THE SPACE OF ALL SELF-HOMEOMORPHISMS
OF A TWO-CELL WHICH FIX THE CELL'S BOUNDARY
IS AN ABSOLUTE RETRACT

BY
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Abstract. The theorem mentioned in the title is proved. A corollary of the title theorem is: any homeomorphism between two compact subsets of the function space mentioned in the title can be extended to a homeomorphism of the function space onto itself.

1. **Introduction.** An interesting question in infinite-dimensional topology is the following: Is $H(D)$, the space of all homeomorphisms of a 2-disk D onto itself which are the identity on the boundary of D, homeomorphic to l_2, the separable Hilbert space of square summable sequences [4, Question 22]; or much more generally, is $H(M)$, the space of homeomorphisms of a compact n-manifold onto itself (under the "sup norm" topology), locally homeomorphic to l_2 [22, Problem M1]? The answer to the last question is yes if $n=1$ [3], unknown if $n>1$. There are several partial results concerning these questions. It is well known that $H(M)$ is a complete separable metric space. The author [16] has shown that if K is a sigma-compact subset of $H(M)$, then $H(M) - K$ is homeomorphic to $H(M)$. Geoghegan [10] has shown that $H(M) \times l_2$ is homeomorphic to $H(M)$. Edwards and Kirby [9] have shown that $H(M)$ is locally contractible. (Earlier, Dyer and Hamstrom [8] showed that $H(M)$ is locally contractible if M is a 2-manifold with boundary. See also [11], [12].)

It is unknown whether or not $H(M)$ is an absolute neighborhood retract. In this paper we prove that $H(D)$ is an absolute retract (Theorem 4).

The basic idea of the proof is to show that for every open covering α of $H(D)$ there is a locally finite polyhedron α-dominating $H(D)$. (See §2.) We make strong use of a selection theorem of Michael (Theorem 6 of this paper) in the main proof of this paper which is contained in §4. The verification that Michael's theorem may be applied is postponed until §5. Possibly readers will supply their own verifications and will not have to read §5. §6 concludes the paper with a corollary.

All homeomorphism spaces mentioned in this paper will be topologized by the "sup norm" distance function (see §2). We shall use without explicit mention elementary results concerning arcwise accessibility and separation of sets in the plane (see Chapters 5–6 of [18], or Chapter 6 of [21]).

2. **Definitions and notation.** The statement that a metric space X is an *absolute retract* (AR) means that whenever X is embedded as a closed subset Z_0 of a metric space Z, there is a retraction of Z onto Z_0. The statement that a metric space X is an *absolute neighborhood retract* (ANR) means that whenever X is embedded as a closed subset Z_0 of a metric space Z, there is a retraction of an open neighborhood of Z_0 onto Z_0.

I or [0, 1] denotes the closed unit interval, and R^2 the plane (Euclidean 2-space). If $f_1, f_2 : A \to B$ are homeomorphisms of the compact metric space A into the metric space B, then dist (f_1, f_2), *the distance between f_1 and f_2*, will be $\sup_{x \in A}$ dist $(f_1(x), f_2(x))$. If A and B are complete metric spaces, with metrics d_1, d_2 resp., then a complete metric d_3 on $A \times B$ is given by

$$d_3((a_1, b_1), (a_2, b_2)) = d_1(a_1, a_2) + d_2(b_1, b_2), \qquad a_1, a_2 \in A, b_1, b_2 \in B.$$

Suppose A and B are point sets. Then $A + B$ denotes the union (sum) of A and B; Bd (A) denotes the set of boundary points of A; Int (A) denotes the set of interior points of A.

Suppose C is a collection of subsets of a set X. Then $|C|$ denotes the set of all points p of X such that p is in some element of C; if $\alpha \in C$, then st (α, C), *the star of α in C*, denotes the collection of all $\gamma \in C$ such that $\alpha \cap \gamma \neq \varnothing$.

A *cover* of a space will always mean an open cover; a cover γ is a *refinement* of a cover α if every element of γ is contained in some element of α.

If γ is a locally finite cover of a space X, then the *nerve of γ* is the simplicial complex whose set of vertices is γ, and where elements $U_1, \ldots, U_n$ of γ are vertices of a (closed) simplex σ of the nerve if and only if $U_1 \cap \cdots \cap U_n \neq \varnothing$. It is well known [7, Theorem 1.1] that there exists a *standard barycentric map b* from X into the nerve of γ such that if $b(x) \in (\sigma - \mathrm{Bd}\,(\sigma))$ for some simplex σ of the nerve, with vertices $U_1, \ldots, U_n$, then $x \in U_1 \cap \cdots \cap U_n$.

By *map* we mean continuous function; an *embedding* of A into B is a homeomorphism from A into B. An *isotopy* λ of a space A into (onto) a space B is a map $\lambda: A \times I \to B$ such that for each $t \in I$, the map λ_t, defined by $\lambda_t(a) = \lambda(a, t)$, all $a \in A$, is a homeomorphism of A into (onto) B.

Suppose X is a space and α is a cover of X. Then a map $f: X \to X$ is *limited by α* if for each $x \in X$ there is an element U of α such that x and $f(x)$ are points of U. A homotopy $\lambda: X \times I \to X$ is an *α-homotopy* if for each $x \in X$ there is an element U of α such that $\lambda(x, t) \in U$ for all $t \in I$. A space Y *α-dominates* X if there are maps $f: X \to Y$ and $g: Y \to X$, and an α-homotopy $\lambda: X \times I \to X$ such that, for each $x \in X$, $\lambda(x, 0) = x$ and $\lambda(x, 1) = gf(x)$.

A space X is *locally n-connected* (LC^n) if for every $x \in X$ and every open set W containing x, there is an open set U containing x such that every map of a k sphere, $k \leq n$, into U is homotopic in W to a constant map.

If X is a space, the *cone over* X, denoted by $C(X)$, is the identification space formed from $X \times I$ by identifying $X \times \{1\}$ to a point.

An arc B is a *spanning arc* of a disk Z if B is contained in Z, the endpoints of B are contained in Bd (Z), and except for its endpoints B misses Bd (Z).

3. $H(D)$ is an absolute retract. Following Alexander [1] let us define the *Alexander map* $AL: H(D) \times H(D) \times I \rightarrow H(D)$ by $AL(f, g, 0)(x) = f(x)$, all $(f, g) \in H(D) \times H(D)$, $x \in D$; and $AL(f, g, t)(x) = f(t \cdot [f^{-1}g(x/t)])$, $0 < t \leq 1$, $x \in D$, $(f, g) \in H(D) \times H(D)$. Here we think of points $x \in R^2$ as being vectors based at the origin, D as being the set $\{x \in R^2 : \text{length } x \leq 1\}$, and we think of maps $f \in H(D)$ as being extended to $R^2 - D$ by the identity.

The following lemma is easily proved.

LEMMA 1. *Let α be a cover of $H(D)$. Then there is a refinement γ of α such that if $\psi: H(D) \rightarrow H(D)$ is a map which is limited by γ then the homotopy $\theta: H(D) \times I \rightarrow H(D)$ defined by $\theta(f, t) = AL(\psi(f), f, t)$ is an α-homotopy between ψ and the identity map.*

(Proof omitted.)

In §4 we shall define a basis HVT (Horizontal and Vertical Tube cover) for $H(D)$. In the present section we shall assume Theorems 2 and 3 concerning HVT. Theorems 2 and 3 will be proved in §4.

THEOREM 2. *Let $O_1, \ldots, O_n$ be a finite set of elements of the basis HVT. Then $\bigcap_{i=1}^{n} O_i$ is an element of HVT.*

THEOREM 3. *Let U be an element of HVT and K a finite-dimensional compact subset of U. Then there is an embedding ψ of the cone over K into U such that $\psi(f, 0) = f$, for all $f \in K$.*

THEOREM 4. *$H(D)$ is an AR.*

Proof. The homotopy $\theta: H(D) \times I \rightarrow H(D)$ defined by $\theta(f, t) = AL(f, \text{Id}, t)$, for all $(f, t) \in H(D) \times I$, contracts $H(D)$ to a point. Therefore, since $H(D)$ is contractible, it suffices, by [6], to prove that $H(D)$ is an ANR. By a theorem of Hanner [13, Theorem 7.2] $H(D)$ is an ANR if for every cover α of $H(D)$ there is a locally finite polyhedron α-dominating $H(D)$. (See §2 for definitions.) By Lemma 1, it suffices to show that for every cover α of $H(D)$ there is a locally finite polyhedron P and maps $b: H(D) \rightarrow P$ and $\psi: P \rightarrow H(D)$ such that ψb is limited by α.

Suppose then that α is a fixed cover of $H(D)$. We may assume that each element of α is also an element of the basis HVT. By [20, p. 980] there is a countable refinement γ of α such that (a) if $h \in \gamma$, then st (h, γ), the star of h in γ, is a finite set, and (b) if $h \in \gamma$ then there is a $V \in \alpha$ such that $|\text{st}(h, \gamma)|$ is contained in V. Let P be the nerve of γ and let $b: H(D) \to P$ be the standard barycentric map. Order the elements of γ, and for each $h_i \in \gamma$, let V_i be an element of α such that $|\text{st}(h_i, \gamma)| \subset V_i$.

We shall define the map $\psi: P \to H(D)$ by induction on the skeletons of P. For each vertex (h_i) of P, let $\psi^0((h_i))$ be any point of the set $h_i \subset H(D)$. Now assume ψ^0 has been extended to a map $\psi^n: P^n \to H(D)$ (P^n denotes n-skeleton of P) such that for each (closed) simplex $\sigma^n = (h_{\lambda_0}, \ldots, h_{\lambda_n})$ we have

(a) $\psi^n(\sigma^n)$ is finite dimensional, and

(b) $\psi^n(\sigma^n) \subset \bigcap \{V_i \in \alpha : h_i \in \bigcap_{j=0}^{n} \text{st}(h_{\lambda_j}, \gamma)\}$.

Now let σ^{n+1} be any closed $n+1$ simplex of P^{n+1}. Assume the vertices of σ^{n+1} are $h_0, \ldots, h_{n+1}$. Let $U = \bigcap \{V_i \in \alpha : h_i \in \bigcap_{j=0}^{n+1} \text{st}(h_j, \gamma)\}$. By Theorem 2, U is an element of HVT. Since the image under ψ^n of each n-simplex in the boundary of σ^{n+1} is a finite-dimensional compact set, the image under ψ^n of the boundary of σ^{n+1} is a finite-dimensional compact set [15, Theorem III2] which we denote by K. Clearly $K \subset U$. By Theorem 3 there is an embedding $\lambda: C(K) \to U$ such that $\lambda(f, 0) = f$ for all $f \in K$. The simplex σ^{n+1} may be thought of as the cone over its boundary, and so for $(x, t) \in \sigma^{n+1}$, let $\psi^{n+1}(x, t) = \lambda(\psi^n(x), t)$.

