TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 163, January 1972

INVERSE H-SEMIGROUPS AND t-SEMISIMPLE INVERSE H-SEMIGROUPS(1)

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Abstract. An H-semigroup is a semigroup such that both its right and left congruences are two-sided. A semigroup is t-semisimple provided the intersection of all its maximal modular congruences is the identity relation. We prove that a semigroup is an inverse H-semigroup if and only if it is a semilattice of disjoint Hamiltonian groups. Using the set E of idempotents of S as the semilattice, we show that an inverse H-semigroup S is t-semisimple if and only if for each pair of groups G_e, G_f in the semilattice, with $f \geqq e$ in E, the homomorphism $\varphi_{f,e}$ on G_f into G_e, defined by $a\varphi_{f,e}=ae$, is a monomorphism; and for each e in E, for each $a \neq e$ in G_e, there exists a subsemigroup T_p of S such that $a \notin T_p$ and, for each f in E, $T_p \cap G_f = H_f$, where $H_f = G_f$ or H_f is a maximal subgroup of prime index p in G_f.

Introduction. In this paper we adopt the definition of a Hamiltonian semigroup presented by R. H. Oehmke [6]. Let σ be an equivalence relation on a semigroup S. If a is equivalent to b we shall write $a \sigma b$. The σ-class containing a will be denoted by σ_a. An equivalence relation σ on a semigroup S is a right (left) congruence provided $a, b, c \in S$ and $a \sigma b$ imply $(ac) \sigma (bc)$ $((ca) \sigma (cb))$. If an equivalence relation is both a right and a left congruence, we shall call it a two-sided congruence or, more briefly, a congruence. We use the natural partial ordering on relations and say that $\sigma \leqq \rho$ if and only if $a, b \in S$ and $a \sigma b$ imply $a \rho b$. Clearly, the identity relation ι and the universal relation ν are congruences and $\iota \leqq \sigma \leqq \nu$ for each congruence σ on S. A congruence $\sigma \neq \nu$ is called maximal if for each congruence σ' on S such that $\sigma \leqq \sigma' \leqq \nu$, either $\sigma = \sigma'$ or $\sigma' = \nu$. An H-semigroup S is defined to be a semigroup such that every right congruence and every left congruence is a two-sided congruence on S. Since a subgroup of a group is normal if and only if its corresponding right (left) congruence is two-sided, then the class of H-semigroups contains the Hamiltonian groups in addition to the commutative semigroups, where we include all commutative groups in the set of all Hamiltonian groups. An inverse H-semigroup is a semigroup that is an inverse semigroup as well as an H-semigroup.

Received by the editors July 16, 1970.

AMS 1969 *subject classifications.* Primary 2092, 2093; Secondary 2058.

Key words and phrases. Semigroup, inverse semigroup, H-semigroup, t-semisimple semigroup, Hamiltonian group, semilattice, maximal congruence, modular congruence.

(1) These results are part of the author's doctoral dissertation written at the University of Iowa under the supervision of Professor Robert H. Oehmke. This research was partially supported by an NSF Science Faculty Fellowship and supported in part under ONR Contract N0014-68-A-0500.

Using the above definitions we prove in §2 that a semigroup is an inverse H-semigroup if and only if it is a semilattice of disjoint Hamiltonian groups.

We define τ to be the intersection of all the maximal modular congruences on a semigroup S, where a congruence σ is called modular if there is an element e of S such that $(ea)\ \sigma\ a$ and $(ae)\ \sigma\ a$ for all a in S. The element e is called an identity for σ. We refer to τ as the t-radical of S. S is said to be t-semisimple if $\tau=\iota$ [7]. In §3, we give necessary and sufficient conditions for an inverse H-semigroup S to be t-semisimple. This result has several nontrivial corollaries.

The author expresses her gratitude to Professor Robert H. Oehmke for his guidance in the preparation of this paper.

1. **Preliminary definitions and results.** An element b of a semigroup S is an inverse of an element a of S provided $aba=a$ and $bab=b$. S is an inverse semigroup provided every element of S has a unique inverse. The inverse of an element a of an inverse semigroup S will be denoted by a^{-1} so that $aa^{-1}a=a$ and $a^{-1}aa^{-1}=a^{-1}$. The preceding definitions are taken from [1].

