sigma^2 \in Q^1_m$, then $\sigma^2 \cap G_m(T^1_{m-1} \times [0, \varepsilon_m])$ lies in $G_m(T^1_{m-1} \times \varepsilon_m)$ and in $\partial \sigma^2$. If $\sigma^1 \in T^1_{m-1}$, then, assuming general position, a component of $\sigma^2 \cap \sigma^1(m-1) = \sigma^2 \cap G_{m-1}(\sigma^1 \times [\varepsilon_m, \varepsilon_{m-1}])$ is a simple closed curve in $\text{Int} \sigma^2 \cap \text{Int} \sigma^1(m-1)$ or is possibly an arc lying in the boundary of each of the 2-cells or a point in the boundary of each. This is a consequence of condition (8) of Lemma 3.

By trading disks we can change $\sigma^1(m-1)$ so that $\sigma^1(m-1) \cap \sigma^2$ contains no simple closed curves. Then $\sigma^1(m-1) \cap \sigma^2$ is a common arc of boundary or a common point in the boundary of each. We do not adjust $\sigma^2$ for fear of uncovering $h(C)$. Suppose $\sigma_1, \sigma_2, \ldots, \sigma_l$ are the 2-simplexes of $Q_m$. First we adjust $T(m-1)$ so that it misses $\text{Int} \sigma_1$ as follows: Let $J$ be any simple closed curve in $\text{Int} \sigma_1 \cap T(m-1)$ that bounds a disk $D$ in $\text{Int} \sigma_1$ such that $\text{Int} D$ does not intersect $T(m-1)$. Replace the disk $J$ bounds in $T(m-1)$ by $D$ and push off $\text{Int} \sigma_1$. Proceeding in this manner $T(m-1)$ may be adjusted so that it misses $\text{Int} \sigma_1$. Note that the
adjusted \(T(m-1)\) (also denoted by \(T(m-1)\)) is homeomorphic to the \(T(m-1)\) we started with. We did not introduce any new self intersections. Now we adjust \(T(m-1)\) to miss \(\text{Int } \sigma_2\), then to miss \(\text{Int } \sigma_3\), and so on. Thus \(T(m-1)\) may be adjusted to miss \(L_m - Q_h^i\). Thus, we have that \(T(m-1) \cap L_m = Q_h^i\).

Before we calculate how much \(T(m-1)\) is moved by the process, let us point out one precaution we wish to make in the “pushing off” part of this disk trading procedure. Each \(H_i^n\) intersects \(L_m\) in a disk \(F_i^n\). From the point of view of the \(H_i^n\), the disk trading occurs only near the \(F_i^n\). By pushing \(F_i^n\) off itself in \(H_i^n\) we get a new disk disjoint from \(F_i^n\) having its boundary in \(\text{Bd } H_i^n - F_i^n\). This new disk together with the \(F_i^n\) and an annulus on \(\text{Bd } H_i^n\) bounds a cell \(K_i^n\) in \(H_i^n\). \(K_i^n\) may be thought of as a cylinder \(F_i^n \times [0, 1]\). In pushing off during the above disk trading procedure, we wish not to push anything into \(H_i^n - \text{Int } K_i^n\). When part of the disk to be pushed off lies in \(F_i^n\) and is to be pushed to the \(H_i^n\) side of \(L_m\), we wish to push along the lines perpendicular to \(F_i^n\) in the representation of \(K_i^n\) as \(F_i^n \times [0, 1]\). Thus, any new disks resulting from such disk trading will intersect \(H_i^n\) as shown in Figure 4. This will be convenient later.

Now to show that the disk trading does not enlarge \(G_m(T_m \times [\zeta_m + 1, \zeta_m]) = T(m)\) too much. Each \(T(m)\) has already had an \(\epsilon_m\)-adjustment to move it off the \(F_i^n\). Recall that \(\rho(h(x), G_m(x, t)) < 1/\eta_m\) for all \(x \in C\) and \(t \in [-\zeta_m, \zeta_m]\) and that \(H_m = G_m\) on \(\text{Bd } C \times [-\zeta_m, \zeta_m]\). This latter condition says \(G_m(\text{Bd } C \times \zeta_m) = L_m\). All this was

![Figure 4](https://www.ams.org/journal-terms-of-use)
true before the alteration of $G_m$ to push $G_m(T^1 \times \xi_m)$ off $F^m_1$. Now $G_m$ is an $(\epsilon_m + 1/n_m)$-homeomorphism of $\{ \bigcup T^1 \times [0, \xi_m] \} \cup \{ \bigcup_{n_m} \Bd C \times \xi_m \}$ instead of a $1/n_m$-homeomorphism. Since $\sigma^2 \in Q^2_m$ is $(\epsilon_m + 1/n_m)$-homeomorphic to some $\sigma \in T^2_{m-1}$, then mesh $Q_m$ is less than

$$\frac{1}{(m-1)} + 2\epsilon_m + 2/n_m.$$ 

Since $1/n_m < \delta_m/3$ and $\delta_m < \epsilon_m/2$ (see McMillan's Theorem 1 and the beginning of this construction) and $\epsilon_m < 1/m < 1/(m-1)$, then mesh $Q_m < 4/(m-1)$. Thus no point of $T(m-1)$ gets moved by more than $4/(m-1)$ in the disk trading procedure.

