ON THE SUMMATION FORMULA OF VORONOI(1)

BY

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Abstract. A formula involving sums of the form \( \sum d(n)f(n) \) and \( \sum d(n)g(n) \) is derived, where \( d(n) \) is the number of divisors of \( n \), and \( f(x) \), \( g(x) \) are Hankel transforms of each other. Many forms of such a formula, generally known as Voronoi's summation formula, are known, but we give a more symmetrical formula. Also, the reciprocal relation between \( f(x) \) and \( g(x) \) is expressed in terms of an elementary kernel, the cosine kernel, by introducing a function of the class \( L^2(0, \infty) \). We use \( L^2 \)-theory of Mellin and Fourier-Watson transformations.

Introduction. In 1904 Voronoi [10] published the following general formula: If \( \tau(n) \) is an arithmetic function and \( f(x) \) is continuous and has a finite number of maxima and minima in \( a < x < b \), then analytic functions \( a(x) \) and \( \delta(x) \), dependent on \( \tau(n) \), can be determined such that

\[
\frac{1}{2} \sum_{n \geq a}^{b} \tau(n)f(n) + \frac{1}{2} \sum_{n \leq a}^{b} \tau(n)f(n) = \int_{a}^{b} f(x) \delta(x) \, dx + 2\pi \sum_{n=1}^{\infty} \tau(n) \int_{a}^{b} f(x)a(nx) \, dx.
\]

One of the better known special cases of this formula is when \( \tau(n) = d(n) \), the number of divisors of \( n \), and

\[
a(x) = (2/\pi)K_0(4\pi x^{1/2}) - Y_0(4\pi x^{1/2}), \quad \delta(x) = \log x + 2\gamma,
\]

\( \gamma \) being Euler's constant and \( Y_0, K_0 \) denote Bessel functions of second and third kinds respectively, of order zero. This special case is generally known as Voronoi's summation formula. Later, this formula received considerable attention as a result of which many modifications were put forth by A. L. Dixon and W. L. Ferrar [2], J. R. Wilton [13], A. P. Guinand [3] and others. Most of the authors used complex analysis and in all the new forms of the Voronoi formula, the kernel used was a combination of the Bessel functions \( Y_0(x) \) and \( K_0(x) \).

Our object in this paper is to obtain a more symmetric and simplified form of Voronoi's formula, which holds under simple conditions. We state below the main result. First, a definition, due to Miller [6] and Guinand [4].

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**Definition.** A function \( f(x) \in G_2^\lambda(0, \infty) \) if and only if, for a fixed \( \lambda > 1/p \) and \( p > 1 \), there exists almost everywhere a function \( f^{(\lambda)}(x) \), such that

(i) \[
f(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} f^{(\lambda)}(t) \, dt, \quad x > 0,
\]
and

(ii) \[
x^\lambda f^{(\lambda)}(x) \in L^p(0, \infty).
\]

The function \( f^{(\lambda)}(x) \) is the \( \lambda \)th derivative (apart from a factor \((-1)^\lambda\)) of \( f(x) \) when \( \lambda \) is an integer. It can be shown that if \( f(x) \in G_2^\lambda(0, \infty) \), then

\[
x^r + \frac{1}{2} f^{(\lambda)}(x) \to 0 \quad \text{as} \quad x \to 0 \text{ or } \infty, \quad 0 \leq r < \lambda,
\]
and that \( G_2^\lambda \) is a subclass of \( L^2 \). In this paper we shall use the class \( G_2^\lambda(0, \infty) \). The properties (i) and (ii), in this case, simply mean that (i) \( f(x) \) is the integral of its derivative \( f'(x) \) (apart from the factor \(-1\)) and (ii) \( xf'(x) \in L^2(0, \infty) \).