We also make use of the following results which have been proved in [1, pp. 23–30]. Let S be an inverse semigroup. The set E of idempotents of S is a semilattice, i.e., a commutative idempotent semigroup with the induced ordering $e \leqq f$ if and only if $ef=e$. If $a, b \in S$ then $(a^{-1})^{-1}=a$ and $(ab)^{-1}=b^{-1}a^{-1}$. Every principal right ideal and every principal left ideal of S has a unique idempotent generator. The idempotent $e=aa^{-1}$ $(f=a^{-1}a)$ is the unique idempotent generator of aS (Sa).

A left (right) zero of a semigroup S is an element a of S such that $as=a$ $(sa=a)$, for each $s \in S$ [1]. Let c be a left (right) zero of an inverse semigroup S. Then for each $s \in S$, $csc=c$ implies $scs=sc$ $(scs=cs)$ is an inverse of c. But c has a unique inverse, namely c, so that, for each $s \in S$, $sc=c$ $(cs=c)$. Hence c is a right (left) zero of S and S has at most one (left, right) zero.

In [6] Oehmke proved the following result which we state as a lemma.

LEMMA 1.1. *If S is an H-semigroup and I is a right (left) ideal of S then, for any b in S, $bI \subseteq I$ $(Ib \subseteq I)$ or $bI=\{c\}$ where c is a left zero ($Ib=\{c\}$ where c is a right zero).*

We use this result to show that every one-sided ideal of an inverse H-semigroup S is two-sided and thus we obtain that S is a semilattice of disjoint groups.

LEMMA 1.2. *A right (left) ideal of an inverse H-semigroup S is two-sided.*

Proof. Let I be a right ideal of S and $b \in S$. By Lemma 1.1, either $bI \subseteq I$ or $bI=\{c\}$ where c is a left zero. If the latter is true, then, since c is also a right zero and I is a right ideal, we have $\{c\}=Ic \subseteq I$ so that $bI=\{c\} \subseteq I$ and I is a left ideal in either case. By a similar proof, any left ideal of S is a right ideal of S.

Let S be an inverse H-semigroup and e an idempotent of S. Since Se is an ideal and $e \in Se$, it follows that $Se=eS$. Then for any $a \in S$ we have $ae=a$ if and only if $ea=a$. But for $a \in S$ there exists a unique element $a^{-1} \in S$ such that aa^{-1} and

$a^{-1}a$ are idempotents. Thus we have $(aa^{-1})a=a$ so that $a(aa^{-1})=a$, and also $a=a(a^{-1}a)$ so that $(a^{-1}a)a=a$. Hence

$$a^{-1}a = a^{-1}(aaa^{-1}) = (a^{-1}aa)a^{-1} = aa^{-1}.$$

It is well known that if every element of an inverse semigroup S commutes with its inverse then S is a union of disjoint groups. Thus we have the following lemma:

LEMMA 1.3. *If S is an inverse H-semigroup then S is a union of disjoint groups.*

If Y is a semilattice such that $S=\bigcup\{S_\alpha : \alpha \in Y\}$ is a decomposition of S such that, for every pair of elements α, β of Y, there is an element γ of Y such that $S_\alpha S_\beta \subseteq S_\gamma$, we say that S is the union of the semilattice Y of semigroups S_α, $\alpha \in Y$. We also abbreviate the expression and say that S is a semilattice of semigroups of type $\mathscr{C}$ to mean that S is the union of the semilattice of semigroups S_α, $\alpha \in Y$, where each S_α is of type $\mathscr{C}$.

Let S be an inverse H-semigroup and let $G_e=\{b \in S : bb^{-1}=e\}$. It readily follows that G_e is a maximal subgroup of S and $S=\bigcup\{G_e : e \in E\}$, where $G_e \cap G_f=\varnothing$ for $e \neq f$. Using Lemma 1.3, we obtain the result that E is contained in the center of S [1, pp. 127–128] so that Theorem 1 follows.