Extending over each σ^{n+1} of P^{n+1} in this way gives $\psi^{n+1}: P^{n+1} \to H(D)$ and completes the induction.

Now suppose $f \in H(D)$, and $h_0, \ldots, h_n$ are the elements of γ which contain f. Let $U' = \bigcap \{V_i \in \alpha : h_i \in \bigcap_{j=0}^{n} \text{st}(h_j, \gamma)\}$. It is easy to check that (1) $\bigcap_{j=0}^{n} h_j \subset U'$, so $f \in U'$, and (2) $\psi b(f) \in U'$. Also, U' is contained in an element of α. Hence ψb is limited by α.

The proof of Theorem 4 is complete.

4. The basis HVT.

In this section we describe the basis HVT and prove Theorems 2 and 3. First, however, we make some preliminary definitions.

It is convenient to think of the disk D now as being a rectangle in R^2 with horizontal and vertical sides. Following [18, Chapter V, §1] a *grating P on D* is formed by drawing a finite number of spanning segments across D, parallel to its sides. These spanning segments will be called the *crosscuts* of P. The 2-cells of P are the closures of the rectangular domains into which the interior of D is divided by the crosscuts of P. The *mesh* of P is the maximum of the diameters of the 2-cells of P. If E is a subset of D, the *star of E in P*, st (E, P), is the collection of all 2-cells of P which intersect E.

Let $P_1, P_2, \ldots$ be a fixed sequence of gratings on D such that (a) the mesh of P_i approaches zero as i approaches infinity, and (b) if l is a crosscut of some P_i, then l is a crosscut of P_j for all $j \geqq i$.
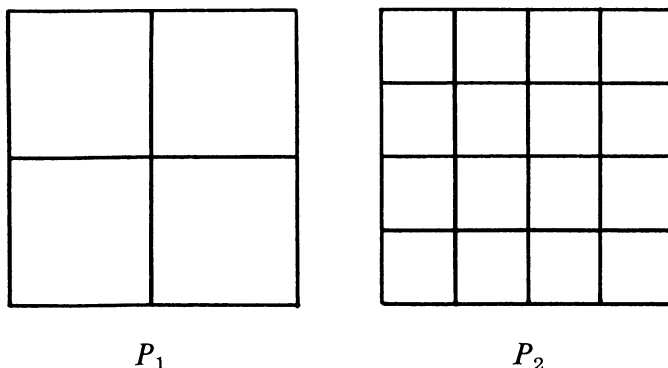
$$P_1 \qquad\qquad P_2$$

DIAGRAM 1

For convenience of notation we shall also assume that, for each i, the set of cross-cuts of P_i is made up of an equal number of vertical and horizontal segments.

Let $\mathscr{H}$ be the collection of all polyhedral disks H contained in D such that Bd (H) is the union of a vertical segment in the left side of Bd (D), a vertical segment in the right side of Bd (D), and two polygonal spanning arcs of D. $\mathscr{H}$ is called the collection of "*horizontal*" *tubes* in D. Let $\mathscr{V}$ be the collection of all polyhedral disks V contained in D such that Bd (V) is the union of a horizontal segment in the top of Bd (D), a horizontal segment in the bottom of Bd (D), and two polygonal spanning arcs of D. $\mathscr{V}$ is called the collection of "*vertical*" *tubes* in D.

We now define the sets $O(P_j; H_1, \ldots, H_n; V_1, \ldots, V_n)$ which make up the basis HVT. Let P_j be any grating in the sequence $P_1, P_2, \ldots$. Let $\{l_1, \ldots, l_n\}$ be the set of horizontal crosscuts of P_j, and $\{m_1, \ldots, m_n\}$ the set of vertical crosscuts of P_j. Let $\{H_1, \ldots, H_n\}$ be a set of n elements of $\mathscr{H}$ such that $H_i \cap H_j = \varnothing$, $i \neq j$; and let $\{V_1, \ldots, V_n\}$ be a set of n elements of $\mathscr{V}$ such that $V_i \cap V_j = \varnothing$, $i \neq j$. Then $O(P_j; H_1, \ldots, H_n; V_1, \ldots, V_n)$ is the set of all $f \in H(D)$ such that $f(l_i) \subset H_i$ $-$closure $(D - H_i)$, $1 \leq i \leq n$, and $f(m_i) \subset V_i -$closure $(D - V_i)$, $1 \leq i \leq n$. It is clear that $O(P_j; H_1, \ldots, H_n; V_1, \ldots, V_n)$ is an open subset of $H(D)$.

All sets of the form $O(P_j; H_1, \ldots, H_n; V_1, \ldots, V_n)$, $P_j \in \{P_1, P_2, \ldots\}$, $\{H_1, \ldots, H_n\}$ $\subset \mathscr{H}$, $\{V_1, \ldots, V_n\} \subset \mathscr{V}$ make up the basis HVT.

LEMMA 5. *HVT is a basis.*

Proof. Suppose we are given $\varepsilon > 0$ and $f \in H(D)$. We want to find an $O \in$ HVT such that $f \in O$, and dist $(f, g) < \varepsilon$ for all $g \in O$.

Choose $\delta > 0$ so that dist $(f(x), f(y)) < \varepsilon$ whenever dist $(x, y) < \delta$. Choose a grating $P_j \in \{P_1, P_2, \ldots\}$ such that if C^2 is any 2-cell of P_j, then diam $|\text{st}\,(C^2, P_j)|$ $< \delta$. Let n denote the number of horizontal crosscuts of P_j. Choose "horizontal" tubes $H_1, \ldots, H_n$ of $\mathscr{H}$ such that (a) $H_i \cap H_j = \varnothing$, $i \neq j$, (b) $f(l_i) \subset H_i$ $-$closure $(D - H_i)$, $1 \leq i \leq n$ and (c) $f^{-1}(H_i) \subset |\text{st}\,(l_i, P_j)|$, $1 \leq i \leq n$. Choose "vertical" tubes $V_1, \ldots, V_n$ of $\mathscr{V}$ similarly. Then the set $O(P_j; H_1, \ldots, H_n; V_1, \ldots, V_n)$ is the required element of HVT. For, let $g \in O(P_j; H_1, \ldots, H_n; V_1, \ldots, V_n)$, and

$x_0 \in D$. Let C^2 be a 2-cell of P_j containing x_0. It is easily seen that $f^{-1}g(\mathrm{Bd}\,(C^2))$ $\subset |\mathrm{st}\,(C^2, P_j)|$, hence $f^{-1}g(C^2) \subset |\mathrm{st}\,(C^2, P_j)|$. Hence $\mathrm{dist}\,(f^{-1}g(x_0), x_0) < \delta$, so $\mathrm{dist}\,(g(x_0), f(x_0)) < \varepsilon$.

The proof of Lemma 5 is complete.

Proof of Theorem 2. Let $U = \bigcap_{i=1}^{n} O_i$. We want to show $U \in \mathrm{HVT}$. Assume $U \neq \varnothing$. To simplify notation assume that P_i is the grating associated with O_i, $1 \leq i \leq n$. Then every crosscut of P_i is a crosscut of P_n, $1 \leq i \leq n$. Let l_1 be a horizontal crosscut of P_n. For each i, $1 \leq i \leq n$, let H_1^i be the "horizontal" tube associated with l_1 and O_i, provided there is one (l_1 may not be a crosscut of some P_i's), otherwise let $H_1^i = D$. Since $U \neq \varnothing$ we may choose $f \in U$. Then $f(l_1)$ is contained in the closure of some component H_1 of $D - \sum_{i=1}^{n} \mathrm{Bd}\,(H_1^i)$. It is then clear that (a) if $g \in U$ then $g(l_1) \subset H_1 - \mathrm{closure}\,(D - H_1)$ and (b) if $h \in H(D)$, and $h(l_1) \subset H_1$ $-\mathrm{closure}\,(D - H_1)$, then $h(l_1) \subset H_1^i - \mathrm{closure}\,(D - H_1^i)$, $1 \leq i \leq n$. Finally, for each i, $1 \leq i \leq n$, $\mathrm{Bd}\,(H_1^i)$ is a polygonal simple closed curve containing the endpoints of l_1, hence by [18, p. 168], $\mathrm{Bd}\,(H_1)$ is a (polygonal) simple closed curve. It follows that $H_1 \in \mathcal{H}$.

Continuing in this way we define a "horizontal" tube H_j for each horizontal crosscut l_j of P_n (assume there are r such crosscuts), and a "vertical" tube V_j for each vertical crosscut m_j, $1 \leq j \leq r$, of P_n. The "horizontal" tubes H_j are pairwise disjoint since each H_j is contained in a tube H_j^n associated with O_n, and similarly for the "vertical" tubes V_j. Thus $U = U(P_n; H_1, \ldots, H_r; V_1, \ldots, V_r)$ is an element of HVT.

The proof of Theorem 2 is complete.

The proof of Theorem 3 makes use of a selection theorem of Michael [17]. The version we state here (Theorem 6) is the one given in [11, p. 32].

THEOREM 6. *Suppose that A and B are metric spaces, A complete and the (covering) dimension of B does not exceed $n+1$, and that Z is a closed subset of B. Suppose, further, that R is an open mapping of A onto B such that the collection of inverses under R is equi-LC^n (see below) and r is a mapping of Z into A such that for $z \in Z$, $r(z) \in R^{-1}(z)$. Then there is a neighborhood U of Z such that r may be extended to a mapping r' of U into A such that for $z \in U$, $r'(z) \in R^{-1}(z)$. If each inverse under R has the property that all its homotopy groups vanish (of order $< n+1$), then U may be taken to be the entire space B.*

The collection of inverses under R is said to be *equi-LC^n* provided that if $b \in B$, $a \in R^{-1}(b)$ and F is a neighborhood of a, then there is a neighborhood G of a such that if $b' \in B$ then every mapping of a k-sphere, $k \leq n$, into $R^{-1}(b') \cap G$ is homotopic in $R^{-1}(b') \cap F$ to a constant map.

Proof of Theorem 3. (See statement in §3.) Let $P \in \{P_1, P_2, \ldots\}$ be the grating associated with U. Let $l_1, \ldots, l_n$ and $m_1, \ldots, m_n$ be the horizontal and vertical crosscuts, resp., of P. Let $H_1, \ldots, H_n$ and $V_1, \ldots, V_n$ be the "horizontal" and "vertical" tubes, resp., associated with U.