**Main Theorem.** Let \( \phi(x) \in G_2^\lambda(0, \infty) \). Then there exist functions \( f(x) \) and \( g(x) \), both \( \in G_2^\lambda(0, \infty) \), defined by

\[
f(x) = 2 \int_0^\infty \phi(t) \cos 2\pi xt \, dt, \quad x > 0,
\]
and

\[
g(x) = 2 \int_0^\infty \frac{1}{t} \phi\left(\frac{1}{t}\right) \cos 2\pi xt \, dt, \quad x > 0,
\]
such that

\[
\lim_{N \to \infty} \left\{ \sum_{n=1}^N d(n)f(n) - \int_0^N (\log t + 2\gamma)f(t) \, dt \right\}
= \lim_{N \to \infty} \left\{ \sum_{n=1}^N d(n)g(n) - \int_0^N (\log t + 2\gamma)g(t) \, dt \right\},
\]
where \( \gamma \) is Euler’s constant.

This symmetric form of Voronoi’s formula could be derived from a general formula [3] of A. P. Guinand, if we had used the kernel \(-Y_0(4\pi x^{1/2}) + (2/\pi)K_0(4\pi x^{1/2})\) and employed sophisticated order results. In our proof we make use of easily derived and elementary results, using the theory of mean convergence of functions of \( L^2(0, \infty) \).

**Definition 2.** A kernel \( k(x) \in D^2 \) if and only if

(i) there is defined a.e. in \((-\infty, \infty)\) a function \( K(\frac{1}{2} + it) \), such that \( |K(\frac{1}{2} + it)| = 1 \), \( K(\frac{1}{2} + it)K(\frac{1}{2} - it) = 1 \);

(ii) the function \( k_1(x) \), defined a.e. by

\[
\frac{k_1(x)}{x} = \frac{1}{2\pi i} \text{l.i.m.} \int_{\frac{1}{2}+it}^{\frac{1}{2}+it} \frac{K(s)}{1-s} x^{-s} \, ds,
\]

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may be chosen, so that
(a) $k_1(x)$ is differentiable, $k_1(x)=\int k(t) dt$,
(b) $k_\infty(x)$ is $O(x^{1/2})$, $x \to \infty$, and $O(x^{1/2})$, $x \to 0$,
(c) $k(x) \in L(1/n, n)$, for all finite $n > 0$.
Such a class of kernels is due to J. B. Miller [7].

The following results can be deduced from the functional relations and expansions of Bessel functions $Y_n(x)$ and $K_n(x)$ [12, pp. 62–80]. If $L_n(x)=-Y_n(x)-(2/\pi)K_n(x)$ and $M_n(x)=-Y_n(x)+(2/\pi)K_n(x)$, then

\begin{equation}
(d/dx)(xL_1(x)) = xM_0(x).
\end{equation}

\begin{equation}
L_1(x) = O(x^{-1/2}), \quad x \to \infty,
\end{equation}

and $=O(x \log x)$, as $x \to 0$.

2. Preliminary results. Consider the function

\begin{equation}
h(x) = \left\{ \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1) \right\} x^{-1}.
\end{equation}

Since [8, p. 262]

\[ \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1) = O(x^{1/2}), \quad x \to \infty, \]

therefore

\begin{equation}
h(x) = O(x^{-1/2}), \quad x \to \infty,
\end{equation}

\begin{equation}
= O(\log x), \quad x \to 0.
\end{equation}

Then its Mellin transform

\[ H(s) = \int_0^\infty h(x) x^{s-1} dx \quad (s = \sigma + it) \]

exists for $0 < \sigma < \frac{1}{2}$. Or

\[ H(s) = \int_1^\infty h(x) x^{s-1} dx + \int_1^\infty h(x) x^{s-1} dx, \quad 0 < \sigma < \frac{1}{2}, \]

\[ = \frac{1}{s^2} - \frac{2\gamma - 1}{s} + \int_1^\infty h(x) x^{s-1} dx, \quad \sigma < \frac{1}{2}. \]

This gives the analytic continuation into $\sigma < 0$. Now

\[ \int_1^\infty h(x) x^{s-1} dx = \int_1^\infty \sum_{n \leq x} d(n) x^{s-2} dx - \int_1^\infty (\log x + 2\gamma - 1) x^{s-1} dx. \]