**Naming the cubes-with-holes.** Now let us do some naming. $T(m-1)$ separates the set $M_{m-1} - \Int L_m$ into little 3-manifolds with connected boundary. See Figure 5. We want to name these manifolds and their sides. Each 3-manifold with boundary is a cube-with-handles with a “top” (which is a disk-with-handles), 3 “sides” (disks) and a “bottom” (disk).

- $\alpha_{m-1} \subset \Bd M_{m-1}$. Let $\sigma \in T^2_{m-1}$. $\sigma$ corresponds to some set $\sigma_{m-1} \subset L_{m-1}$ under $G_{m-1}$, namely $G_{m-1}(\sigma \times \xi_{m-1})$. It does not correspond to an element of $Q_{m-1}$, for each such element is the image of elements of $T^2_{m-2}$ under $G_{m-1}$. However, each element of $Q_{m-1}$ is a union of such $\sigma_{m-1}$'s. Let $\alpha_{o,m}$ be the disk-with-handles obtained by replacing those $F_i^{-1}$ in $\sigma_{m-1}$ with $\Bd F_i^{-1}$. Let $\alpha_{o,m} \subset \Bd L_m$. Let $\sigma \in T^2_{m-1}$. Under $G_m$ there corresponds some $\sigma_m \in Q^2_m$, namely $G_m(\sigma \times \xi_m)$. Let $\beta_{o,m}$ be this $\sigma_m$.

- $\alpha_{o,m} \subset \Bd F_i^{-1}$. Let $\sigma_i \in T^1_{m-1}$, let $\Bd \sigma_i$ be $\sigma_i \cup \sigma_2 \cup \sigma_3$, where $\sigma_i \in T^1_{m-1}$, $i = 1, 2, 3$. Define $\gamma_{o,m} = \sigma_{m-1}$. Each of the $\beta_{o,m}$ and $\gamma_{o,m}$ ($i = 1, 2, 3$) is a disk. If any two among $\alpha_{o,m}, \beta_{o,m}$ and $\gamma_{o,m}$ intersect, it is along an arc of boundary. Recall that

$$T(m-1) = \bigcup \{ \gamma_{o,m} : \sigma^1 \in T^1_{m-1} \} = G_{m-1}(T^1_{m-1} \times [\xi_m, \xi_{m-1}]).$$

Then $\alpha_{o,m} \cup \beta_{o,m} \cup \gamma_{o,m} \cup \gamma_{o,m} \cup \gamma_{o,m}$ is a 2-manifold separating $S^3$. Denote

Vertical pieces are part of

$$\text{Front face is } \gamma_{o,m}$$

**Figure 5**
by $\Gamma_{\sigma,m}$ the closure of that component of $S^3$ minus this 2-manifold in $\text{Int} \ M_{m-1}$ (see Figure 6). Then $M_{m-1} - \text{Int} \ L_m = \bigcup \{\Gamma_{\sigma,m} : \sigma \in T_m^{m-1}\}$. See Figure 5. For if $p \in \text{Int} (M_{m-1} - \text{Int} \ L_m)$ take an arc $pq$ from $p$ to $\text{Bd} \ M_{m-1} \cup \text{Bd} \ L_m$ such that $\text{Int} (pq) \subseteq \text{Int} (M_{m-1} - \text{Int} \ L_m)$ and $pq \cap T(m-1) = \emptyset$. Then $q \in \alpha_{\sigma,m}$ or $\beta_{\sigma,m}$ for some $\sigma \in T_m^{m-1}$. Since $pq-q$ misses $\text{Bd} \ \Gamma_{\sigma,m}$, and points near $q$ on the other side of $\text{Bd} \ \Gamma_{\sigma,m}$ can be joined by an arc to $S^3 - M_{m-1}$ missing $\text{Bd} \ \Gamma_{\sigma,m}$, then $p \in \Gamma_{\sigma,m}$. Hence, $M_{m-1} - \text{Int} \ L_m \subseteq \bigcup \{\Gamma_{\sigma,m} : \sigma \in T_m^{m-1}\}$. The other inclusion is obvious.

We must now alter the $\gamma$ so that we can replace $\beta_{\tau,m}$ with the union of the appropriate $\alpha_{\tau,m+1}$'s—that is, replace disks on $\beta_{\sigma,m}$ with $\text{Bd} \ H_i^{m-1} - \text{Int} \ F_i^{m-1}$'s in the manner we did to make the $\alpha_{\sigma,m}$. To do this we adjust the $\gamma$ to miss the $H_i^n$. We cannot do this, however, while the $\gamma$ remain disks, so we add handles to the $\gamma$.

Simplifying intersections with $H_i^n$. Before we can adjust $T(m-1)$ so that no $H_i^n$ can intersect it, we must be sure that no handle of $H_i^n$ loops a "fence post" $G_{m-1}(v \times [\xi_m, \zeta_m])$, where $v$ is a vertex of $T_{m-1}$. In Figure 7, $H_i^n$ is shown as a torus growing out of Bd $L_m$. Bd $L_m$ is shown jutting up through two "walls" in $T(m-1)$. The walls are shown as they were adjusted to remove $T(m-1)$ from Bd $L_m$.