Step A. There is a small square N contained in a 2-cell of the grating P and an isotopy $\theta_1: K \times I \to U$ such that (a) $N \cap \text{Bd}(D)$ is a segment, (b) $N \cap \sum_{i=1}^n (H_i + V_i)$ is empty, (c) $\theta_1(f, 0) = f$ for all $f \in K$, (d) if $f \in K$ then $\theta_1(f, 1)|\text{Bd}(N) = \text{Id}|\text{Bd}(N)$, and (e) if $f, g \in K$, $f \neq g$, then $\theta_1(f, 1)|N \neq \theta_1(g, 1)|N$. θ_1 and N are obtained as follows. Let $AL: H(D) \times H(D) \times I \to H(D)$ be the Alexander map of Lemma 1. Let $\mu_1: H(D) \times I \to H(D)$ be defined by $\mu_1(f, t) = AL(\text{Id}, f, 1-t)$ all $(f, t) \in H(D) \times I$. By the compactness of K, there is a $t_0 \in (0, 1)$ such that if $f \in K$, and $0 \leq t \leq t_0$, then $\mu_1(f, t) \in U$. We may choose a square N, satisfying (a) and (b), small enough so that $\mu_1(f, t_0)|N = \text{Id}|N$ for all $f \in H(D)$. We may also assume, by the compactness of K, that N is small enough so that if $f \in \mu_1(K \times \{t_0\})$, $g \in H(D)$, and $f = g$ on $D - N$, then $g \in U$. Let $F: N \to D$ be a fixed homeomorphism of N onto D. Let $\mu_2: \mu_1(K \times \{t_0\}) \times I \to U$ be the isotopy defined by

$$\mu_2(f, t)(x) = f(x) \qquad \text{for } x \in D - N,$$
$$= (F^{-1} \circ AL(\text{Id}, f, t) \circ F)(x) \text{ for } x \in N.$$

Then $\mu_2(f, 0) = f$ for all $f \in \mu_1(K \times \{t_0\})$, since $f|N = \text{Id}$; and if $f \neq g$, then $\mu_2(f, 1) \neq \mu_2(g, 1)$ on N. Then, for $f \in K$, $t \in I$, let $\theta_1(f, t) = \mu_1(f, t)$ if $0 \leq t \leq t_0$, and $\theta_1(f, t) = \mu_2(\mu_1(f, t_0), (t - t_0)/(1 - t_0))$ if $t_0 < t \leq 1$.

Let $K_1 = \theta_1(K \times \{1\})$. Then K_1 has the property that if $\lambda: K_1 \times I \to U$ is a homotopy, and $\lambda(f, t)|N = f|N$ for all $t \in I$, $f \in K_1$, then λ is an isotopy.

Step B. Our goal now is to shrink K_1 so that all the homeomorphisms in the image of K_1 agree when restricted to the horizontal crosscut l_1.

Let the arcs making up the boundary of the "horizontal" tube H_1 be denoted by L, R, T, B, where L is in the left side of $\text{Bd}(D)$, R is in the right side, T is the "top" of H_1, and B is the "bottom". Since the set $|\{f(l_1) : f \in K_1\}|$ is a compact subset of D disjoint from $T + B$, we may adjust T and B slightly by "pushing them in", so that every component of $H_1 \cap \sum_{i=1}^n \text{Bd}(V_i)$ is a (polygonal) spanning arc of H_1. Since $\text{Bd}(H_1)$ and $\sum_{i=1}^n \text{Bd}(V_i)$ are polygonal we see that $H_1 \cap \sum_{i=1}^n \text{Bd}(V_i)$ has only a finite number of components. (The "pushing in" step is not absolutely necessary, but it simplifies the remaining arguments.) The components of $H_1 \cap \sum_{i=1}^n \text{Bd}(V_i)$ may be partitioned into two collections, denoted by *SEP* and *NSEP*. The elements of *SEP* are arcs which separate L and R in H_1 (and thus have endpoints in both T and B). The elements of *NSEP* are arcs which do not separate L and R in H_1 (and thus have either both endpoints in T or both in B).

We want to get an isotopy $\theta_2: K_1 \times I \to U$ such that (a) $\theta_2(f, 0) = f$, all $f \in K_1$, (b) $\theta_2(f, t) = f$ outside $f^{-1}(H_1)$, all $(f, t) \in K_1 \times I$, and (c) $\theta_2(f, 1)(l_1) \cap |NSEP| = \varnothing$, all $f \in K_1$.

Let B_1 be an element of *NSEP*. We first push $f(l_1)$ off B_1 for each $f \in K_1$.

Let *MON* be a fixed subset of R^2 consisting of (a) three disks D_1, D_2, D_3, with disjoint interiors, such that $D_1 \cap D_2$ and $D_2 \cap D_3$ are arcs, and $D_1 \cap D_2 \cap D_3$ is two points e_1, e_2, plus (b) two arcs ("stickers") M_1, M_2 such that $M_i \cap (D_1 + D_2 + D_3) = \{e_i\}$, $i = 1, 2$. (See Diagram 2.)
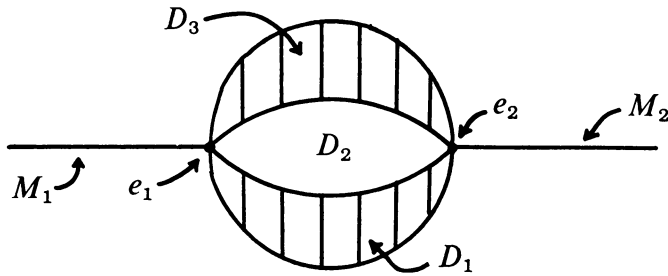
DIAGRAM 2

Let M_3 be the arc $M_1 + (D_2 \cap D_3) + M_2$. Let $H(MON, H_1)$ be the space of embeddings of MON into the tube H_1. Let $Q \subset K_1 \times H(MON, H_1)$ be the set of all (f, j) in $K_1 \times H(MON, H_1)$ such that

(1) $j(M_3) = B_1, j$ (endpoint of M_3 in M_1) = fixed endpoint u of B_1,

(2) $j(D_1 + D_2 + D_3) \subset \text{Int}(H_1)$,

(3) $j(D_3) \cap |NSEP - B_1| = \varnothing$,

(4) $f(\sum_{i=1}^{n} m_i) \cap j(MON) = \varnothing$ (recall that the m_i's are the vertical crosscuts of P),

(5) $f(l_1) \cap j(D_1 + M_1 + M_2) = \varnothing$,

(6) B_1 separates $j(\text{Int}(D_1))$ and $L + R$ in H_1.

The set of all (f, j) satisfying (2)–(5) is open, and the set satisfying (1) and (6) is closed, in $K_1 \times H(MON, H_1)$, hence Q is a G_δ in a complete metric space, hence Q is a complete metric space [14, Theorem 2-76].

Let $\pi_1: Q \to K_1$ be defined by $\pi_1(f, j) = f$, all $(f, j) \in Q$.

LEMMA 7. *There is a mapping $r': K_1 \to Q$ such that $\pi_1 r' = \text{Id}: K_1 \to K_1$.*

Lemma 7 will be proved in §5. The proof consists of verifying that the hypotheses of Theorem 6 hold for the spaces Q, K_1, and the map π_1.

Let $\pi_2: Q \to H(MON, H_1)$ be the map defined by $\pi_2(f, j) = j$, all $(f, j) \in Q$. Let $r: K_1 \to H(MON, H_1)$ be the map $r = \pi_2 r'$. For each $f \in K_1$, we shall use the embedding $r(f)$ to guide us in pushing $f(l_1)$ off B_1. Let $G: MON \times I \to MON$ be an isotopy of MON onto itself such that $G_0 = \text{Id}$, $G_t = \text{Id}$ on Bd (MON), all $t \in I$, and $G_1(D_1) = D_1 + D_2$. Let $F: K_1 \times I \to H(D)$ be the map defined by

$$F(f, t)(x) = x \qquad\qquad \text{if } x \notin r(f)(MON),$$
$$= (r(f) \circ G_t \circ r(f)^{-1})(x) \qquad \text{if } x \in r(f)(MON)$$

for all $(f, t) \in K_1 \times I$, $x \in D$.

Finally, let $\theta_2: K_1 \times I \to U$ be the isotopy defined by

$$\theta_2(f, t) = F(f, t) \circ f, \quad \text{all } (f, t) \in K_1 \times I.$$

Then $\theta_2(f, t) \in U$, all $(f, t) \in K_1 \times I$, because (a) $F(f, t) = \text{Id}$ on $\sum_{i=1}^{n} f(m_i)$, since $r(f)(MON) \cap \sum_{i=1}^{n} f(m_i) = \varnothing$ (property (4) of Q), and (b) $F(f, t) = \text{Id}$ outside H_1. Since $f(l_1) \cap r(f)(D_1 + M_1 + M_2) = \varnothing$, we see that $\theta_2(f, 1)(l_1) \cap r(f)(M_3) =$

$\theta_2(f, 1)(l_1) \cap B_1 = \varnothing$. Also by property (3) of Q, $\theta_2(f, 1)(l_1)$ intersects no new elements of $NSEP$. By induction then we may assume that $\theta_2(f, 1)(l_1) \cap |NSEP| = \varnothing$, for all $f \in K_1$. Also, θ_2 is an *isotopy* since $\theta_2(f, t) = f$ on the square N, all $(f, t) \in K_1 \times I$.

Let $K_2 = \theta_2(K_1 \times \{1\})$.

Step C. Let H_1' be the closure of the component of $H_1 - |NSEP|$ which has the vertical segments L and R in its boundary. Then

(a) Bd (H_1') is a polygonal simple closed curve [18, Theorem 11-8, p. 119],

(b) if $f \in K_2$, then $f(l_1)$ is a spanning arc of H_1' from L to R, and

(c) the components of Int $(H_1') \cap \sum_{i=1}^n$ Bd (V_i) are exactly the (open) arcs in SEP.

Let A be a fixed spanning arc in H_1', with the same endpoints as l_1, such that A intersects each arc in SEP in exactly one point. We want to get a fixed embedding $j_0: l_1 \to A$ and an isotopy $\theta_3: K_2 \times I \to U$ such that $\theta_3(f, 1)|l_1 = j_0$ for all $f \in K_2$.

For each i, $1 \leq i \leq n$, let p_i denote the point of intersection of l_1 and the vertical crosscut m_i. One may show that for each i, $1 \leq i \leq n$, the set $\{f(p_i) : f \in K_2\}$ is contained in just one component of $H_1' - |SEP|$; denote this component by S_i. Let $j_0: l_1 \to A$ be any fixed embedding such that $j_0(p_i) \in S_i$, $1 \leq i \leq n$.