By splitting the range of integration $(1, \infty)$ into $(1, 2), (2, 3), \ldots$ and solving, we get

\[ \int_1^\infty \sum_{n \leq x} d(n) x^{s-2} dx = \frac{1}{1-s} \sum_{n=1}^\infty d(n) n^{s-1} = \frac{\zeta^2(1-s)}{1-s}, \]

where $\zeta(x)$ is the Riemann-zeta function.
Now, for \( \sigma < 0 \),

\[
\int_1^\infty (\log x + 2\gamma - 1)x^{\sigma-1} \, dx = \frac{1}{s^2} - \frac{2\gamma - 1}{s}.
\]

Hence, by analytic continuation, we obtain

\[
(2.3) \quad H(s) = \zeta^2(1-s)/(1-s) \quad (0 < \sigma < \frac{1}{2}).
\]

Since \( x^{-\sigma}h(x) \in L^2(0, \infty) \), \( 0 < \sigma < \frac{1}{2} \), by Mellin’s inversion formula

\[
\frac{1}{2}\{h(x+0) + h(x-0)\} = \lim_{\gamma \to \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} \zeta^2(s) x^{-s} \, ds.
\]

Next we shall show that \( H(s) \in L^2(-\infty, \infty) \) on \( s = \frac{1}{2} + it \) and deduce that

\[ h(x) \in L^2(0, \infty). \]

Now [8, p. 92]

\[ \zeta^{1/2 + it} = O(t^{1/8} \log t), \quad t \to \infty. \]

Therefore \( \zeta^2(1-s)/(1-s) \in L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty) \) and has a Mellin transform \( h_1(x) \), say, belonging to \( L^2(0, \infty) \), defined by

\[ h_1(x) = \frac{1}{2\pi i} \lim_{\gamma \to \infty} \int_{1/2-iT}^{1/2+iT} \frac{\zeta^2(1-s)}{1-s} x^{-s} \, ds \]

a.e. for \( x > 0 \). Let \( C \) be the contour \( (\sigma - iT, \frac{1}{2} - iT, \frac{1}{2} + iT, \sigma + iT, \sigma - iT) \). By Cauchy’s Theorem

\[ \int_C \frac{\zeta^2(1-s)}{1-s} x^{-s} \, ds = 0, \quad 0 < \sigma < \frac{1}{2}, \]

the integrals along the lines \( (\sigma - iT, \frac{1}{2} - iT) \) and \( (\frac{1}{2} + iT, \sigma + iT) \) vanish as \( T \to \infty \), since [8, p. 82] \( \zeta(\sigma + it) = O(t^{1/2-\sigma/2}) \), \( 0 < \sigma < 1 \).

We have then

\[ \lim_{T \to \infty} \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} \frac{\zeta^2(1-s)}{1-s} x^{-s} \, ds = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \frac{\zeta^2(1-s)}{1-s} x^{-s} \, ds \]

a.e. Or, \( h_1(x) = h(x) \) a.e. and hence \( h(x) \in L^2(0, \infty) \).

Let us define a function

\[
(2.4) \quad A(x) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{1/2-iT}^{1/2+iT} \frac{\phi(s)}{1-s} x^{s-1} \, ds,
\]

where \( \phi(s) = \psi(1-s)/\psi(s) \) and \( \psi(s) = \sum_{n=1}^{\infty} d(n)n^{-s} \).

Thus \( \psi(s) = \zeta^2(s) \) and using the functional equation

\[ \zeta(s) = 2^{s\pi^{-1}}(\sin \frac{1}{2}s\pi)\Gamma(1-s)\zeta(1-s) \]

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we obtain
\[(2.5) \quad \mathcal{X}(s) = 4(2\pi)^{-2s} \Gamma^2(s) \cos^2 \frac{1}{2} s \pi.\]

Now
\[(2.6) \quad |\mathcal{X}(\frac{1}{2} + it)| = 1, \quad |\mathcal{X}(\frac{1}{2} + it)| \mathcal{X}'(\frac{1}{2} - it) = 1\]

and consequently, on the line \(s = \frac{1}{2} + it, |\mathcal{X}(s)/(1-s)| = O(t^{-1})\) and thus belongs to \(L^2(-\infty, \infty)\) when integrated with respect to \(t\). Hence the integral (2.4) converges in mean square. Also, \(x^{-1} A(x) \in L^2(0, \infty)\) and \(A(x)\) is a Fourier kernel in Watson's sense [11].