Let $N_i^n$ be a regular neighborhood in $\text{Cl} (S^3 - L_m)$ of $H_i^n$. We want the $N_i^n$ to be mutually disjoint, each $N_i^n$ to be an $e_m$-set, and $E^n_i = N_i^n \cap L_m$ to be a disk. We want each $E^n_i$, and hence each $N_i^n$, to miss $G_m(T_m^n \times [0, \xi_m])$. This can be done, because the $F_i^n$ have this property. In particular, we want each $E^n_i$ to miss the 1-skeleton of $Q_m$. We also want $N_i^n \subseteq \text{Int} \ M_{m-1}$.

We want to look at each $H_i^n$ as a fattened up wedge of simple closed curves $J_{r_i,m}$, $r_i=1,2,\ldots,R_{i,m}$, with $J_{r_i,m}$ in general position with respect to $\bigcup \gamma_{\sigma,m}$, with the wedge point $x_{i,m}$ in $\text{Int} F_i^n$, and with $J_{r_i,m} - x_{i,m} \subseteq \text{Int} H_i^n$. Furthermore, we
want to choose the $\gamma_i^m$ so that they intersect the disks $\gamma \cap K_i^m$ exactly twice (see Figure 4). Define a pseudo-isotopy $f_i^m: \mathbb{N}_i^m \times I \to \mathbb{N}_i^m$ such that

1. $f_i^m(x, 0) = x$, for all $x \in \mathbb{N}_i^m$,
2. $f_i^m(x, t) = x$, for all $x \in \gamma_i^m$, $r=1, 2, \ldots, R_i,m$, and for all $x \in \partial \mathbb{N}_i^m$,
3. $f_i^m((\mathbb{N}_i^m - (H_i^m - F_i^m)) \times t$ is a homeomorphism for all $t \in [0, 1]$,
4. $f_i^m(H_i^m, 1) = \bigcup \{J_i^m : r=1, 2, \ldots, R_i,m\} \cup F_i^m$,
5. $f_i^m(N_i^m, t) = N_i^m$ for all $t \in [0, 1]$.

The plan is to adjust the $\gamma_s,m$'s to miss $\bigcup J_i^m$ and use $f_i^m$ to push them off all of $H_i^m$. By choosing a stage $t$ of $f_i^m$ close enough to the end stage that $f_i^m(H_i^m, t)$ lies so close to $((\bigcup J_i^m) \cup F_i^m$ that it misses all the $\gamma_s,m$'s, too, we can use

$$(f_i^m \mid \mathbb{N}_i^m \times t)^{-1}$$

to push all the $\gamma_s,m$'s off $H_i^m$.

First, we make a few definitions. Consider each $J_i^m$ and each $\gamma_s,m$ as being oriented. Let $J$ be any $J_i^m$ and $\gamma$ any $\gamma_s,m$. Let $p(J)$ be the number of positive crossings of $J$ through $\gamma$ and let $n(J)$ be the number of negative crossings of $J$ through $\gamma$. Define the piercing number of $J$ with respect to $\gamma$ to be

$$p \# J = p(J) - n(J)$$

and the intersection number of $J$ with respect to $\gamma$ to be

$$I(J) = p(J) + n(J).$$

We will refer to an arc of boundary of a $\gamma_s,m$ spanning from $\partial L_{m-1}$ to $\partial L_m$ as a post. We will refer to $\gamma_s,m$ as a fin from each of its posts. We will assume that each $J_i^m$ is in general position with respect to $T(m-1)$ so that $J \cap T(m-1)$ is finite, misses all the posts of $T(m-1)$, and crosses at each point of intersection.
Let us look back to $T(m-1)$, which is the union of the $\gamma_{e,m}$'s. We made two adjustments to $T(m-1)$ to get the $\gamma_{o,m}$'s. Before the adjustments, each $H^n_m$ was an $e_m$-set so that, by our choice of $e_m$ at the very beginning of this proof, $H^n_m \cap T(m-1)$ lay in the union of the fins radiating from some post $P$. Thus $p \# J^{l,m}_m = 0$ with respect to all $\gamma'$ not radiating from this post $P$, because $I(J^{l,m}_m) = 0$ with respect to such $\gamma'$. The first adjustment (moving $G_{m-1}(T_{m-1}m_1 \times [0, \zeta_{m-1}])$ off the $F^n_m$ and $F^{n-1}_m$) does not change this. The next adjustment, the disk trading, did. It caused new $\gamma'$ to hit $H^n_m$ in $K^n_m$. But $p \# J$ with respect to $\gamma''$ is the linking number of $J$ and $\partial \gamma''$. Since $\partial \gamma''$ and $J$ were not moved in the disk trading procedure, $p \# J$ with respect to any $\gamma''$ which is not a fin of $P$ remained 0. Hence neither adjustment made $p \# J$ with respect to those $\gamma$ which are not fins from $P$ nonzero.