Let $H(A, H_1')$ denote the space of all embeddings of A into H_1' which are fixed on the endpoints of A and which take $A \cap$ Int (H_1') into Int (H_1'). By Lemma 11(b) there is a homotopy $\lambda: K_2 \times I \to H(A, H_1')$ such that for all $f \in K_2$, (a) $\lambda(f, 0) = f \circ j_0^{-1}$, (b) $\lambda(f, 1) = \mathrm{Id}: A \to A$, and (c) if A_i is the subarc of A with endpoints $j_0(p_i)$ and $j_0(p_{i+1})$, and α is any arc in SEP, then $\lambda(f, t)(A_i) \cap \alpha = \varnothing$ if and only if $A_i \cap \alpha = \varnothing$, all $t \in I$.

For each $f \in K_2$, $\lambda(f, t) \circ j_0$ will tell us where $f(l_1)$ is located during the isotopy θ_3. We want to extend $\lambda(f, t)$ to an element of $H(H_1')$ for each $(f, t) \in K_2 \times I$. (Here $H(H_1')$ denotes the space of all homeomorphisms of D onto itself which are the identity outside H_1'.) Because of certain technical difficulties we must obtain this extension in two stages.

Let LEV be the set of all (f, h) in $K_2 \times H(H_1')$ such that

(1) $\lambda(f, 0) = h|A$ $(= f \circ j_0^{-1})$,

(2) $h^{-1}(f(\sum_{i=1}^n m_i)) \cap |SEP| = \varnothing$.

Let $\pi_1: LEV \to K_2$, $\pi_2: LEV \to H(H_1')$ be projection on the first and second coordinates resp.

LEMMA 8. *There is a map $r_1': K_2 \to LEV$ such that $\pi_1 r_1' = \mathrm{Id}: K_2 \to K_2$.*

The proof of Lemma 8 will be discussed in §5. Let $r_1 = \pi_2 r_1'$. For each $f \in K_2$, let MIS $(f) = r_1(f)^{-1}(f(\sum_{i=1}^n m_i))$. Note that MIS $(f) \cap |SEP| = \varnothing$.

Let LIF be the set of all $(f, t, h) \in K_2 \times I \times H(H_1')$ such that

(3) $\lambda(f, t) = h|A$,

(4) $h(\mathrm{MIS}(f)) \cap |SEP| = \varnothing$.

Let $\pi_{12}: LIF \to K_2 \times I$, $\pi_3: LIF \to H(H_1')$ be projection on the first two and third

coordinates resp. Let $\gamma: K_2 \times \{0\} \to LIF$ be defined by $\gamma(f, 0) = (f, 0, r_1(f))$, all $f \in K_2$.

LEMMA 9. *There is a map* $r_2': K_2 \times I \to LIF$, *which extends* γ, *such that* $\pi_{12}r_2' = \text{Id}: K_2 \times I \to K_2 \times I$.

Lemma 9 will be proved in §5. Let $r_2 = \pi_3 r_2'$.

Finally, define the isotopy $\theta_3: K_2 \times I \to U$ by $\theta_3(f, t) = r_2(f, t) \circ r_1(f)^{-1} \circ f$, all $(f, t) \in K_2 \times I$.

θ_3 is an *isotopy* since $\theta_3(f, t) = f$ on N, all $(f, t) \in K_2 \times I$. Also, $\theta_3(f, 0) = r_1(f) \circ r_1(f)^{-1} \circ f = f$, all $f \in K_2$. $\theta_3(f, t) \in U$, for all $(f, t) \in K_2 \times I$, since $\theta_3(f, t)(\sum_{i=1}^n m_i)$ misses $|SEP|$. Finally, $\theta_3(f, 1)|l_1 = r_2(f, 1) \circ (j_0 \circ f^{-1}) \circ f|l_1 = r_2(f, 1) \circ j_0 = \lambda(f, 1) \circ j_0 = j_0$, all $f \in K_2$.

Let $K_3 = \theta_3(K_2 \times \{1\})$. Since $\theta_3(f, 1) = f$ outside $f^{-1}(H_1)$, we may assume by induction that if $f, g \in K_3$, then $g = f$ on *all* horizontal crosscuts l_i, $1 \le i \le n$.

Step D. Our goal next is to shrink K_3 so that all homeomorphisms in the image of K_3 agree when restricted to any crosscut of the grating P.

Let m_1' be the subarc of the vertical crosscut m_1 such that the endpoints of m_1' are points of the horizontal crosscuts l_1 and l_2 (and m_1' interesects no other horizontal crosscuts). Thus, if $f \in K_3$, then $f(m_1')$ is in the "vertical" tube V_1, and $f(m_1')$ is a spanning arc of the disk D_1' bounded by $f(l_1) + f(l_2)$ plus two arcs in Bd (D). Let D_1 be the closure of the component of $D_1' - $ Bd (V_1) which has $f(m_1')$ as a spanning arc. By [18, Theorem 11.8, p. 119], D_1 is a disk. Note that if $g \in K_3$, then $g(m_1')$ is a spanning arc of D_1 with the same endpoints as $f(m_1')$, since $f = g$ on $l_1 + l_2$.

Let $j_1: m_1' \to D_1$ be a fixed embedding such that $j_1 = f|m_1'$ for some $f \in K_3$. We want to get an isotopy $\theta_4: K_3 \times I \to U$ such that for all $(f, t) \in K_3 \times I$, $\theta_4(f, t) = f$ outside $f^{-1}(D_1)$, and $\theta_4(f, 1)|m_1' = j_1$.

Let $H(D_1)$ denote the space of all homeomorphisms of D_1 onto itself which are fixed on Bd (D_1). Let VER be the set of all $(f, h) \in K_3 \times H(D_1)$ such that $h|A_1 = f \circ j^{-1}$, where $A_1 = j_1(m_1')$. Let $\pi_1: VER \to K_3$, and $\pi_2: VER \to H(D_1)$ be the usual projections.

LEMMA 10. *There is a map* $r_3': K_3 \to VER$ *such that* $\pi_1 r_3' = \text{Id}: K_3 \to K_3$.

Lemma 10 will be proved in §5. Let $r_3: K_3 \to H(D_1)$ be the map defined by $r_3 = \pi_2 r_3'$. Let $AL': H(D_1) \times H(D_1) \times I \to H(D_1)$ be a map defined analogously to the Alexander map AL of Lemma 1, except that we replace the disk D in the definition of AL by the disk D_1.

Define the isotopy $\theta_4: K_3 \times I \to U$ by $\theta_4(f, t) = AL'(\text{Id}, r_3(f)^{-1}, t) \circ f$. (We extend $AL'(\text{Id}, r_3(f)^{-1}, t)$ outside D_1 by the identity.)

Then $\theta_4(f, t) \in U$ for all $(f, t) \in K_3 \times I$, since $\theta_4(f, t) = f$ outside $f^{-1}(D_1)$. And $\theta_4(f, 1)|m_1' = r_3(f)^{-1} \circ f|m_1' = (j_1 \circ f^{-1}) \circ f = j_1$. θ_4 is an isotopy, since $\theta_4(f, t) = f$ on N for all $(f, t) \in K_3 \times I$.

Let $K_4 = \theta_4(K_3 \times \{1\})$. Since $\theta_4(f, 1) = f$ outside $f^{-1}(D_1)$, all $f \in K_3$, we may assume

by induction that if f, $g \in K_4$, then $f=g$ on all vertical crosscuts m_i (and all horizontal crosscuts l_i), $1 \leq i \leq n$.

Step E. We shrink K_4 to a point in U. Let g_0 be a fixed element of K_4. Let $C_1^2, \ldots, C_q^2$ be the 2-cells of the grating P. For each i, $1 \leq i \leq q$, let $F_i \colon C_i^2 \to D$ be a fixed homeomorphism of C_i^2 onto D, and let AL be the Alexander map of Lemma 1. Define a homotopy $\theta_5 \colon K_4 \times I \to U$ by

$$\theta_5(f, t)(x) = g_0 \circ F_i^{-1} \circ [AL(F_i g_0^{-1} f F_i^{-1}, \text{Id}, t)] \circ F_i(x)$$

for all $(f, t) \in K_4 \times I$, $x \in C_i^2$, $1 \leq i \leq q$.

Note that if $x \in \text{Bd}\,(C_i^2)$, $1 \leq i \leq q$, and $f \in K_4$, then $g_0(x) = f(x)$, thus θ_5 is well defined. Also, $\theta_5(f, 1) = g_0$, $\theta_5(f, 0) = f$, for all $f \in K_4$, and if $t \neq 1$, then $\theta_5 | K_4 \times \{t\}$ is a homeomorphism.

Finally, combining the homotopies $\theta_1, \ldots, \theta_5$ (an equation for this may be obtained as in [4, §5]) gives us a map $\psi_1 \colon C(K) \to U$ such that $\psi_1(f, 0) = f$, all $f \in K$, and for each $t \in I$, $\psi_1 | K \times \{t\}$ is a homeomorphism. But then, by Lemma 14 (see §5), there is an *embedding* $\psi \colon C(K) \to U$ such that $\psi(f, 0) = f$ for all $f \in K$.

The proof of Theorem 3 is complete.

5. **Some lemmas.**   In this section we prove the lemmas used in Theorem 3. First, however, we bring together in Lemmas 11 and 12 some ways of shrinking sets of homeomorphisms. The proofs of Lemmas 11 and 12 are based on various modifications of the Alexander map AL of Lemma 1.

In Lemmas 11 and 12, Z will denote a fixed rectangle with vertical and horizontal sides, Z_1 a closed subset of Z and $H(Z_1, Z)$ will be the space of all embeddings of Z_1 into Z *which are the identity on* $Z_1 \cap \text{Bd}\,(Z)$, and which take $Z_1 \cap \text{Int}\,(Z)$ into $\text{Int}\,(Z)$. A subset X of $H(Z_1, Z)$ has the *compact shrinking property* (C.S.P.) if given any compact subset Y of X, there is a map $\lambda \colon C(Y) \to X$ such that $\lambda(f, 0) = f$, all $f \in Y$, and $\lambda(f, 1)$ is the inclusion map of Z_1 into Z, all $f \in Y$.