Substituting the value of \(\mathcal{X}(s)\), obtained above, in (2.4), we have
\[(2.7) \quad A(x) = \lim_{t \to \infty} \frac{1}{2\pi i} \int_{1/2 - i \infty}^{1/2 + i \infty} 4(2\pi)^{-2s} \Gamma(s) \Gamma(s-1) \cos^2 \frac{\pi}{2} s \pi \cdot x^{1-s} \, ds.\]

We shall now evaluate the above integral. It is known [9, p. 195] that for \(1 < \sigma < \frac{3}{4}\)
\[\frac{1}{2\pi i} \int_{a - i \infty}^{a + i \infty} (\pi)^{-1} (2\pi)^{1-2s} \Gamma(s) \Gamma(s-1) \cos \pi s \cdot x^{-s} \, ds = x^{-1/2} Y_1(4\pi x^{1/2}).\]

Moving the line of integration to \(\sigma = \frac{1}{2}\) and by applying the theory of residues we get
\[(2.8) \quad \frac{1}{2\pi i} \int_{1/2 - i \infty}^{1/2 + i \infty} (\pi)^{-1} (2\pi)^{1-2s} \Gamma(s) \Gamma(s-1) \cos \pi s \cdot x^{-s} \, ds = x^{-1/2} \left( \frac{2}{\pi} K_1(4\pi x^{1/2}) + (2\pi x)^{-1} \right).\]

Also, [9, p. 197] for \(\sigma > 1\)
\[\frac{1}{2\pi i} \int_{a - i \infty}^{a + i \infty} (\pi)^{-1} (2\pi)^{1-2s} \Gamma(s) \Gamma(s-1) x^{-s} \, ds = \frac{2}{\pi} x^{-1/2} K_1(4\pi x^{1/2}).\]

Moving the line of integration to \(\sigma = \frac{1}{2}\), we have
\[(2.9) \quad \frac{1}{2\pi i} \int_{1/2 - i \infty}^{1/2 + i \infty} (\pi)^{-1} (2\pi)^{1-2s} \Gamma(s) \Gamma(s-1) x^{-s} \, ds = \frac{2}{\pi} x^{-1/2} K_1(4\pi x^{1/2}) - (2\pi x)^{-1}.\]

Now from (2.8) and (2.9),
\[-\frac{1}{2\pi i} \int_{1/2 - i \infty}^{1/2 + i \infty} \frac{2}{\pi} (2\pi)^{1-2s} \Gamma(s) \Gamma(s-1) \cos^2 \pi s \cdot x^{-s} \, ds = - x^{-1/2} \left( Y_1(4\pi x^{1/2}) + (2/\pi) K_1(4\pi x^{1/2}) \right).\]

Thus (2.7) yields \(A(x) = x^{1/2} L_1(4\pi x^{1/2}).\)

Note that \(A(x)\) is differentiable, and let \(A(x) = \int_0^x \chi(t) \, dt\), from whence \(\chi(x) = 2\pi M_0(4\pi x^{1/2})\) by (1.2). From (1.3) and (2.6) we see that all relevant conditions are satisfied and therefore \(\chi(x)\) belongs to the kernel class \(D^2\).

Further, let
\[(2.10) \quad F(x) = \lim_{t \to \infty} \frac{1}{2\pi i} \int_{1/2 - i t}^{1/2 + i t} \frac{s \mathcal{X}(s)}{(1-s)(2-s)} x^{1-s} \, ds.\]
From (2.6), \(|s\mathcal{H}(s)/(1-s)(2-s)| = O(t^{-1})\), therefore the integral (2.10) converges in mean square and \(x^{-1}F(x) \in L^2(0, \infty)\). Thus \(F(x)\) is a generalized Hankel kernel [11].