THEOREM 1. *If S is an inverse H-semigroup then S is a semilattice of disjoint groups, and if $f \geqq e$ in E, the mapping $\varphi_{f,e}$, defined by $a\varphi_{f,e}=ae$ where $a \in G_f$, is a homomorphism of G_f into G_e. Also, $\varphi_{f,f}$ is the identity mapping of G_f and if $f \geqq e \geqq g$, then $\varphi_{f,e}\varphi_{e,g}=\varphi_{f,g}$. Moreover, every product in S is known, since for $a \in G_f$ and $b \in G_e$, $ab=(a\varphi_{f,fe})(b\varphi_{e,fe})$.*

2. In this section we shall obtain a characterization of inverse H-semigroups, namely:

THEOREM 2. *A semigroup S is an inverse H-semigroup if and only if S is a semilattice of disjoint Hamiltonian groups.*

Proof. Let δ be a right congruence on an inverse H-semigroup S. Let G_e be a maximal subgroup of S and let δ' be the restriction of δ to G_e. By a straightforward argument, it can be shown that there is a subgroup H_e of G_e such that δ' is the right congruence on G_e induced by H_e. On the other hand, for any e in E, let H_e be any subgroup of G_e. Let σ be the right congruence induced by H_e on G_e. If $f<e$, let $H_f=G_f$. If e and f are not comparable, written $e\ ?\ f$, let $H_f=G_f$. If $f \geqq e$, let $H_f=(H_e)\varphi_{f,e}$ where $\varphi_{f,e}$ is the homomorphism on G_f into G_e. Let $a, b \in S$. Write

$$a\,\sigma'\,b \Leftrightarrow a, b \in G_f \text{ and } ab^{-1} \in H_f \quad \text{for some } f \in E.$$

It readily follows that σ' is an equivalence relation on S. Assume $a\,\sigma'\,b$ and let $c \in G_k$. If (1) $k<e, f<e$ or (2) $k<e, f\ ?\ e$ then either $fk<e$ or $fk\ ?\ e$ so that $H_{fk}=G_{fk}$. In these cases $H_f=G_f$ and $H_k=G_k$. Hence $H_f \cdot H_k \subseteq H_{fk}$. A similar argument obtains the same result in each of the remaining cases so that $(ac)\,\sigma'\,(bc)$ and σ'

is a (right) congruence on S. Therefore, if $a, b \in G_e$ and $a \sigma b$, then $ab^{-1} \in H_e$ implies σ is the restriction of σ' to G_e so that σ is a congruence on G_e. Then H_e is a normal subgroup of G_e and G_e is Hamiltonian. Hence, if S is an inverse H-semigroup, then S is a semilattice of disjoint Hamiltonian groups.

Let Y be any semilattice and to each element α of Y assign a group G_α such that G_α and G_β are disjoint if $\alpha \neq \beta$ in Y. To each pair of elements α, β of Y such that $\alpha > \beta$, assign a homomorphism $\varphi_{\alpha,\beta}$ of G_α into G_β such that if $\alpha > \beta > \gamma$, then $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$. Let $\varphi_{\alpha,\alpha}$ be the identity automorphism of G_α. Let S be the union of all the groups G_α, $\alpha \in Y$, and define the product of any two elements a_α, b_β of S (a_α in G_α, b_β in G_β) by $a_\alpha b_\beta = (a_\alpha \varphi_{\alpha,\alpha\beta})(b_\beta \varphi_{\beta,\alpha\beta})$. Then S is an inverse semigroup which is a union of groups [1, p. 128].