If $P$ is a post in $T(m-1)$, $\gamma$ and $\gamma'$ are fins from $P$, and $(H^n_m - K^n_m) \cap T(m-1)$ lies in the union of the fins from $P$, then, for any $J^{l,m}_m$ in $H^n_m$, $p \# J^{l,m}_m = 0$ with respect to $\gamma$ implies $p \# J^{l,m}_m = 0$ with respect to $\gamma'$. For consider the set

$$X = \bigcup \{ \Gamma_{e,m} : P \subset \Gamma_{e,m} \} \cup K^n_m.$$  

Then $\gamma \cup \gamma'$ separates $X$, $K^n_m$ lies in one component, and $J^{l,m}_m \subset X$. Since $J^{l,m}_m$ is a simple closed curve, it crosses $\gamma \cup \gamma'$ as many times in one direction as another. Since $p \# J^{l,m}_m = 0$ with respect to $\gamma$, it crosses $\gamma$ as many times in one direction as another. Thus it must cross $\gamma'$ as many times in one direction as another. Hence $p \# J^{l,m}_m = 0$ with respect to $\gamma'$. This shows, in fact, that any simple closed curve in

$$\bigcup \{ \Gamma_{e,m} : P \subset \Gamma_{e,m} \}$$

links $\partial \gamma$ iff it links $\partial \gamma'$.

We want $p \# J = 0$ with respect to all $\gamma$ making up $T(m-1)$. To accomplish this we must adjust $T(m-1)$. For each post $P$ we choose a fin $\gamma$ such that $\partial \gamma \subset \partial \gamma'$, and for each $J$ such that $p \# J \neq 0$ with respect to $\gamma$ and $J \subset H^n_m$ such that $(H^n_m - K^n_m) \cap T(m-1)$ lies in those fins radiating from $P$, we will make $I(J) = 0$ with respect to $\gamma$ by an adjustment of $T(m-1)$. The manner in which we do this says that $p \# J = 0$ with respect to all $\gamma$ radiating from $P$ and hence for all $\gamma$ making up $T(m-1)$.

We take a collection of disjoint polygonal arcs from $P$ minus its endpoints to points of $J \cap \gamma$ such that each arc misses $J'$ for all $J' \neq J$, each arc intersects $J$ at only one point, and each arc lies, except for its endpoint on $P$, in $\text{Int} \gamma$.

Choose a disjoint collection of neighborhoods $N_1, N_2, \ldots, N_k$ of the arcs joining $P$ to $J \cap \gamma$. We choose the $N_i$ so that none contains but one such arc, none contains a point of $J'$ for any $J' \neq J$, none gets outside of $\text{Int} (M_{m-1} - L_m)$, none intersects any post other than $P$, none intersects any $\gamma_{e,m}$ not radiating from $P$, none gets outside $\text{Int} (\bigcup \{ \Gamma_{e,m} : P \subset \Gamma_{e,m} \})$ and none gets outside the $e_m$-neighborhood of the arc it contains. With a homeomorphism of $S^3$, fixed outside $\bigcup_{i=1}^k N_i$, and taking $\gamma \cup \gamma'$ onto $\gamma \cup \gamma'$ (here $\gamma'$ is any other fin from $P$), we move $P$ so that all the arcs lie in $\gamma'$. We also want the new $\gamma'$ to contain all the old $\gamma'$. This homeomorphism adjusts $T(m-1)$ so that $I(J) = 0$ with respect to $\gamma$. 

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We do the above process to all the $J$ of the type under consideration intersecting that particular $\gamma$ so that no such $J$ has nonzero piercing number with respect to any fin from $P$. We do this for every post $P$. After doing this, no $J$ will loop any post $P$. In other words, $p \# J = 0$ with respect to any $\gamma$ making up $T(m-1)$ and for all $J^{i, m}$ for all $H^m_i$.

We are now in a position to alter the $\gamma$ so that they miss the $J^{i, m}$. We choose such a $J$ and show how it is done. Let $x_0$ be the point at which $J$ is attached to $L_m$. Suppose for the time being that $J \subset H^m_i$ and $K^{i, m}_i$ misses all the walls $\gamma$. Proceed along $J$ to the first point of intersection with a $\gamma$, say $\gamma_0$. Now proceed to the first point $q_0$ at which $J$ pierces this $\gamma_0$ in the opposite direction and back up to the last point $p_0$ at which $J$ pierced $\gamma_0$ in the original direction. We would like for the arc $p_0q_0$ to be disjoint from all the $\gamma$'s except for its endpoints $p_0$ and $q_0$. If not we connect $q_0$ to $p_0$ by an arc $A_0$ lying in $\gamma_0$ and push the resultant simple closed curve $J_0$ off $\gamma_0$. $J_0 \cap \gamma_0 = \emptyset$ so that $J_0$ does not link $Bd \gamma_0$ and hence does not link $Bd \gamma$ for any $\gamma$ radiating from $P$. Hence there are points $p_1, q_1$ of $p_0q_0$ lying in some fin $\gamma_1$ such that $p_1q_1 \cap \gamma_1 = \{p_1, q_1\}$. Continuing in this way we can find two points $p_n, q_n$ such that $p_nq_n$ is a subarc of $J$, $p_n$ and $q_n$ lie on some wall $\gamma_n$, $J$ pierces this wall $\gamma_n$ in different directions at $p_n$ and $q_n$, and $Int (p_nq_n)$ misses all the $\gamma$.