**Lemma 2.1.** Let \(h(x)\) be defined by (2.1). Then

\[
\int_0^x t h(t) \, dt = x \int_0^\infty h(t) \frac{F(xt)}{t} \, dt,
\]

where \(F(x)\) is the generalized Hankel kernel defined by (2.10).

**Proof.** Applying Parseval's theorem to \(L^2\)-functions \(h(x)\) and \(x^{-1}F(x)\), we have

\[
x \int_0^\infty h(t) \frac{F(xt)}{t} \, dt = \frac{x}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{H(1-s) \mathcal{H}(s)}{(1-s)(2-s)} \, x^{1-s} \, ds,
\]

which, by (2.3) and (2.5), is

\[
\frac{1}{2\pi} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\zeta^2(1-s)}{(1-s)(2-s)} \, x^{2-s} \, ds = \int_0^x t h(t) \, dt,
\]

as required.

Thus we can say that \(h(x)\) is the \(F\)-transform of itself.

**Lemma 2.2.** Let \(f(x) \in G^2_1(0, \infty)\). Then there exists \(g(x) \in G^2_1(0, \infty)\), such that

\[
g(x) = 2\pi \int_0^\infty f(t) \chi(xt) \, dt, \quad x > 0,
\]

and

\[
f(x) = 2\pi \int_0^\infty g(t) \chi(xt) \, dt, \quad x > 0.
\]

Further \(xf'(x)\) and \(xg'(x)\) are \(F\)-transforms of each other. Here \(\chi(x) = 2\pi M_0(4\pi x^{1/2})\).

**Proof.** The first part is immediate by a result due to J. B. Miller [6], since the kernel \(\chi(x) \in D^2\). The second part can be proved by the same method as used in the proof of Lemma 2.1.

**Lemma 2.3.** Let \(\phi(x) \in G^2_1(0, \infty)\) and define \(f(x)\) by the equation

\[
(2.11) \quad f(x) = 2 \int_0^\infty \phi(t) \cos 2\pi xt \, dt, \quad x > 0.
\]

Then \(f(x) \in G^2_1(0, \infty)\). Further, if a function \(g(x)\) is defined by

\[
(2.12) \quad g(x) = 2 \int_0^\infty \frac{1}{t} \phi\left(\frac{1}{t}\right) \cos 2\pi xt \, dt, \quad x > 0,
\]

then \(g(x) \in G^2_1(0, \infty)\).

**Proof.** It can be seen that \(2 \cos 2\pi x \in D^2\). Thus by Theorem I of J. B. Miller [6], \(f(x) \in G^2_1(0, \infty)\), since \(\phi(x) \in G^2_1(0, \infty)\). Similarly \(g(x) \in G^2_1(0, \infty)\), provided we
can show that $(1/x)\phi(1/x) \in G_2^2(0, \infty)$ when $\phi(x)$ does. Now,

$$\frac{d}{dx} \left\{ \frac{1}{x} \phi \left( \frac{1}{x} \right) \right\} = -\frac{1}{x} \phi \left( \frac{1}{x} \right) - \frac{1}{x^2} \phi \left( \frac{1}{x} \right).$$

Since $\phi(x) \in G_2^2$, by property (ii), $(1/x)\phi(1/x)$ and $(1/x^2)\phi'(1/x)$ belong to $L^2(0, \infty)$, and using Minkowski's inequality, we can show that $x(d/dx)((1/x)\phi(1/x))$ also belongs to $L^2(0, \infty)$.

Also,

$$\phi(x) = \frac{1}{x} \int_0^x \frac{d}{dt} \{t\phi(t)\} \, dt.$$

Or,

$$\frac{1}{x} \phi \left( \frac{1}{x} \right) = \int_0^{1/x} \{\phi(t) + t\phi'(t)\} \, dt$$

$$= \int_x^{\infty} \left\{ \frac{1}{u^2} \phi \left( \frac{1}{u} \right) + \frac{1}{u^3} \phi' \left( \frac{1}{u} \right) \right\} \, du = -\int_x^{\infty} \frac{d}{du} \left\{ \frac{1}{u} \phi \left( \frac{1}{u} \right) \right\} \, du.$$

Thus $(1/x)\phi(1/x)$ is the integral of its derivative, and hence $(1/x)\phi(1/x) \in G_2^2(0, \infty)$. This proves the lemma.