Assume the groups G_α, $\alpha \in Y$, are Hamiltonian. It remains to show that S is an H-semigroup. Let σ be a right congruence on S. For each G_α, σ restricted to G_α induces a right congruence σ_α on G_α. Since G_α is Hamiltonian then σ_α is two-sided so that σ determines a normal subgroup H_α of G_α. Let e_α be the identity of G_α. Recall that E is in the center of S where E is the semilattice of idempotents of S. Then we have

$$\begin{aligned} a_\alpha \,\sigma\, b_\beta &\Rightarrow (a_\alpha e_\alpha e_\beta) \,\sigma\, (b_\beta e_\alpha e_\beta) \Rightarrow (a_\alpha e_\beta) \,\sigma\, (b_\beta e_\alpha) \\ &\Rightarrow (a_\alpha e_\beta) \,\sigma_{\alpha\beta}\, (b_\beta e_\alpha) \Rightarrow a_\alpha e_\beta (b_\beta e_\alpha)^{-1} = a_\alpha b_\beta^{-1} \in H_{\alpha\beta}. \end{aligned}$$

Further,

$$a_\alpha \,\sigma\, b_\beta \Rightarrow (a_\alpha b_\beta^{-1}) \,\sigma\, e_\beta \Rightarrow (a_\alpha b_\beta^{-1}) \,\sigma\, e_{\alpha\beta}$$

so that $e_{\alpha\beta} \,\sigma\, e_\beta$ and, by symmetry, $e_{\alpha\beta} \,\sigma\, e_\alpha$ so that $e_\beta \,\sigma\, e_{\alpha\beta} \,\sigma\, e_\alpha$.

Conversely,

$$\begin{aligned} a_\alpha b_\beta^{-1} \in H_{\alpha\beta} \text{ and } e_\beta \,\sigma\, e_{\alpha\beta} \,\sigma\, e_\alpha &\Rightarrow (a_\alpha b_\beta^{-1}) \,\sigma\, e_{\alpha\beta} \,\sigma\, e_\beta \text{ and } a_\alpha \,\sigma\, (a_\alpha e_\beta) \\ &\Rightarrow (a_\alpha e_\beta) \,\sigma\, b_\beta \text{ and } a_\alpha \,\sigma\, b_\beta. \end{aligned}$$

Let $c_\gamma \in S$ and assume $a_\alpha \,\sigma\, b_\beta$.

$$\begin{aligned} a_\alpha \,\sigma\, b_\beta &\Rightarrow a_\alpha b_\beta^{-1} \in H_{\alpha\beta} \Rightarrow (a_\alpha b_\beta^{-1}) \,\sigma_{\alpha\beta}\, e_{\alpha\beta} \Rightarrow (a_\alpha b_\beta^{-1}) \,\sigma\, e_{\alpha\beta} \\ &\Rightarrow (a_\alpha b_\beta^{-1} e_\gamma) \,\sigma\, (e_{\alpha\beta} e_\gamma) \Rightarrow (a_\alpha b_\beta^{-1} e_\gamma) \,\sigma_{\alpha\beta\gamma}\, e_{\alpha\beta\gamma} \Rightarrow a_\alpha b_\beta^{-1} e_\gamma \in H_{\alpha\beta\gamma}. \end{aligned}$$

Since $H_{\alpha\beta\gamma}$ is a normal subgroup of $G_{\alpha\beta\gamma}$ then

$$(c_\gamma e_\alpha e_\beta) a_\alpha b_\beta^{-1} e_\gamma (c_\gamma e_\alpha e_\beta)^{-1} = c_\gamma a_\alpha b_\beta^{-1} c_\gamma^{-1} = (c_\gamma a_\alpha)(c_\gamma b_\beta)^{-1} \in H_{\alpha\beta\gamma}$$

and

$$(c_\gamma a_\alpha)^{-1} (c_\gamma b_\beta) \in H_{\alpha\beta\gamma}.$$

Further, $a_\alpha \,\sigma\, b_\beta \Rightarrow e_\beta \,\sigma\, e_{\alpha\beta} \,\sigma\, e_\alpha \Rightarrow e_{\gamma\alpha} \,\sigma\, e_{\gamma\alpha\beta} \,\sigma\, e_{\gamma\beta}$. Thus we have $(c_\gamma a_\alpha) \,\sigma\, (c_\gamma b_\beta)$ and σ is also a left congruence. By an analogous proof, if σ is a left congruence on S, then σ is a right congruence on S. Hence S is an inverse H-semigroup.

3. In this section we first identify the maximal modular congruences of an inverse H-semigroup S, and then obtain necessary and sufficient conditions for S to be t-semisimple.