Take a small regular neighborhood $R$ of $p_nq_n$ in the $\Gamma_{s, m}$ containing $p_nq_n$, remove it from that $\Gamma_{s, m}$ and attach it to the $\Gamma_{s, m}$ which $J_n$ leaves at $p_n$ and enters at $q_n$. Replace the two disks of $R \cap \gamma_n$ with $Bd R$ minus the interiors of those disks to get a new $\gamma_n$ (the new $\gamma_n$ now has an oriented handle). The number of points at which $J$ hits the $\gamma$ is now reduced by two.

We must, of course, take $R$ sufficiently close to $p_nq_n$ so it misses all of $J - p_nq_n$ as well as all the other $J^{i, m}$ and lies inside the $H^m_i$ containing $J$. Note that the size of a $\gamma$ after a finite number of changes of this sort is not increased as much as $2e_m$.

Now consider the case in which $J \subset H^m_i$ and $K^{i, m}_i$ intersects some wall $\gamma$. Then $H^m_i \cap \gamma = K^{i, m}_i \cap \gamma$ is a collection of mutually disjoint disks in $K^{i, m}_i$, each of which $J$ intersects once in each direction. Since the component disks are linearly ordered from $X_0$, they may now be treated in a manner similar to the above.

Repeating this process a finite number of times, adjusts the walls $\gamma$ so that no $J^{i, m}$ hits any of them. Thus a wedge of simple closed curves $J^{i, m}_r$, $r = 1, 2, \ldots, R_{i, m}$, lies in the interior of the $\Gamma_{s, m}$ containing $x_{i, m}$ (except that $x_{i, m}$ lies on $Bd \Gamma_{s, m}$).

By choosing a number $\xi_{i, m} > 0$ close enough to 1 we have that

$$f^m_i (H^m_i, \xi_{i, m}) \subset \Gamma_{s, m}$$

and lies so close to $\bigcup J^{i, m}_r$ that it misses all the walls $\gamma_{s, m}$ of $\Gamma_{s, m}$. Then

$$(f^m_i \mid N^m_i \times \xi_{i, m})^{-1}$$

pushes all the walls off of $H^m_i$, is fixed outside $N^m_i$, and moves no point more than diameter $N^m_i \leq e_m$. Piecing together all these $$(f^m_i \mid N^m_i \times \xi_{i, m})^{-1}$$'s we move all the walls $\gamma$ off all the $H^m_i$ by a space homeomorphism fixed outside $\bigcup_{i=1}^{n} N^m_i$. 

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Naming new $\Gamma_{\sigma,m}$. At present, a $\Gamma_{\sigma,m}$ is a cube-with-holes with $\partial \Gamma_{\sigma,m} = \alpha_{\sigma,m} \cup \beta_{\sigma,m} \cup \gamma_{\sigma_1,m} \cup \gamma_{\sigma_2,m} \cup \gamma_{\sigma_3,m}$, where the $\gamma_{\sigma_i,m}$ are disks-with-handles as obtained in the previous section. The $\beta_{\sigma,m}$ is a disk lying in $\partial L_m$. We want to change $\beta_{\sigma,m}$ to a disk-with-handles by replacing each $F^m$ lying in $\beta_{\sigma,m}$ with $\partial F^m_{\sigma,m} = \Gamma_{\sigma,m} \cup \alpha_{\sigma,m} \cup \gamma_{\sigma_1,m} \cup \gamma_{\sigma_2,m} \cup \gamma_{\sigma_3,m}$ where the $\gamma_{\sigma_i,m}$ is the new $\gamma_{\sigma_i,m}$. Now we have that $\cup_{\sigma \in T^2_{m-1}} \Gamma_{\sigma,m} = M_{m-1} - \text{Int } M_m$ rather than $M_{m-1} - \text{Int } L_m$ as before. Thus $\cup \{\Gamma_{\sigma,m} : \sigma \in T^2_{m-1}, m = 2, 3, \ldots\} = M_1 - h(C)$. Define $\Gamma_1$ to be the closure of $S^3 - M^1$. Then

$$\Gamma_1 \cup (\cup \Gamma_{\sigma,m}) = S^3 - h(C).$$

We wish now to calculate diameter $\Gamma_{\sigma,m}$ for $m > 1$. Clearly, diameter $\Gamma_{\sigma,m}$ equals diameter $\partial \Gamma_{\sigma,m}$. Recall

$$\partial \Gamma_{\sigma,m} = \alpha_{\sigma,m} \cup \beta_{\sigma,m} \cup \gamma_{\sigma_1,m} \cup \gamma_{\sigma_2,m} \cup \gamma_{\sigma_3,m},$$

where $\sigma \in T^2_{m-1}$ and $\sigma_1, \sigma_2, \sigma_3$ are the 1-faces of $\sigma$.

(i) diameter $\alpha_{\sigma,m} < 4/(m-1)$.


diameter $\alpha_{\sigma,m} < \text{mesh } T_{m-1} + \text{diameter } G_m + \text{diameter } H^m_i$

+ adjustment to move $Q_{m-1}$ off of $F^m_{i-1}$

< $1/(m-1) + 2/n_{m-1} + e_m + 2e_{m-1} < 6/(m-1)$.