3. The Main Theorem. Applying Parseval's theorem [1] for the two pairs $h(x), h(x)$ and $xf'(x), xg'(x)$ of $F$-transforms of the class $L^2(0, \infty)$, we have

$$(3.1) \int_0^\infty xh(x)f'(x) \, dx = \int_0^\infty xh(x)g'(x) \, dx.$$

The left-hand side is

$$\int_0^{\infty} \left\{ \sum_{n \geq x} d(n) - x(\log x + 2\gamma - 1) \right\} f'(x) \, dx$$

$$= \lim_{N \to \infty} \left\{ \left[ \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1) \right] f(x) \right\} \bigg|_0^N$$

$$- \int_0^N f(x) \, d \left( \sum_{n \leq x} d(n) \right) + \int_0^N (\log x + 2\gamma) f(x) \, dx.$$

Since $f(x)$ and $h(x)$ satisfy (1.1) and (2.2) respectively, the integrated term vanishes at both limits, and the above expression reduces to

$$\lim_{N \to \infty} \left\{ - \sum_{n=1}^N d(n)f(n) + \int_0^N (\log x + 2\gamma)f(x) \, dx \right\}.$$

Treating the right-hand side of (3.1) in the same manner, we obtain

**Theorem 3.1.** Let $f(x) \in G_2^2(0, \infty)$. If $g(x)$ is defined by

$$g(x) = 2\pi \int_0^\infty f(t)x(t) \, dt$$
then \( g(x) \) belongs to \( G_2^2(0, \infty) \), where \( \chi(x) = 2\pi M_0(4\pi x^{1/2}) \). Further

\[
\lim_{N \to \infty} \left\{ \sum_{n=1}^{N} d(n)f(n) - \int_0^{\infty} (\log x + 2\gamma)f(x) \, dx \right\} = \lim_{N \to \infty} \left\{ \sum_{n=1}^{N} d(n)g(n) - \int_0^{\infty} (\log x + 2\gamma)g(x) \, dx \right\}.
\]

**Theorem 3.2.** Let \( \phi(x) \in G_2^2(0, \infty) \). If there exist functions \( f(x) \) and \( g(x) \) defined by the equations (2.11) and (2.12), then the equations

\[
f(x) = \int_0^{\infty} g(t)x(\pi t) \, dt, \quad g(x) = \int_0^{\infty} f'(t)x(\pi t) \, dt
\]

hold for \( x > 0 \), where \( \chi(x) = 2\pi M_0(4\pi x^{1/2}) \).

**Proof.** Integrating by parts the integral in (2.11), we get

\[
f(x) = \left[ \phi(t) \frac{\sin 2\pi xt}{\pi x} \right]_{-\infty}^{\infty} - \int_0^{\infty} \phi'(t) \frac{\sin 2\pi xt}{\pi x} \, dt
\]

(3.2)

\[
= -\int_0^{\infty} t\phi'(t) \frac{\sin 2\pi xt}{\pi xt} \, dt.
\]

The integrated term vanishes by (1.1) since \( \phi(x) \in G_2^2(0, \infty) \). If \( \Phi(s) \) denotes the Mellin transform of \( \phi(x) \), then \(-s\Phi(s)\) is the Mellin transform of \( x\phi'(x) \). Now, we know that \( t\phi'(t) \) and \( (\sin 2\pi xt)/\pi xt \) both belong to \( L^2(0, \infty) \). Therefore by applying, to the right side of (3.2), the Parseval theorem for Mellin transforms of \( L^2 \)-functions, we obtain

\[
f(x) = -\frac{1}{\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} s\Phi(s)(2\pi x)^{s-1} \Gamma(-s) \sin \frac{s\pi}{2} \, ds.
\]

(3.3)

Now, from (2.12),

\[
\int_0^{\infty} g(u) \, du = \frac{1}{\pi} \int_0^{\infty} \phi \left( \frac{1}{t^2} \right) \sin \frac{2\pi xt}{t^2} \, dt.
\]