A homomorphic image of an inverse semigroup is an inverse semigroup. Moreover, in any homomorphism, the inverse of an element is mapped onto the inverse of the image of that element [2, p. 57].

As one might also expect, the homomorphic image of an H-semigroup S is an H-semigroup. For if ψ is a homomorphism from S onto S', S' is obviously a semigroup. And if μ' is any right (left) congruence on S', we can define μ on S by

$$a\ \mu\ b \Leftrightarrow (a\psi)\ \mu'\ (b\psi).$$

Then μ is a congruence on S and from this it follows that μ' is a left (right) congruence on S'.

Let S be an H-semigroup. Let I be an ideal of S and T a subsemigroup of S such that $I \cup T = S$ and $I \cap T = \varnothing$. Write

$$a\ \rho\ b \Leftrightarrow a, b \in I \text{ or } a, b \in T.$$

Then ρ is a maximal modular congruence on S, where each element of T is an identity for ρ and ρ is not cancellative.

Let S be an inverse H-semigroup. For $e \in E$, let $T_e = \bigcup \{G_f : e \leqq f\}$. Define the relation $\rho^{(e)}$ on S by

$$a\ \rho^{(e)}\ b \Leftrightarrow a, b \in T_e \text{ or } a, b \in T'_e$$

where $T'_e = S - T_e$. If e is not a minimum idempotent in S, we claim that $\rho^{(e)}$ is a maximal modular congruence on S with identity e and $\rho^{(e)}$ is not cancellative. Since $e \in T_e$, $T_e \neq \varnothing$. Let $d, b \in T_e$, say $d \in G_f$, $b \in G_k$. Then $e \leqq f$ and $e \leqq k$ imply $e \leqq fk$ so that $db \in G_{fk} \subseteq T_e$ and T_e is a subsemigroup of S. Assume e is not minimum in E so that $T'_e \neq \varnothing$. Let $d \in T'_e$, $b \in S$, say $d \in G_f$ and $b \in G_k$. Now $d \in T'_e$ implies $f < e$ or $f\ ?\ e$. If $f < e$ then $fk < e$ and $db \in G_{fk} \subseteq T'_e$. If $f\ ?\ e$ then $fk < e$ or $fk\ ?\ e$ and $db \in T'_e$. Thus T'_e is a right ideal of S. By Lemma 1.2, T'_e is an ideal of S. It follows that $\rho^{(e)}$ is a maximal congruence on S with identity e and $\rho^{(e)}$ is not cancellative.

Let σ be a maximal modular congruence on an inverse H-semigroup S. For each $e \in E$, let H_e be the subgroup of G_e induced by σ. Then, as in the proof of Theorem 2, we know that for $a, b \in S$, say $a \in G_f$, $b \in G_k$, $a\ \sigma\ b \Leftrightarrow ab^{-1} \in H_{fk}$ and $f\ \sigma\ (fk)\ \sigma\ k$. Let a be an identity for σ, say $a \in G_f$. Then for each $s \in S$, $(as)\ \sigma\ s$ implies $(fas)\ \sigma\ (fs)$ so that $(as)\ \sigma\ (fs)\ \sigma\ s$. Thus f is an identity for σ.

It is generally known that σ is cancellative if and only if $E \subseteq \sigma_e$.

Suppose σ is not cancellative and let $e \in E$ be an identity for σ. If $h \in E$ is an identity for σ, then $h\ \sigma\ (eh)\ \sigma\ e$ and $h \in \sigma_e$. Since σ is not cancellative there exists $f \in E$ such that $f \notin \sigma_e$, so that f is not an identity for σ. Let $I = \{f \in E : f \text{ is not an identity for } \sigma\}$. Then I is an ideal in E. Let $J = \bigcup \{G_f : f \in I\}$. Then J is an ideal of S and J' is a semigroup of S. Oehmke [7] has shown that if σ is a maximal congruence and J any ideal of S, then either J is contained in a σ-class S_0 (which is also an ideal of S) or J contains an element of each σ-class. If $x \in \sigma_e \cap J$, then $x\ \sigma\ e$ and $x \in G_f$, for some $f \in I$. But then $x\ \sigma\ (ef)$ so that $e\ \sigma\ (ef)$ and, since also