We call the reader's attention to the notation change from $G_{n,m}$ to $G_m$ and $\xi_{m}$ to $\xi_m$ following the construction of $M_m$.

(ii) diameter $\beta_{\sigma,m} < 4/(m-1)$. We have here


diameter $\beta_{\sigma,m} < \text{mesh } Q_m + 2e_m < \text{mesh } Q_m + 2/(m-1) < 6/(m-1)$.

The mesh $Q_m$ is calculated prior to the first naming of the $\Gamma_{\sigma,m}$.

(iii) diameter $\gamma_{\sigma_i,m} < 16/(m-1)$. We have diameter $\alpha_i < 1/(m-1)$, which is the mesh $T_{n-1}$; each point of the original $\gamma_{\sigma_i,m}$ is within $1/n_{m-1}$ of a point of $\alpha_i$. We moved $\gamma_{\sigma_i,m}$ by $e_m$ and $e_{m-1}$, respectively, to get it off of $F^m_i$ and $F^m_{i-1}$, by $4/(m-1)$ in the disk trading and by $e_m$ in pushing them off of the $H^m_i$. Thus


diameter $\gamma_{\sigma_i,m} < 1/(m-1) + 2/n_{m-1} + e_m + 2e_{m-1} + 8/(m-1) + 2e_m$

< $1/(m-1) + 1/(m-1) + 2/(m-1) + 2/(m-1) + 8/(m-1) + 2e_m$

< $16/(m-1)$.

Putting all these parts together, we have that

$$\text{diameter } \partial \Gamma_{\sigma,m} < 6/(m-1) + 6/(m-1) + 3(16/(m-1)) = 60/(m-1).$$

We see then that diameter $\Gamma_{\sigma,m} < 60/(m-1)$ and, thus, diameter $\Gamma_{\sigma,m} \to 0$ as $m \to \infty$.

4. Repairing embeddings. The following lemma will be useful in proving Theorem 1:
Lemma 4. Let $N$ be a connected closed 2-manifold and let $K_1, K_2, \ldots, K_n$ be a finite collection of disjoint connected 1-complexes in $N$. Suppose there is a map $h$ taking $N$ onto a 2-sphere $S$ whose nondegenerate point inverses are the $K_i$. Then there is an extension $f$ of $h$ taking $N \times [0, 1]$ onto $S \times [0, 1]$ such that

(a) $f(x, 0) = h(x)$ for all $x \in N$,
(b) $f^{-1}(S \times t) = N \times t$,
(c) $f|N \times 1$ has just one nondegenerate point inverse, $K$, and $K$ is a connected 1-complex,
(d) each nondegenerate point inverse of $f$ is a connected 1-complex,
(e) the image of the nondegenerate point inverses under $f$ is $n$ arcs, disjoint except a common endpoint $f(K)$.

Proof. Since each $K_i$ goes to a point in $S$ under $h$, the $K_i$ do not separate $N$. Thus there is a collection of disjoint polygonal arcs $A_1, A_2, \ldots, A_{n-1}$ in $N$ such that $A_i$ joins $K_i$ to $K_{i+1}$ and $\text{Int } A_i$ lies in $N - \bigcup_{i=1}^{n-1} K_i$. Set $K^0 = (\bigcup K_i) \cup (\bigcup A_i)$. Then $h(K)$ is a wedge of $n-1$ arcs in $S$ emanating from $h(K_1)$. Define a pseudo-isotopy $H: S \times [0, 1] \to S$ that shrinks $h(K)$ to a point in $S$. Then

$$f: N \times [0, 1] \to S \times [0, 1]$$

is the required mapping.

The following result is our main theorem. It says that the embedding of a 3-cell in $S^9$ can be repaired.

Theorem 1. Let $C$ be a (wild) 3-cell in $S^3$ and let $h: C \to S^3$ be an embedding of $C$ such that $h(C)$ is tame. Then $h$ can be extended to a monotone map $f$ of $S^3$ onto itself such that $f(S^3 - C) = S^3 - f(C)$. Furthermore, each nondegenerate point inverse can be taken to be a finite 1-complex.

Proof. We may suppose that $h(C)$ is the round unit ball in $S^3$. Now consider a sequence of triangulations $T_i$ of $\text{Bd } C$ as given in §3. Then $h(T_i)$ is a sequence of triangulations of $h(\text{Bd } C)$. Let $H$ carry $\text{Bd } C \times [0, \frac{1}{2}]$ homeomorphically into $S^3 - \text{Int } h(C)$ such that, for all $x \in \text{Bd } C$, $H(x, 0) = h(x)$, $H(\text{Bd } C \times (0, \frac{1}{2})) \cap h(C) = \emptyset$, and $H(\text{Bd } C \times t)$ is a round sphere concentric with $h(\text{Bd } C)$. Let $C_{\sigma, m}$ be the 3-cell $H(\sigma \times [\frac{1}{2}m+1, \frac{1}{2}m])$ for each $\sigma \in T^3_m, m \geq 1$. Let $C_1$ be the closure of that component of $S^3 - H(\text{Bd } C \times \frac{1}{2})$ not containing $h(C)$. We want to map $\Gamma_{\sigma, m}$ (from §3) onto $C_{\sigma, m}$ in such a way as to extend $h$. First, we define the map on $\text{Bd } \Gamma_{\sigma, m}$. Recall from §3 that