Let \( G(s) \) be the Mellin transform of \( g(x) \). It can be shown easily that \( \phi(1-s) \) is the Mellin transform of \( (1/x)\phi(1/x) \) and \( x^s/s \) is the Mellin transform of the function \( 1, 0 < u < x; 0, u > x \). Applying the Parseval theorem for Mellin transforms to both sides of the last equation, we get

\[
\frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} G(s) \frac{x^{1-s}}{1-s} \, ds = -\frac{1}{\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \Phi(s)(2\pi)^{-s} \Gamma(s-1) \cos \frac{s\pi}{2} x^{1-s} \, ds.
\]

Or,

\[
\frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \{G(s) - 2(2\pi)^{-s}\Phi(s)\Gamma(s)\cos \frac{s\pi}{2}\} \frac{x^{1-s}}{1-s} \, ds = 0,
\]

and, by Mellin inversion formula,

\[
G(s) = 2(2\pi)^{-s}\Phi(s)\Gamma(s) \cos \frac{s\pi}{2},
\]

(3.4)
a.e. on $R(s) = \frac{1}{2}$. Substituting the value of $\Phi(s)$ in (3.4) and using the functional equation $\Gamma(1-s) = \pi \csc \pi s$, we obtain from (3.3)

$$f(x) = -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{2}{(2\pi)^{2s-1}} \Gamma(-s) \Gamma(1-s) \sin^{\frac{1}{2}} 2\pi x^2 \, sG(s) \, ds$$

$$= -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} sG(s) \mathcal{L}(1-s) \, ds,$$

say, where

$$\mathcal{L}(s) = (2/\pi)(2\pi)^{1-2s} \Gamma(s-1) \cos^{2\frac{1}{2}} 2\pi x^{-s}.$$

It can be easily deduced from the value of the integral in (2.7) that $\mathcal{L}(s)$ is the Mellin transform of $-(x/)^{1/2} L_1(4\pi(x)^{1/2})$, when considered as a function of $t$. Now $xg(x)$ and $x^{-1/2} L_1(4\pi x^{1/2})$ both belong to $L^2(0, \infty)$ due to (1.1), as $g(x) \in G^2$, and (1.3). Thus applying Parseval's theorem to the above pair of $L^2$-functions, we obtain

$$\int_0^\infty \int_{1/2-i\infty}^{1/2+i\infty} sG(s) \mathcal{L}(1-s) \, ds = f(x).$$

Integrating the left-hand side by parts, we can write (3.5) as

$$f(x) = -\int_0^\infty t g'(t)(xt^{-1/2}L_1(4\pi(x)^{1/2})) \, dt = -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} sG(s) \mathcal{L}(1-s) \, ds = f(x).$$

The integrated term vanishes at both the limits by (1.1) and (1.3). Hence

$$f(x) = 2\pi \int_0^\infty g(t)M_0(4\pi(x)^{1/2}) \, dt, \quad x > 0,$$

as required. Similarly

$$g(x) = \int_0^\infty f(t)x(t) \, dt, \quad x > 0.$$

Finally, the main theorem stated in the introduction follows by combining the results obtained in Theorems 3.1 and 3.2.