$(ef)\ \sigma\ f$, then $e\ \sigma\ f$ and $f \in I$, which is a contradiction. Hence $\sigma_e \cap J = \varnothing$ and there exists a σ-class S_0 such that $J \subseteq S_0$. Suppose there is some $b \in S_0$ such that $b \notin J$, that is, $b \in G_h$ where $h\ \sigma\ e$. Let $f \in I$. Then $b\ \sigma\ f$ implies $bf \in H_{hf}$ and $h\ \sigma\ (hf)\ \sigma\ f$, so that $f\ \sigma\ e$. Contradiction. Therefore $J = S_0$. Since J is an ideal and J' is a semigroup, we have the maximal modular congruence σ^* defined by $a\ \sigma^*\ b \Leftrightarrow a, b \in J$ or $a, b \in J'$, where each element of J is an identity for σ^*. Clearly $\sigma \leqq \sigma^*$. Hence $\sigma = \sigma^*$ and we have proved the following lemma.

LEMMA 3.1. *If σ is a maximal modular congruence on an inverse H-semigroup S, then σ is cancellative or σ has exactly two congruence classes, namely the semigroup of identities for σ and the ideal of nonidentities for σ.*

Suppose σ is cancellative. Then $E \subseteq \sigma_e$ where $\sigma_e = \bigcup \{H_f : f \in E\}$. Since σ is maximal, then S/σ has no nontrivial congruences. Therefore S/σ is the semigroup $\{0, 1\}$ or S/σ is simple. Since σ is cancellative, it follows that $S/\sigma \neq \{0, 1\}$. Since S is an inverse H-semigroup, then S/σ is an inverse H-semigroup. Therefore, by Lemma 1.2, every one-sided ideal of S/σ is a two-sided ideal. Hence S/σ is both left and right simple, so that S/σ is a Hamiltonian group [1, p. 6]. Since S/σ has no nontrivial congruences, then S/σ has no nontrivial homomorphisms, so that S/σ is a simple group. Since S/σ is Hamiltonian, then S/σ has no nontrivial subgroups. Hence S/σ is a cyclic group of prime order. Thus there exists a prime number p such that, for every $a \notin \sigma_e$, the σ-classes may be written as $\sigma_a, \sigma_{a^2}, \ldots, \sigma_{a^p}$, where $\sigma_{a^p} = \sigma_e$ and a^p is an identity for σ. In fact, for every $c \in \sigma_e$, $c\ \sigma\ e$ implies $(cs)\ \sigma\ (es)\ \sigma\ s$ and $(sc)\ \sigma\ (se)\ \sigma\ s$ so that c is an identity for σ. By a similar argument, if $a \notin \sigma_e$, then a is not an identity for σ. If $G_e = H_e$ then $G_e \subseteq \sigma_e$. If $G_e \neq H_e$ then σ partitions G_e into cosets of H_e. The number of these cosets must be p, for otherwise S/σ would contain a proper subgroup. Hence the cosets of H_e must form a cyclic group of prime order and H_e must be a maximal subgroup of G_e. We state these results in the following lemma:

LEMMA 3.2. *If σ is a maximal modular cancellative congruence on an inverse H-semigroup S, then S/σ is a cyclic group of prime order p such that for each nonidentity element g for σ, the cosets of S/σ are $\sigma_e, \sigma_g, \ldots, \sigma_{g^{p-1}}$. Moreover, if σ' is the restriction of σ to G_e, then for each e in E, $\sigma' = \nu$ or σ' induces a maximal subgroup H_e of G_e where the cosets of H_e form a cyclic group of prime order p.*

LEMMA 3.3. *If T is a proper subsemigroup of S such that for each $e \in E$, $T \cap G_e = H_e$, where $H_e = G_e$ or H_e is a maximal subgroup of index p in G_e, and for each pair of groups G_e, G_f in the semilattice S, where $e \leqq f$, the homomorphism $\varphi_{f,e}$ on G_f into G_e defined by $a\varphi_{f,e} = ae$ is a monomorphism, then T induces a maximal modular cancellative congruence on S.*