$$\text{Bd } \Gamma_{\sigma, m} = \alpha_{\sigma, m} \cup \beta_{\sigma, m} \cup \gamma_{\sigma_1, m} \cup \gamma_{\sigma_2, m} \cup \gamma_{\sigma_3, m}.$$ Each of $\alpha_{\sigma, m}$ and $\gamma_{\sigma, m}$ is a disk-with-handles, so there is a 1-complex on each missing its boundary such that modding out this 1-complex gives a decomposition
space homeomorphic to a disk. Call these 1-complexes $K(\alpha_{m}, m)$ and $K(\gamma_{m}, m)$ ($i=1, 2, 3$), respectively.

Since $\bigcup \{Bd \alpha_{m} : \sigma \in T_{m-1}^{2}\}$ is a copy of $T_{m-1}^{2}$, there is a natural homeomorphism from this set onto $H(T_{m-1}^{2} \times 1/2^{m})$. This homeomorphism can be extended to a monotone mapping of $\bigcup \{\alpha_{m} : \sigma \in T_{m-1}^{2}\}$ onto $H(Bd C \times 1/2^{m})$ collapsing only the $K(\alpha_{m}, m)$ to a point. Piecing together these maps we have an extension of $h$ to $C \cup (\bigcup \{\alpha_{m} : \sigma \in T_{m-1}^{2}, m=2, 3, 4, \ldots\})$. This map is clearly continuous at $Bd C$ by construction of the $\alpha_{m}$'s. Now we have $h$ defined on two disjoint arcs of boundary of each $\gamma_{m}$. Extend to all the $\gamma_{m}$ in such a manner that only the $K(\gamma_{m}, m)$ get collapsed to a point and so that $\gamma_{m}$ gets taken onto $H(\sigma_{i} \times [1/2^{m+1}, 1/2^{m}])$. Thus $h$ is extended to $\bigcup \{Bd \Gamma_{m} : \sigma \in T_{m-1}^{2}, m=2, 3, 4, \ldots\} \cup C$, because $\beta_{m}$ is the union of $\alpha_{m+1}$'s.

Now we extend the map to collars in each $\Gamma_{m}$ of the boundary of $\Gamma_{m}$ by using Lemma 4. On the inside of these collars the map collapses a connected 1-complex to a point and there is only one nondegenerate point inverse on the inside of the collar of a $\Gamma_{m}$. By Theorem 6.2 of [4], the map can be extended to the rest of $\Gamma_{m}$ onto $\Gamma_{m}$ in such a manner that each point inverse is a connected 1-complex.

The extension to $\Gamma_{1}$ onto $C_{1}$ is done in the same manner. The extended map is the required mapping $f$.

Remark. In Theorem 1, if $C$ is locally tame at each point of an open set $U$ of $Bd C$, then $f$ can be taken to be a homeomorphism on some neighborhood in $S^{3}$ of $U$. Just push $Bd C$ into $S^{3} - C$ at all points of $U$ and apply the technique of Theorem 1 to the new 3-cell $C'$ so formed.

A crumpled cube $C$ is a space homeomorphic to the union of a 2-sphere and its interior in $E^{3}$. If $C$ is a crumpled cube, $Int C$ means the set of all points having a neighborhood homeomorphic to $E^{3}$ and $Bd C$ means $C - Int C$. Thus $Bd C$ is a 2-sphere and $Int C$ is homeomorphic to the interior of $Bd C$ under some embedding in $E^{3}$. If $C_{1}$ and $C_{2}$ are crumpled cubes and $h$ is a homeomorphism of $Bd C_{1}$ onto $Bd C_{2}$, then $C=C_{1} \cup_{h} C_{2}$ is the space $C_{1} \cup C_{2}$ with $x \in Bd C_{1}$ identified with $h(x) \in Bd C_{2}$. $C_{1}$ and $C_{2}$ are said to be sewn along their boundary by $h$ and $C$ is called the sum of $C_{1}$ and $C_{2}$. The following theorem is an immediate corollary to Theorem 2 and a result due to Hosay [11] and to Lininger [14], which says that any crumpled cube may be sewed to a 3-cell in such a way that the sewing gives $S^{3}$. (For a relatively easy proof of this theorem, the reader is referred to [7].)

Corollary 1. If $C$ is a crumpled cube and $K$ is a 3-cell, then any homeomorphism of $Bd C$ onto $Bd K$ can be extended to a monotone mapping $f$ of $C$ onto $K$ such that $f(Int C)=Int K$ and $f(Bd C)=Bd K$.

Proof. Sew $C$, $K$ to 3-cells $C'$, $K'$, respectively, to get two copies of $S^{3}$. Then use Theorem 1 to get a map of $S^{3}$ onto itself taking $C'$ to $K'$ extending the given homeomorphism. The restriction of this map to $C$ is the required extension.