4. An example. Let

$$f(x) = K_0(2\pi x), \quad R(z) > 0.$$ 

Then

$$\phi(x) = \int_0^\infty K_0(2\pi xt) \cos 2\pi xt \, dt$$

$$= \frac{1}{2}(x^2 + x^2)^{-1/2}, \quad \text{cf. [12, p. 388].}$$
Now define a function
\[ g(x) = 2 \int_0^\infty \frac{1}{t} \phi \left( \frac{1}{t} \right) \cos 2\pi x t \, dt \]
\[ = \int_0^\infty t^{-1} (z^2 + t^{-2})^{-1/2} \cos 2\pi x t \, dt = \int_0^\infty (1 + z^2 t^2)^{-1/2} \cos 2\pi x t \, dt \]
\[ = z^{-1} \int_0^\infty (1 + u^2)^{-1/2} \cos \frac{2\pi xu}{z} \, du = z^{-1} K_0 \left( \frac{2\pi x}{z} \right), \quad R(z) > 0, \]
cf. [12, p. 434]. Also,
\[
K_0(x) = O(x^{-1/2} e^{-x}), \quad x \to \infty, \\
= O(\log x), \quad x \to 0. 
\]
Thus \( K_0(2\pi x) \) and \( z^{-1} K_0(2\pi x/z) \), as function of \( x \), satisfy the conditions of the main theorem, which yields the formula
\[
\sum_{n=1}^\infty d(n) K_0(2\pi zn) - \int_0^\infty (\log t + 2\gamma) K_0(2\pi z t) \, dt \\
= z^{-1} \sum_{n=1}^\infty d(n) K_0 \left( \frac{2\pi n}{z} \right) - z^{-1} \int_0^\infty (\log t + 2\gamma) K_0 \left( \frac{2\pi t}{z} \right) \, dt. 
\]
We shall now evaluate the two integrals in (4.2). First consider
\[
I_1 = \int_0^\infty (\log t + 2\gamma) K_0(2\pi z t) \, dt \\
= \frac{1}{2\pi z} \left\{ (2\gamma - \log 2\pi z) \int_0^\infty K_0(u) \, du + \int_0^\infty \log u K_0(u) \, du \right\}. 
\]
Now [12, p. 388]
\[
\int_0^\infty K_0(u) \, du = \frac{\pi}{2}. 
\]
Let \( \int_0^\infty \log u K_0(u) \, du = I \), say.
It is known that [12, p. 172] \( K_0(z) = \int_1^\infty e^{-zt} (t^2 - 1)^{-1/2} \, dt \). Therefore
\[
I = \int_0^\infty \log u \, du \int_1^\infty e^{-zt} (t^2 - 1)^{-1/2} \, dt = \int_1^\infty (t^2 - 1)^{-1/2} \, dt \int_0^\infty \log u \, e^{-ut} \, du. 
\]
The inversion of order of integration is justified by absolute convergence. 
Now
\[ \int_0^\infty \log u \, e^{-ut} \, du = -t^{-1} \log (e^t), \]
y being Euler's constant. Thus
\[
I = -\int_1^\infty t^{-1} (t^2 - 1)^{-1/2} \log (e^t) \, dt \\
= -\gamma \int_1^\infty t^{-1} (t^2 - 1)^{-1/2} \, dt - \int_1^\infty t^{-1} (t^2 - 1)^{-1/2} \log t \, dt \\
= -\gamma \frac{\pi}{2} - \frac{\pi}{2} \log 2. 
\]
Hence from (4.3) and (4.4)

\[ I_1 = \frac{1}{2\pi z} \left( \frac{2\gamma - \log 2\pi z}{2} \pi \frac{\pi}{2} (\gamma + \log 2) \right) = (4\pi)^{-1}(\gamma - \log 4\pi z). \]

Next consider

\[ I_2 = z^{-1} \int_0^\infty (\log t + 2\gamma) K_0 \left( \frac{2\pi t}{z} \right) dt \]

\[ = \frac{1}{2\pi} \left( 2\gamma - \log \frac{2\pi}{z} \right) \int_0^\infty K_0(u) \, du + \frac{1}{2\pi} \int_0^\infty \log u K_0(u) \, du \]

\[ = \frac{1}{4}(2\gamma - \log (2\pi/z)) - \frac{1}{4}(\gamma + \log 2), \]

by (4.3) and (4.4). Thus \( I_2 = \frac{1}{4}(\gamma - \log (4\pi/z)). \) Substituting the values of the integrals \( I_1 \) and \( I_2 \) in (4.2) and rearranging the terms, we obtain

\[ \sum_{n=1}^{\infty} d(n)K(2\pi zn) - z^{-1} \sum_{n=1}^{\infty} d(n)K_0 \left( \frac{2\pi n}{z} \right) = \frac{1}{4}z^{-1}(\gamma - \log 4\pi z) - \frac{1}{4}(\gamma - \log (4\pi/z)), \]

which is a known formula due to N. S. Koshliakov [5].

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