Proof. Define σ on S by $a\ \sigma\ b \Leftrightarrow ab^{-1} \in H_{kf}$, where $a \in G_k$, $b \in G_f$. Clearly, σ is reflexive, symmetric and compatible. That σ is transitive follows from the hypothesis that for each e, f in E where $f \geqq e$, the homomorphism $\varphi_{f,e}$ is a monomorphism. Therefore σ is a congruence on S. It follows immediately from the definition of σ that $T=\sigma_e$ and σ is modular. Then $E \subseteq \sigma_e$ and σ is cancellative. If $\sigma < \sigma'$ where, for each $e \in E$, σ' induces the subgroup K_e in G_e, then there exists $a, b \in S$ such that $a\, \sigma'\, b$ and $a \not\sigma b$. Say $a \in G_f$, $b \in G_h$. Then $ab^{-1} \in K_{fh}$ and $ab^{-1} \notin H_{fh}$ which implies that $H_{fh} \subset K_{fh}$ so that $K_{fh}=G_{fh}$, since H_{fh} is then maximal in G_{fh}. If $k > fh$, then $G_k=K_k$. For $k < fh$, $H_k \subset (G_f)\varphi_{fh,k} \subseteq K_k \subseteq G_k$. Assume $K_k \subset G_k$. Since H_k has index p in G_k, then K_k has finite index j in G_k and the index m of H_k in K_k is such that $p=mj$ [4, p. 63]. Then either $m=p$ and $j=1$, or $m=1$ and $j=p$. If $m=1$, then $H_k=K_k$, which is a contradiction. If $j=1$, then $G_k=K_k$, contrary to the assumption. Thus, for $k<fh$ we must have $G_k=K_k$. If $k\ ?\ fh$ then $fhk<fh$ and $G_{fhk}=K_{fhk}$. Since $\varphi_{k,fhk}$ is a monomorphism then $G_k=K_k$. Therefore $\sigma'=\nu$ and σ is maximal. This completes the proof.

Define the relation ρ on S as follows:

$$x\, \rho\, y \Leftrightarrow \text{there exists } e \in E \text{ such that } ex = ey.$$

Clearly ρ is a congruence on S. Then for each e, f in E, $e\, \rho\, f$ since $(ef)e=ef=(ef)f$. Thus $E \subseteq \rho_e$. Once again we note that S/ρ is an inverse H-semigroup containing exactly one idempotent so that S/ρ is a Hamiltonian group. Let σ be any maximal modular cancellative congruence on S and let $a, b \in S$ such that $a\, \rho\, b$. Then

$$\begin{aligned} a\, \rho\, b &\Rightarrow \text{there exists } e \text{ in } E \text{ such that } ea = eb \\ &\Rightarrow (ea)\, \sigma\, (eb) \Rightarrow a\, \sigma\, b. \end{aligned}$$

Thus the intersection α of all the maximal modular cancellative congruences of S is greater than or equal to ρ. Now for any f, e in E, where $f<e$, if the homomorphism $\varphi_{e,f}$ is not a monomorphism then there exist $a \neq b$ in G_e with $fa=fb$ so that $a\, \rho\, b \Rightarrow a\, \alpha\, b$. Suppose that S is t-semisimple. From Lemma 3.1, it is clear that the intersection β of all the maximal modular noncancellative congruences of S separates S into its maximal subgroups. Thus, if $\varphi_{e,f}$ is not a monomorphism then $a\, \alpha\, b$ and $a\, \beta\, b$ imply $a\, \tau\, b$ so that $\tau \neq \iota$, contrary to the supposition. Let $a \neq e$, $a \in G_e$, $e \in E$. Since $a\, \beta\, e$ then there must be a maximal modular cancellative congruence σ on S such that $a \not\sigma e$. From Lemma 3.2 it follows that there exists a maximal subgroup H_e of index p in G_e such that $a \notin H_e$. Since we know that, for each $f \in E$, the restriction of σ to G_f induces a subgroup H_f of G_f such that H_f is of index p in G_f or $H_f=G_f$, and since $E \subseteq \sigma_e$, then the union of these subgroups is a proper inverse subsemigroup of S. Let $\mathscr{T}$ be the collection of all inverse subsemigroups T_p of S such that, for each e in E, $T_p \cap G_e=H_e$, where $H_e=G_e$ or H_e is a maximal subgroup of prime index p in G_e. Then we may say that, if S is t-semisimple, then for each e in E, for each $a \neq e$ in G_e, there exists $T_p \in \mathscr{T}$ such that $a \notin T_p$. Conversely, assume that for each f, e in E, where $e<f$, $\varphi_{f,e}$ is a monomorphism; and for each e in E, for each $a \neq e$ in G_e, there exists $T_p \in \mathscr{T}$ such that $a \in T_p$. Suppose $a\, \tau\, b$, where $a \in G_e$, $b \in G_f$. Since β separates S into its maximal