LEMMA 11. *Let A be a horizontal spanning arc of Z, and $A_1, \ldots, A_n$ a subdivision of A into subarcs with disjoint interiors. Let VAN be a subset of $H(Z_1, Z)$. Then VAN has the C.S.P. if one of* (a), (b), (c) *holds.*

(a) *Z_1 is the closure of a component of $Z - A$, and $VAN = H(Z_1, Z)$.*

(b) *There are vertical spanning segments $\gamma_1, \ldots, \gamma_m$ of Z such that, for all i, j, $1 \leq i \leq n$, $1 \leq j \leq m$, γ_j misses the endpoints of A_i. $Z_1 = A$, and*

$$VAN = \{h \in H(Z_1, Z) : h(A_i) \cap \gamma_j = \varnothing$$
$$\text{if and only if } A_i \cap \gamma_j = \varnothing, \text{ all } i, j, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

(c) *There are two collections $\{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\{\gamma_1, \ldots, \gamma_m\}$ of pairwise disjoint spanning arcs of Z, each arc meeting the top and bottom of $\text{Bd}\,(Z)$, such that for all $i, j, \gamma_j \cap \alpha_i = \varnothing$, and for all i, $1 \leq i \leq n-1$, $\alpha_i \cap A$ is the right-hand endpoint of A_i. $Z_1 = Z$ and $VAN = \{h \in H(Z_1, Z) : h | A = \text{Id}, \text{ and } h(\alpha_i) \cap \gamma_j = \varnothing, \text{ all } i, j\}.$*

Proof. *Case* (a). Identify Z with the disk D used in defining the Alexander map AL of Lemma 1. We may assume that if x is a point (vector) of Z_1 then x/t is either a point of Z_1 or a point outside Z, all $t \in (0, 1]$. But then the map $\lambda(f, t) = AL(\text{Id}, f, 1-t)$ is well defined for all $(f, t) \in C(Y)$ and is the map required.

The proof for conditions (b) and (c) makes use of a canonical family of isotopies $R(\alpha, \beta, \gamma, t)$ of the plane onto itself. Let α, β, γ be vertical lines such that α is to the left of β and β is to the left of γ. Let $\{R(\alpha, \beta, \gamma, t)\}_{t \in [0,1]}$ be a continuous (in t) family of homeomorphisms of R^2 onto R^2 such that (1) if l is any horizontal line in R^2, then $R(\alpha, \beta, \gamma, t)(l) = l$ for all $t \in I$, (2) if v is any vertical line in R^2, then $R(\alpha, \beta, \gamma, t)(v)$ is a vertical line, all $t \in I$, (3) $R(\alpha, \beta, \gamma, 0) = \text{Id}$, (4) $R(\alpha, \beta, \gamma, t)(p) = p$ if p is on or to the right of the vertical line γ, and (5) $R(\alpha, \beta, \gamma, 1)(\alpha) = \beta$.

Case (b). Assume inductively that we have a homotopy $\lambda_{n-2} \colon Y \times I \to VAN$ such that $\lambda_{n-2}(f, 0) = f$, all $f \in Y$; $\lambda_{n-2}(f, 1) = \text{Id}$ on $A_1 + \cdots + A_{n-2}$, all $f \in Y$; and if x is a point of $A_{n-1} + A_n$, then $\lambda_{n-2}(f, 1)(x)$ is on or to the right of a vertical line ω_{n-2} through the right-hand endpoint of A_{n-2}, all $f \in Y$. Let ω_{n-1} be a vertical line through the right-hand endpoint of A_{n-1}. Let $\omega'_{n-1}, \omega''_{n-1}$ be vertical lines to the left and right, resp., of ω_{n-1} such that none of the segments γ_j, $1 \leq j \leq m$, intersects the (closed) strip between ω'_{n-1} and ω''_{n-1}.

For $t \in I$, let

$$Q_t = R(\omega_{n-2}, \omega'_{n-1}, \omega_{n-1}, t),$$

$$S_t = R(\omega'_{n-1}, \omega_{n-1}, \omega''_{n-1}, t),$$

$$\lambda'_{n-1}(f, t) = Q_t \circ \lambda_{n-2}(f, 1) \circ Q_t^{-1}, \qquad \text{all } f \in Y,$$

$$\lambda_{n-1}(f, t) = \lambda_{n-2}(f, 3t), \qquad 0 \leq t \leq \tfrac{1}{3},$$

$$= \lambda'_{n-1}(f, 3t-1), \qquad \tfrac{1}{3} < t \leq \tfrac{2}{3},$$

$$= S_{(3t-2)} \circ \lambda'_{n-1}(f, 1) \circ S_{(3t-2)}^{-1}, \qquad \tfrac{2}{3} < t \leq 1$$

(elements of $H(Z_1, Z)$ extend outside Z by the identity).

As t moves from $\tfrac{1}{3}$ to $\tfrac{2}{3}$, $\lambda_{n-1}(f, t)$ becomes the identity on the part of A_{n-1} between ω_{n-2} and ω'_{n-1}, then as t moves from $\tfrac{2}{3}$ to 1, $\lambda_{n-1}(f, t)$ becomes the identity on all of A_{n-1}. Note that Q_t moves no point of A_n, thus $\lambda_{n-1}(f, t) \in VAN$, $\tfrac{1}{3} \leq t \leq \tfrac{2}{3}$; and if $x \in A$ lies to the right of ω'_{n-1}, then $\lambda'_{n-1}(f, 1)(x)$ lies to the right of ω'_{n-1}, thus $\lambda_{n-1}(f, t) \in VAN$, $\tfrac{2}{3} < t \leq 1$. Thus λ_{n-1} satisfies our inductive hypothesis.

Finally, we change $\lambda_{n-1}(f, 1)$ to the identity on A_n (and thus on all of A), for each $f \in Y$. Identify the disk bounded by ω_{n-1}, the right side of Bd (Z), and two arcs in the top and bottom of Bd (Z) with the disk D used in defining the Alexander map AL of Lemma 1. We may assume that the arc A_n is a diameter of D. Then let $\lambda(f, t) = \lambda_{n-1}(f, 2t)$, $0 \leq t \leq \tfrac{1}{2}$, and $\lambda(f, t) = AL(\text{Id}, \lambda_{n-1}(f, 1), 2-2t)$, $\tfrac{1}{2} < t \leq 1$, all $f \in Y$. λ is the required map.

Case (c). Let Y be a given compact subset of VAN. For each i, $1 \leq i \leq n-1$, the compact set $\alpha_i + |\{f(\alpha_i) : f \in Y\}|$ is disjoint from all arcs γ_j, $1 \leq j \leq m$. Hence we may choose spanning arcs α'_i, α''_i of Z separating $\alpha_i + |\{f(\alpha_i) : f \in Y\}|$ from the left

and right sides, resp., of Z such that each of α_i', α_i'' intersects A in one point, and such that the disk bounded by $\alpha_i' + \alpha_i''$ plus two arcs of Bd (Z) intersects none of the arcs γ_j, $1 \leq j \leq m$. We assume that the arcs α_1', α_1'', ..., α_{n-1}'' are pairwise disjoint. We may also assume that the arcs α_i', α_i, α_i'' are vertical segments, for all i, $1 \leq i \leq n-1$ (there is a homeomorphism of Z onto itself taking the arcs α_i', α_i, α_i'' onto vertical segments, for all i, $1 \leq i \leq n-1$, while keeping the arc A horizontal). Then for each i, $1 \leq i \leq n-1$, let ω_i', ω_i, ω_i'' be vertical lines extending α_i', α_i, α_i'' resp. Assume inductively that we have a homotopy $\lambda_{n-2} : Y \times I \to VAN$ such that $\lambda_{n-2}(f, 0) = f$, all $f \in Y$, and $\lambda_{n-2}(f, 1)(p) = p$ if p lies to the left of the vertical line ω_{n-2}, all $f \in Y$. Define λ_{n-1}', λ_{n-1} exactly as in Case (b). Complete the argument as in Case (b).
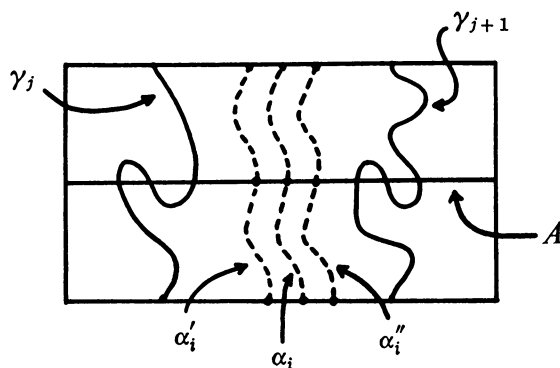
The proof of Lemma 11 is complete.

DIAGRAM 3

LEMMA 12. (For definitions of $Z, Z_1, H(Z_1, Z)$, see beginning of §5.) *Suppose Z_2 is a disk contained in Z_1 such that $Z_2 \cap$ Bd (Z) is an arc, and every point of Bd $(Z_2) - (Z_2 \cap$ Bd $(Z))$ is an interior point of Z_1. Suppose Z_3 is a closed subset of Z_1 such that $Z_2 \cap Z_3$ is at most the endpoints of $Z_2 \cap$ Bd (Z). Suppose VAN is a subset of $H(Z_1, Z)$ satisfying either* (a) *or* (b) *below. Then if Y is any compact subset of VAN, there is a homotopy $\lambda : Y \times I \to VAN$ such that for all $f \in Y$, $\lambda(f, 0) = f$, and $\lambda(f, 1)|Z_2 = \mathrm{Id}$.*

(a) $VAN = \{f \in H(Z_1, Z) : f(Z_2) \cap Z_3 = Z_2 \cap Z_3\}$,

(b) $VAN = \{f \in H(Z_1, Z) : Z_2 \cap f(Z_3) = Z_2 \cap Z_3\}$.

In addition, λ may be chosen so that if N is a preassigned neighborhood (in Z) of Z_2, then for all $x \in Z_1 - N$, either $\lambda(f, t)(x) = f(x)$ or $\lambda(f, t)(x) \in N$, for all $(f, t) \in Y \times I$.

Proof. "Push out" slightly on the disk Z_2 to obtain a new disk Z_2' in Z_1 such that Bd $(Z_2') \cap$ Bd $(Z_2) = Z_2 \cap$ Bd $(Z) = Z_2' \cap$ Bd (Z). (Thus Bd $(Z_2') +$ Bd (Z) is a θ curve.) We may assume that $Z_2' \cap Z_3 = Z_2 \cap Z_3$. By the compactness of Y, Z_2' may be chosen close enough to Z_2 so that either $f(Z_2') \cap Z_3 = Z_2 \cap Z_3$ (Case (a)) or $Z_2' \cap f(Z_3) = Z_2 \cap Z_3$ (Case (b)), for all $f \in Y$. Let Z_2'' be the disk such that $Z_2' = Z_2 + Z_2''$ and Int $(Z_2) \cap$ Int $(Z_2'') = \varnothing$. Let $\{R_t\}_{t \in I}$ be a continuous (in t) family of

homeomorphisms of Z *into* Z such that $R_0 = \mathrm{Id}$; $R_t(p) = p$ if $p \notin Z_2'$, for all $t \in I$; and $R_1(Z_2') = Z_2''$. Define $\lambda: Y \times I \to VAN$ by

$$\lambda(f, t)(x) = x \qquad \text{if } x \notin R_t(Z),$$
$$= R_t f R_t^{-1}(x) \quad \text{if } x \in R_t(Z)$$

for all $(f, t) \in Y \times I$. Then λ is the required homotopy. λ satisfies the last condition of the lemma if we choose Z_2' inside the neighborhood N.