Corollary 2 says any embedding of a 2-sphere in $S^{3}$ can be repaired:
Corollary 2. If $S_1$ and $S_2$ are 2-spheres in $S^3$, $S_2$ tame, and $h$ is a homeomorphism of $S_1$ onto $S_2$, then $h$ can be extended to a monotone mapping $f$ of $S^3$ onto itself such that $f(S^3 - S_1) = S^3 - S_2$.

Proof. Consider $S^3$ as the sewing of two crumpled cubes, $C_1$ and $C_2$, along $S_1$ and also as a sewing of two 3-cells, $K_1$ and $K_2$, along $S_2$, and use Corollary 1.

Professors Daverman and Eaton have pointed out that the following theorem, which says that the embedding of any disk in $S^3$ can be repaired, is easily proved using a result of theirs and Theorem 1:

Corollary 3. If $D_1$ and $D_2$ are disks in $S^3$, $D_2$ is tame, and $h$ is a homeomorphism of $D_1$ onto $D_2$, then there is an extension of $h$ to a monotone mapping $f$ of $S^3$ onto itself such that $f(S^3 - D_1) = S^3 - D_2$.

Proof. We may suppose that $D_2$ is the disk $\{(x, y, 0) : x^2 + y^2 \leq 1\}$. Let $C$ be the cell $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$. In Theorem 7 of [8], Daverman and Eaton have shown that there is a 3-cell $K$ in $S^3$ and a monotone mapping $g$ of $S^3$ onto itself such that $g(K) = D_1$, $g|S^3 - K$ is a homeomorphism of $S^3 - K$ onto $S^3 - D_1$, and the following diagram commutes for some homeomorphism $h$:

\[
\begin{array}{ccc}
K & \xrightarrow{h_1} & C \\
\downarrow{g|K} & & \downarrow{g_1} \\
D_1 & \xrightarrow{h} & D_2
\end{array}
\]

where $g_1 : C \to D_2$ is given by $g_1(x, y, z) = (x, y, 0)$. By Theorem 1, $h_1$ can be extended to a monotone mapping $h_2$ of $S^3$ onto itself such that $h_2(S^3 - K) = S^3 - C$. Clearly, $g_1$ can be extended to a mapping $g_2$ of $S^3$ onto itself such that $g_2|S^3 - C$ is a homeomorphism of $S^3 - C$ onto $S^3 - D_2$. Set $f = g_2 \circ h_2 \circ g_1^{-1}$. Then $f$ is the required monotone mapping.

In general, it is not known whether an embedding of an arc or a simple closed curve in $S^3$ can be repaired. However, Theorem 3 of the previously mentioned paper of Daverman and Eaton says that, if $C$ is a 3-cell in $S^3$, there is a map $f$ of $S^3$ onto itself such that $f(C)$ is an arc and $f|S^3 - C$ is a homeomorphism of $S^3 - C$ onto $S^3 - f(C)$. Since this can be done for wild 3-cells $C$ in such a manner that $f(C)$ is also wild, then it follows that certain wild arcs in $S^3$, namely those obtained by squeezing a 3-cell in $S^3$, can be repaired. However, a converse of this result does not exist so that the following question is still open: Can an embedding of an arc (simple closed curve) in $S^3$ be repaired?

The above theorems completely repair an embedding. But are questions such as the following also true? If $S$ is a wild sphere in $S^3$ and $U$ an open subset of $S$, is there a monotone map $f : S^3 \to S^3$ such that $f|S$ is a homeomorphism, $f(S)$ is made locally tame only at each point of $f(U)$, and $f(S^3 - S) = S^3 - f(S)$? And if
so, does such a map change the wildness of points on $S$ that are well away from $U$?
The proof of Theorem 1 required the extension of the map to $\Gamma_1$, the closure of the
complement in $S^3$ of a cube-with-handles. For surfaces in 3-manifolds or for spheres-
with-handles in $S^3$, we do not give the theorem analogous to Theorem 1 because
of the difficulty of extending a map of $\partial S_1$ into a 3-manifold other than a cell.
See Lambert [13] and Jaco and McMillan [12].

These results enable us to extend monotone upper semicontinuous decompositions
of the following variety. Let $S$ be a wild 2-sphere in $S^3$. Let $G_1$ be an upper semi-
continuous decomposition of $S$ into continua not separating $S$. By a well known
theorem of R. L. Moore [16], $S/G_1$ is homeomorphic to $S$. By Corollary 2, there is
a monotone decomposition $G_2$ of $S^3$ whose nondegenerate elements are disjoint
from $S$, $S^3/G_2 = S^3$, and $S$ goes to a tame 2-sphere in $S^3$. If $G$ is the decomposition
whose nondegenerate elements are those of $G_1$ together with those of $G_2$, then,
by [9, Theorem 8], and the preceding statement, $S^3/G = S^3$ and $S$ goes to a tame
2-sphere in $S^3/G$. For an example of a decomposition in $S^3$ that cannot be ex-
tended to a decomposition of $S^3$ giving back $S^3$, the reader is referred to §8 of [4].
For a theorem analogous to Corollary 2, in the sense that it shows how to unknot
simple closed curves in $E^3$ see Theorem 5 of [10].

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