The proof of Lemma 12 is complete.

We now return to the lemmas used in proving Theorem 3.

Proof of Lemma 7. (We assume the reader is familiar with the notation used in Step B of the proof of Theorem 3.) We want to show that the spaces Q, K_1, and the map $\pi_1: Q \to K_1$ satisfy the hypotheses of Theorem 6. We know that Q is complete. K_1 is finite dimensional since it is homeomorphic to K.

π_1 is onto. Fix $f \in K_1$. Let HOL be the disk bounded by B_1 plus a subarc of either the "top" or "bottom" of Bd (H_1). Let p, q be points of $B_1 \cap \mathrm{Int}\,(H_1)$ such that $B_1 \cap f(l_1) \subset F - \{p, q\}$, where F is the subarc of B_1 with endpoints p, q. It is clear that $f(l_1 \cap \sum_{i=1}^{n} m_i)$ lies outside HOL, so $f(l_1) \cap HOL$ and $f(\sum m_i) \cap HOL$ are disjoint closed subsets of HOL. By [18, 16.2, p. 127] there are two spanning arcs F_1, F_2 of HOL from p to q such that $F_1 \cap F_2 = \{p, q\}$, and the disk bounded by $F_1 + F_2$ misses $f(l_1 + \sum m_i)$ (and thus separates $f(l_1) \cap HOL$ and $f(\sum m_i) \cap HOL$ in HOL). Let F_3 be a spanning arc of the closure of $H_1 - HOL$, from p to q, chosen close enough to F so that the disk bounded by $F + F_3$ misses $|NSEP - B_1| + f(\sum m_i)$. Combining the disks bounded by $F + F_3$, $F + F_2$, $F_1 + F_2$ with the arc B_1 gives us the homeomorphic image of MON which we require.
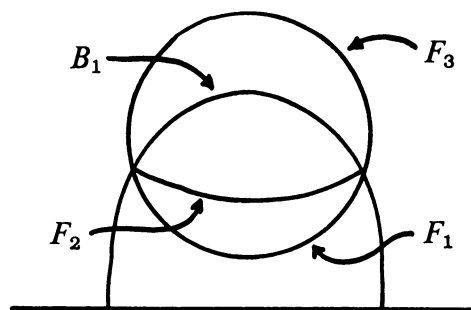

DIAGRAM 4

π_1 is open. Suppose $(f, j_0) \in W$, and W is open in Q. Then for sufficiently small ε, if dist $(f, g) < \varepsilon$, then $(g, j_0) \in W$.

Each inverse set is homotopically trivial. Fix $f \in K_1$. Let α be a map of an n-sphere S^n into $\pi_1^{-1}(f)$. Let

$$X = \{j \in H(MON, H_1) : (f, j) \in \alpha(S^n)\}.$$

It suffices to construct a map $\lambda: C(X) \to H(MON, H_1)$ such that $\lambda(j, 0) = j$, all $j \in X$, and $\{f\} \times \lambda(C(X) \subset Q$.

Let HOL be the disk defined above. We note that if $j \in X$, then $j(D_1) \subset HOL$, rather than $j(D_3) \subset HOL$ (by property (6) in the definition of Q). Since the compact set $|\{j(D_3) : j \in X\}|$ is disjoint from $f(\sum m_i) + |NSEP - B_1|$ there is a spanning arc in the closure of $H_1 - HOL$, joining the endpoints of B_1, such that the disk OUT bounded by this arc and B_1 contains $|\{j(D_3) : j \in X\}|$ but misses $f(\sum m_i) + |NSEP - B_1|$ (by [18, 16.2, p. 127]). Similarly there is a spanning arc in HOL joining the endpoints of B_1 such that the disk IN bounded by this arc and B_1 contains $|\{j(D_1 + D_2) : j \in X\}|$ but misses $f(\sum m_i)$. We shall shrink X to a point so that if i is in the image of $C(X)$ then $i(MON) \subset IN + OUT$, thus $i(D_3) \cap |NSEP - B_1| = \varnothing$, and $i(MON)$ misses $f(\sum m_i)$.

We shrink X in four stages.

Stage 1. Let F be an open subarc of B_1 such that $f(l_1) \cap B_1 \subset F$, and

$$F \cap |\{j(M_1 + M_2) : j \in X\}| = \varnothing$$

(recall that M_1, M_2 were the "stickers" attached to $D_1 + D_2 + D_3$). (We assume here that $f(l_1) \cap B_1 \neq \varnothing$. If $f(l_1) \cap B_1 = \varnothing$ we choose the points b_1, b_2, mentioned below, to be any two points in $B_1 \cap \text{Int}(H_1)$ and define isotopies $G_t^{p,q}$ similar to those defined below.) Let b_1, b_2 be fixed points of $\text{Int}(H_1)$ in different components of $B_1 - F$. For all pairs of points p, q of $\text{Int}(H_1)$, p, q in different components of $B_1 - F$, we may construct a canonically defined isotopy $G_t^{p,q}$ of $IN + OUT$ onto itself which is fixed on $\text{Bd}(IN + OUT)$, which takes B_1 onto itself, which moves the points p, q onto the points b_1, b_2, and which is the identity outside a preassigned neighborhood of $B_1 - F$. Using the isotopies $G_t^{p,q}$ we may obtain a homotopy $\lambda_1: X \times I \to H(MON, H_1)$ such that for all $(j, t) \in X \times I$ we have $\lambda_1(j, 0) = j$, $\lambda_1(j, t)(MON) \subset IN + OUT$, $\{f\} \times \lambda_1(X \times I) \subset Q$, and $\lambda_1(j, 1)(e_i) = b_i$, $i = 1, 2$ (recall that e_1, e_2 are the endpoints of the arc $D_2 \cap D_3$). Let $X_1 = \lambda_1(X \times \{1\})$.

Stage 2. Let j_1 be a fixed element of X_1. We want to get a homotopy $\lambda_2: X_1 \times I \to H(MON, H_1)$ such that for all $(j, t) \in X_1 \times I$, we have $\lambda_2(j, 0) = j$, $\lambda_2(j, t)(MON) \subset IN + OUT$, $\{f\} \times \lambda_2(X_1 \times I) \subset Q$, and $\lambda_2(j, 1)|M_3 = j_1|M_3$.

Since the compact set $|\{j(D_1) : j \in X_1\}|$ is disjoint from $f(l_1)$ there is a spanning arc CUT in HOL, from b_1 to b_2, such that CUT separates $f(l_1)$ and $|\{j(D_1) : j \in X_1\}| - \{b_1 + b_2\}$ in HOL. Let $H(B_1)$ denote the space of homeomorphisms of B_1 onto itself which are fixed on the endpoints of B_1 and on b_1, b_2. There is a map $\mu: H(B_1) \times I \to H(B_1)$, defined similarly to the Alexander map AL of Lemma 1, which contracts $H(B_1)$ to $\{\text{Id}\} \subset H(B_1)$. Also, for each $(h, t) \in H(B_1) \times I$, we may extend $\mu(h, t)$ in a canonical fashion to a homeomorphism of $IN + OUT$ onto itself which is the identity on $\text{Bd}(IN + OUT) + CUT$. Identify $\mu(h, t)$ and its extension. Then the homotopy $\lambda_2: X_1 \times I \to H(MON, H_1)$ is defined by

$$\lambda_2(j, t) = \mu(j_1 j^{-1}|B_1, 1-t) \circ j.$$

Note that for all $(j, t) \in X_1 \times I$, $\lambda_2(j, t)(D_1)$ misses $f(l_1)$, since $\mu(j_1 j^{-1} | B_1, 1 - t) = \text{Id}$ on CUT. Let $X_2 = \lambda_2(X_1 \times \{1\})$.

Stage 3. Let FIX be the disk bounded by CUT plus a subarc of B_1. We want a homotopy $\lambda_3 \colon X_2 \times I \to H(MON, H_1)$ which satisfies the usual three conditions plus $\lambda_3(j, 1) = j_1$ on $M_3 + j_1^{-1}(FIX)$ (j_1 is the fixed element mentioned in Stage 2). If we let Z, Z_1, Z_2, Z_3, Y be represented by IN, $j_1(M_3 + D_1 + D_2)$, FIX, $j_1(D_1)$, $\{jj_1^{-1} : j \in X_2\}$, respectively, then by Lemma 12(b), there is a homotopy $\mu \colon X_2 \times I \to H(Z_1, Z)$ such that $\mu(j, 0) = jj_1^{-1}$ and $\mu(j, 1) = \text{Id}$ on FIX, all $j \in X_2$. Define $\lambda_3 \colon X_2 \times I \to H(MON, H_1)$ by $\lambda_3(j, t) = j$ on D_3, $\lambda_3(j, t) = \mu(j, t) \circ j_1$ on $M_3 + D_1 + D_2$. Note that for all $(j, t) \in X_2 \times I$, $\lambda_3(j, t)(D_1)$ misses $f(l_1)$, since, by the defining property of VAN, $\lambda_3(j, t)(D_1)$ misses the interior of FIX. Let $X_3 = \lambda_3(X_2 \times \{1\})$.

Stage 4. We complete the shrinking. If we let Z, Z_1 be represented by OUT, $j_1(D_3)$, resp., then by Lemma 11(a), there is a homotopy $\psi \colon X_3 \times I \to H(Z_1, Z)$ such that $\psi(j, 0) = jj_1^{-1}$ and $\psi(j, 1) = \text{Id}$, all $j \in X_3$. Define a homotopy $\lambda_4 \colon X_3 \times I \to H(MON, H_1)$ by $\lambda_4(j, t) = j$ on $M_3 + D_1 + D_2$, $\lambda_4(j, t) = \psi(j, t) \circ j_1$ on D_3. Then for all $j \in X_3$, $\lambda_4(j, 1) = j_1$ except on $(D_1 + D_2) - j_1^{-1}(FIX)$. Let $X_3 = \lambda_4(X_3 \times \{1\})$. If we let Z, Z_1 now be represented by the closures of $IN - FIX$ and $j_1(D_1 + D_2) - FIX$, resp., then another application of Lemma 11(a) shrinks X_4 to the point $j_1 \in X_4$. This completes the shrinking and demonstrates that $\pi_1^{-1}(f)$ is homotopically trivial.

An analogous, though easier (omits Stage 3), proof shows that $\pi_1^{-1}(f)$ is LC^n for all n (see definition of LC^n in §2), and all $f \in K_1$.

$\{\pi_1^{-1}(f)\}_{f \in K_1}$ is equi-LC^n for all n. Fix $(f, j_0) \in Q$, an integer n, and $\varepsilon > 0$. Let W be an $\varepsilon/2$ neighborhood of (f, j_0) in $K_1 \times H(MON, H_1)$. Assume ε is small enough so that if $(h, j) \in K_1 \times H(MON, H_1)$, and $j(M_3) = B_1$, and dist $((h, j), (f, j_0)) < \varepsilon$, then $(h, j) \in Q$. Since $\pi_1^{-1}(f)$ is LC^n we may choose a δ-neighborhood U' of (f, j_0), $\delta < \varepsilon/2$, such that any mapping of a k-sphere, $k \leq n$, into $\pi_1^{-1}(f) \cap U'$ is homotopic in $\pi_1^{-1}(f) \cap W$ to a constant. Let U'' be a $\delta/2$ neighborhood of (f, j_0). Let α be a map of a k-sphere S^k into $\pi_1^{-1}(g) \cap U''$ for some $g \in K_1$. Define $\gamma \colon \pi_1^{-1}(g) \to \pi_1^{-1}(f)$ by $\gamma(g, j) = (f, j)$. Then $\gamma\alpha$ maps S^k into $\pi_1^{-1}(f) \cap U'$, so $\gamma\alpha$ has an extension $\bar{\alpha} \colon C(S^k) \to \pi_1^{-1}(f) \cap W$. But then $\gamma^{-1}\bar{\alpha}$ is the required map of $C(S^k)$ into $\pi_1^{-1}(g) \cap \varepsilon$-neighborhood of (f, j_0).

Thus by Theorem 6, π_1 has a right inverse.

The proof of Lemma 7 is complete.

The following lemma may be proved using the classical Schoenflies theorem and elementary plane topology (see [18]).

LEMMA 13. *Let A be a spanning arc of a disk Z, ε a positive number, and h a homeomorphism of Z onto Z which is fixed on* Bd (Z). *Then there is a positive number δ such that given any homeomorphism j of A onto a spanning arc of Z having the same endpoints as A, with* dist $(j, h|A) < \delta$, *there is a homeomorphism g of Z onto Z, fixed on* Bd (Z), *such that $g|A = j$, and* dist $(g, h) < \varepsilon$.

(Proof omitted.)

We use Lemma 13 in the

Proof of Lemmas 8 and 9. (We assume the reader is familiar with the notation used in Step C of the proof of Theorem 3.) The proofs of Lemmas 8 and 9 are almost identical. Therefore, we shall assume Lemma 8 has been proved and proceed with the proof of Lemma 9. The proof consists of verifying that the hypotheses of Theorem 6 hold for the spaces LIF, $K_2 \times I$, and the map $\pi_{12}: LIF \to K_2 \times I$. LIF is complete, since the set of all (f, t, h) satisfying $\lambda(f, t) = h|A$ is closed, and all (f, t, h) satisfying $h(\mathrm{MIS}\,(f)) \cap |SEP| = \varnothing$ is open in $K_2 \times I \times H(H_1')$. ($LIF$ is a G_δ in a complete metric space.) Since K_2 is the homeomorphic image of K, we see that $K_2 \times I$ is finite dimensional. Before verifying the rest of the conditions we must stop and analyze the sets $\mathrm{MIS}\,(f)$, $f \in K_2$ (if we were proving Lemma 8, we would analyze $f(\sum m_i)$, $f \in K_2$). Fix $f \in K_2$. Since $(r_1(f)^{-1} \circ f)(l_1) = j_0(l_1) = A$, we see that for each vertical crosscut m_i, $1 \le i \le n$, $(r_1(f)^{-1} \circ f)(m_i)$ intersects the arc A in exactly one point (the point $j_0(p_i)$, which is the "left" endpoint of the arc A_i). Thus there is only one component of $(r_1(f)^{-1} \circ f)(m_i) \cap \mathrm{Int}\,(H_1')$ which intersects A. Denote this component (open arc) by E_i. Since A separates the "top" and "bottom" of H_1', all other components of $(r_1(f)^{-1} \circ f)(m_i) \cap \mathrm{Int}\,(H_1')$ have either both endpoints in the "top" or both endpoints in the "bottom" of H_1'. Let PI denote the set $\{E_1, \ldots, E_n\}$. Since the set $|SEP|$ separates $f(m_i)$ and $f(m_j)$, $i \ne j$, in H_1', and since $\mathrm{MIS}\,(f) \cap |SEP| = \varnothing$, we see that $|SEP|$ separates E_i and E_j, $i \ne j$. Thus each component of $H_1' - |SEP|$ can contain at most one element of PI.

We now choose a finite collection $\{F_\alpha\}$ of spanning arcs of H_1' which will separate $\mathrm{MIS}\,(f) - |PI|$ from $A + |SEP|$. Let G_α be one of the (finitely many) components of $\mathrm{Int}\,(H_1') - (|SEP| + |PI| + A)$. Then $\mathrm{Bd}\,(G_\alpha)$ is a simple closed curve, $\mathrm{Bd}\,(G_\alpha) \cap \mathrm{Bd}\,(H_1')$ is an arc, and the only limit points in $\mathrm{Bd}\,(G_\alpha)$ of $\mathrm{MIS}\,(f) \cap G_\alpha$ lie in the arc $\mathrm{Bd}\,(G_\alpha) \cap \mathrm{Bd}\,(H_1')$. Let F_α' be a subarc of $\mathrm{Bd}\,(G_\alpha) \cap \mathrm{Bd}\,(H_1')$ which contains all limit points in $\mathrm{Bd}\,(G_\alpha)$ of $\mathrm{MIS}\,(f) \cap G_\alpha$, and which misses $|SEP|$. Finally, choose F_α to be a spanning arc of G_α, with the same endpoints as F_α', such that
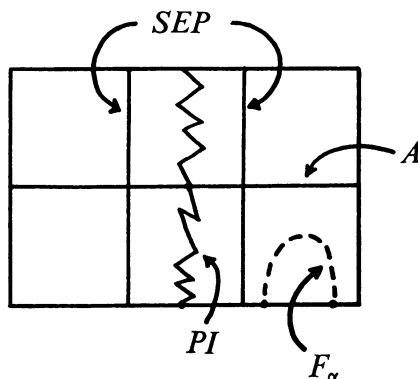

DIAGRAM 5

$F_\alpha \cap$ MIS $(f) \cap G_\alpha = \varnothing$. This defines the (finite) collection $\{F_\alpha\}$. For future reference note that each F_α can be chosen to lie in a preassigned neighborhood of $F'_\alpha + (G_\alpha \cap$ MIS $(f))$. (If we were proving Lemma 8, we would replace MIS (f) above by $f(\sum m_i)$, A by $f(l_1)$, and choose similar F_α's.)

π_{12} is onto. Fix $(f, t) \in K_2 \times I$. Let PI and $\{F_\alpha\}$ be defined as above for our given $f \in K_2$. We want to find $h \in H(H'_1)$ so that $(f, t, h) \in LIF$. We start by defining h on A as equal to $\lambda(f, t)$. Now, if α is an arc of SEP, then $A_i \cap \alpha = \varnothing$ if and only if $\lambda(f, t)(A_i) \cap \alpha = \varnothing$, $1 \le i \le n$ (by our choice of λ). Therefore, for each arc $E_i \in PI$, we may choose a corresponding spanning arc E''_i, with the same endpoints as E_i, such that $E''_i \cap |SEP| = \varnothing$, and $E''_i \cap h(A) = h(E_i \cap A) = h(j_0(p_i))$. We may then extend h to a homeomorphism of Bd $(H'_1) + A + \sum E_i$ onto Bd $(H'_1) + h(A) + \sum E''_i$ in the obvious way. Now, for each arc F_α we pick a corresponding spanning arc F''_α, with the same endpoints as F_α, such that $F''_\alpha \cap$ Int (H'_1) misses $|SEP| + h(A) + \sum E''_i$. Extend h in the obvious way to a homeomorphism of Bd $(H'_1) + A + \sum E_i + \sum F_\alpha$ onto Bd $(H'_1) + h(A) + \sum E''_i + \sum F''_\alpha$. Complete the extension by finitely many applications of the Schoenflies theorem. Now $(f, t, h) \in LIF$, since $h(\sum E_i) \cap |SEP| = \varnothing$, and the disks bounded by the F''_α's plus the appropriate subarcs of the "top" and "bottom" of H'_1 contain $h($MIS $(f) - |PI|)$, and do not intersect $|SEP|$.

Each inverse set is homotopically trivial. Fix $(f, t) \in K_2 \times I$. Let α be a map of an n-sphere S^n into $\pi_{12}^{-1}(f, t)$. Let $X = \{h \in H(H'_1) : (f, t, h) \in \alpha(S^n)\}$. It suffices to construct a map $\mu : C(X) \to H(H'_1)$ such that $\mu(h, 0) = h$ for all $h \in X$, and $\{(f, t)\} \times \mu(C(X)) \subset LIF$.

Fix an element $h_1 \in H(H'_1)$. Suppose we can obtain a homotopy $\psi : X \times I \to H(H'_1)$ such that for all $(h, t) \in X \times I$, (a) $\psi(h, 0) = h_1^{-1}h$, $\psi(h, 1) = \text{Id}$, (b) $\psi(h, t)|A = \text{Id}$, and (c) $\psi(h, t)($MIS $(f)) \cap h_1^{-1}(|SEP|) = \varnothing$. Then the required shrinking μ will be defined by $\mu(h, t) = h_1 \circ \psi(h, t)$, all $(h, t) \in X \times I$.

It remains to construct ψ. Let PI, $\{F_\alpha\}$ be defined as above for our given $f \in K_2$. For each F_α, let F'_α be the subarc of either "top" or "bottom" of Bd (H'_1) cut off by the endpoints of F_α. Note that $F'_\alpha \cap |SEP| = \varnothing$. We may choose F_α so that every point of F_α is within a preassigned distance of $F'_\alpha +$ MIS (f). Therefore, since $\{h_1^{-1}h : h \in X\}$ is a compact set of homeomorphisms, and the compact set $|\{h_1^{-1}h($MIS $(f)) : h \in X\}|$ is disjoint from $h_1^{-1}(|SEP|)$, we may assume that the set $|\{h_1^{-1}h(F_\alpha) : h \in X\}|$ is disjoint from $h_1^{-1}(|SEP|)$, for each F_α. Hence, if J_α is the disk bounded by $F_\alpha + F'_\alpha$, we have that $|\{h_1^{-1}h(J_\alpha) : h \in X\}|$ misses $h_1^{-1}(|SEP|)$, for each J_α. Fix a J_α. Now if we let Z, Z_1, Z_2, Z_3, Y be represented by H'_1, H'_1, J_α, $h_1^{-1}(|SEP|)$, $\{h_1^{-1}h : h \in X\}$, resp., then by Lemma 12(a), there is a homotopy $\psi_1 : X \times I \to H(H'_1)$ such that for all $h \in X$, $\psi_1(h, 0) = h_1^{-1}h$, $\psi_1(h, 1)|J_\alpha = \text{Id}$, and ψ_1 has the other two required properties. Let $X_1 = \psi_1(X \times \{1\})$. We may assume by induction that if $h \in X_1$, then $h|J_\alpha = \text{Id}$ for all J_α. Let H''_1 be the closure of $H'_1 - \sum J_\alpha$. Then $|PI| + A + h_1^{-1}(|SEP|) \subset H''_1$, and $h = \text{Id}$ on Bd (H''_1), all $h \in X_1$. Now if we let Z, A, $\{\alpha_1, \ldots, \alpha_{n-1}\}$, $\{\gamma_1, \ldots, \gamma_m\}$ be represented by H''_1, A, PI, $\{h_1^{-1}(\beta) : \beta \in SEP\}$,

resp., then by Lemma 11(c) the set X_1 may be shrunk to the Id in the required way. Thus $\pi_{12}^{-1}(f, t)$ is homotopically trivial. (This step for Lemma 8 is somewhat easier; $h_1^{-1}(|SEP|)$ may be replaced by $|SEP|$.)

A proof using only the Alexander map AL of Lemma 1 shows that $\pi_{12}^{-1}(f, t)$ is LC^n for all n, all $(f, t) \in K_2 \times I$.

We prove that π_{12} is open and equi-LC^n, for all n, at the same time. Fix $(f, t_1, h_1) \in LIF$, an integer n, and a positive number ε. Assume ε is small enough so that if $(g, t, h) \in K_2 \times I \times H(H_1')$, and dist $((f, t_1, h_1), (g, t, h)) < \varepsilon$, then $h(\text{MIS}(g)) \cap |SEP| = \varnothing$. Let W be an $\varepsilon/4$ neighborhood of (f, t_1, h_1). Since $\pi_{12}^{-1}(f, t_1)$ is LC^n, we may choose a δ neighborhood U' of (f, t_1, h_1), $\delta < \varepsilon/4$, so that any map of a k-sphere, $k \leq n$, into $U' \cap \pi_{12}^{-1}(f, t_1)$ is homotopic in $W \cap \pi_{12}^{-1}(f, t_1)$ to a constant. By Lemma 13 and the continuity of λ, there is a δ_1 neighborhood U'' of (f, t_1, h_1), $\delta_1 < \delta/4$, such that if $(g, t) \in K_2 \times I$, and dist $((g, t), (f, t_1)) < \delta_1$, then there is a map $h_2 \in H(H_1')$ such that $h_2|A = \lambda(g, t)$, and dist $(h_1, h_2) < \delta/2$. (Thus $(g, t, h_2) \in LIF$, and $(g, t, h_2) \in U'$, hence π_{12} is an open map.) Now let α be a map of a k-sphere S^k into $\pi_{12}^{-1}(g, t) \cap U''$ for some $(g, t) \in K_2 \times I$. Choose h_2 for (g, t) as above. Define a map $\gamma: \pi_{12}^{-1}(g, t) \to \pi_{12}^{-1}(f, t_1)$ by $\gamma(g, t, h) = (f, t_1, h_1 h_2^{-1} h)$. Since dist $(h_1 h_2^{-1}(x), x) < \delta/2$ for all $x \in H_1'$, we see that $\gamma\alpha$ maps S^k into $U' \cap \pi_{12}^{-1}(f, t_1)$. Hence there is an extension $\bar{\alpha}$ mapping $C(S^k)$ into $W \cap \pi_{12}^{-1}(f, t_1)$. But then $\gamma^{-1}\bar{\alpha}$ is the required mapping into $\pi_{12}^{-1}(g, t)$ and the ε neighborhood of (f, t_1, h_1).

Hence, by Theorem 6, the map π_{12} has a right inverse.

The proof of Lemma 9 is complete.

Proof of Lemma 10. (We assume the reader is familiar with the notation used in Step D of the proof of Theorem 3.) That the spaces VER, K_3, and the map π_1 satisfy the hypotheses of Theorem 6 is easily verified. K_3, being the homeomorphic image of K, is finite dimensional. VER, being a closed subset of the complete metric space $K_3 \times H(D_1)$, is a complete metric space. π_1 is onto by the Schoenflies theorem. For each $f \in K_3$, $\pi_1^{-1}(f)$ is LC^n, all n, and homotopically trivial by application of the Alexander map AL of Lemma 1. π_1 is open, and $\{\pi_1^{-1}(f)\}_{f \in K_3}$ is equi-LC^n, all n, by Lemma 13 and the fact that $\pi_1^{-1}(f)$ is LC^n, all $f \in K_3$. This completes the proof.

LEMMA 14. *Suppose K is a compact subset of $H(D)$, ε a positive number, and $\psi_1: C(K) \to H(D)$ a map such that $\psi_1(f, 0) = f$, all $f \in K$, and for each $t \in [0, 1)$, $\psi_1|K \times \{t\}$ is a homeomorphism. Then there is an embedding $\psi: C(K) \to H(D)$ such that $\psi(f, 0) = f$, all $f \in K$, and dist $(\psi_1(f, t), \psi(f, t)) < \varepsilon$ for all $(f, t) \in C(K)$.*

Proof. Let $X = \psi_1(C(K))$. X is compact, so by Ascoli's theorem [19, Chapter 4, §25] there is a positive number δ, $\delta < \varepsilon/2$, so that if $x, y \in D$ and dist $(x, y) < \delta$, then dist $(f(x), f(y)) < \varepsilon/2$ for all $f \in X$. We think of the disk D as being the set $\{(x, y) \in R^2 : 0 \leq x \leq 1, -1 \leq y \leq 1\}$, and denote the open arc $\{(x, 0) : 0 < x < \delta/3\}$ by A. Choose a sequence $\{p_n\}_{n=1}^{\infty}$ of points of A such that $\pi_1(p_1) > \pi_1(p_2) > \cdots$, and $\{\pi_1(p_n)\} \to 0$ ($\pi_1: D \to I$ is projection on first coordinate). Let $F \in H(D)$ be a

homeomorphism such that dist $(F, \text{Id}) < \delta/3$, and for each integer n, $0 < \pi_1(F(p_n))$ $< \inf \{\pi_1(f^{-1}(p_n)) : f \in X\}$. (Define F in an order preserving manner from $\{p_n\}$ into A; extend linearly to a homeomorphism of A onto itself; extend to D by "coning".) Then for all $f \in X$, $f \circ F$ is not the identity in any neighborhood of the left side of Bd (D).

Let $S: R^1 \to R^1$ be the homeomorphism defined by $S(x) = x$ if $\delta/3 \leq x$, $S(x) = 2x - \delta/3$ if $\delta/6 \leq x < \delta/3$, $S(x) = x - \delta/6$ if $x < \delta/6$. Let $S_t: R^1 \to R^1$ be the isotopy defined by $S_t(x) = tS(x) + (1-t)x$, all $t \in I$. Finally, let $R_t: R^2 \to R^2$ be the isotopy defined by $R_t(x, y) = (S_t(x), y)$, all $t \in I$. Note that dist $(R_t, \text{Id}) < \delta/3$, all $t \in I$.

The required embedding $\psi: C(K) \to H(D)$ is given by

$$\psi(f, t) = R_t^{-1} \circ \psi_1(f, t) \circ F \circ R_t \circ F^{-1}, \quad \text{all } (f, t) \in C(K).$$

(We extend $\psi_1(f, t) \circ F$ outside D by the identity.) It is easily checked that dist $(\psi_1(f, t), \psi(f, t)) < \varepsilon$, all $(f, t) \in C(K)$. If $f, g \in K$, $f \neq g$, $t \in [0, 1)$, then $\psi_1(f, t) \neq \psi_1(g, t)$, so $\psi(f, t) \neq \psi(g, t)$. Suppose now $t, s \in I$, $f, g \in K$, and $s < t$. Let l be the left side of Bd (D); let $f_1 = R_s^{-1} \circ \psi_1(f, s) \circ F \circ R_s$; and let $g_1 = R_t^{-1} \circ \psi_1(g, t) \circ F \circ R_t$. Then $f_1 \neq \text{Id}$ in any neighborhood of $R_s^{-1}(l)$, but $g_1 = \text{Id}$ in some neighborhood of $R_s^{-1}(l)$. Hence $f_1 \neq g_1$ hence $\psi(f, s) \neq \psi(g, t)$. Hence ψ is an embedding and the proof of Lemma 14 is complete.

6. A corollary.   An interesting consequence of Theorem 4 is

COROLLARY 15. *Suppose K_1 and K_2 are compact subsets of $H(D)$, and α is a homeomorphism of K_1 onto K_2. Then α may be extended to a homeomorphism of $H(D)$ onto itself.*

We sketch the proof. Let l_2 be separable Hilbert space. By [10], $H(D)$ is homeomorphic to $l_2 \times l_2 \times H(D)$. Let p be a fixed point of $l_2 \times H(D)$. For each i, $i = 1, 2$, let f_i be a homeomorphism of $l_2 \times l_2 \times H(D)$ onto itself such that $f_i(K_i) \subset \{0\} \times l_2 \times H(D)$. The existence of f_i follows from [2, Corollary 10.2], since $\pi_{12}(K_i)$ is a compact subset of $l_2 \times l_2$ (π_{12} is projection on first two coordinates). For each i, $i = 1, 2$, there is an embedding $g_i : f_i(K_i) \to l_2 \times \{p\}$ [14, Theorem 2-46]. Since l_2 is an AR, g_i may be extended to a *map* of $\{0\} \times l_2 \times H(D)$ into $l_2 \times \{p\}$. By Theorem 4 of this paper $g_i^{-1} : g_i f_i(K_i) \to \{0\} \times l_2 \times H(D)$ may be extended to a map of $l_2 \times \{p\}$ into $\{0\} \times l_2 \times H(D)$. But then, by [5, Lemma 1, Theorem 1], g_i may be extended to a homeomorphism of $l_2 \times l_2 \times H(D)$ onto itself. Identify g_i and its extension. By [2, Corollary 10.3], there is a homeomorphism h of $l_2 \times \{p\}$ onto itself such that $h | g_1 f_1(K_1) = g_2 f_2 \alpha f_1^{-1} g_1^{-1}$. Extend h to a homeomorphism of $l_2 \times l_2 \times H(D)$ onto itself, and identify h with its extension. Then the required homeomorphism of $l_2 \times l_2 \times H(D)$ onto itself is $f_2^{-1} g_2^{-1} h g_1 f_1$.

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