subgroups, then $e=f$. If $a \neq b$ then $ab^{-1} \neq e$, and by assumption there exists $T_p \in \mathscr{T}$ such that $ab^{-1} \notin T_p$. By Lemma 3.3, it follows that there exists a maximal modular cancellative congruence σ on S separating a and b. Thus if $a \tau b$, then $a=b$ and S is t-semisimple. We can now state the main result of this section.

THEOREM 3. *An inverse H-semigroup S is t-semisimple if and only if for each pair of groups* G_e, G_f *in the semilattice, with* $f \geqq e$, *the homomorphism* $\varphi_{f,e}$ *on* G_f *into* G_e, *defined by* $a\varphi_{f,e}=ae$, *is a monomorphism, and for each e in E, for each* $a \neq e$ *in* G_e, *there exists a subsemigroup* T_p *of S such that* $a \notin T_p$ *and, for each f in E,* $T_p \cap G_f = H_f$, *where* $H_f = G_f$ *or* H_f *is a maximal subgroup of prime index p in* G_f.

COROLLARY 3.1. *S is an inverse H-semigroup all of whose maximal modular congruences are cancellative if and only if S is a Hamiltonian group.*

COROLLARY 3.2. *S is a t-semisimple inverse H-semigroup all of whose nontrivial maximal modular congruences are not cancellative if and only if S is a semilattice.*

COROLLARY 3.3. *If S is a t-semisimple inverse H-semigroup, then S is a semilattice of disjoint t-semisimple Hamiltonian groups.*

Proof. Let $f \in E$. It suffices to show that G_f is t-semisimple. Let a, b be distinct elements of G_f. Since $a \beta b$ then there must be a maximal modular cancellative congruence σ on S such that $a \not\sigma b$. Let the σ-classes be $\sigma_e, \sigma_g, \ldots, \sigma_{g^{p-1}}$. Let σ' be the restriction of σ to G_f. Then $a \not\sigma' b$. Now $a \not\sigma b$ implies either $a \notin \sigma_e$ or $b \notin \sigma_e$. Say $a \notin \sigma_e$. Then the σ-classes may be written as $\sigma_f, \sigma_a, \ldots, \sigma_{a^{p-1}}$ so that the σ'-classes are $\sigma'_f, \sigma'_a, \ldots, \sigma'_{a^{p-1}}$. Further, G_f/σ' is a cyclic group of prime order so that σ' is a maximal congruence on G_f separating a and b. Hence G_f is t-semisimple.

LEMMA 3.4.1. *A t-semisimple Hamiltonian group is commutative.*

Proof. Let G be a t-semisimple Hamiltonian group and assume G is not commutative. Then $G = Q \times A \times B$ where Q is a quaternion group, A a commutative group of exponent two, B a commutative group where each element has odd order. Since Q is a finite p-group then Q is nilpotent, and since A and B are commutative then A and B are nilpotent [3, p. 155, p. 149]. Therefore G is nilpotent [5, p. 212]. Now every maximal subgroup of a nilpotent group is normal, is of prime index, and contains the derived group [3, p. 154]. Hence the intersection Φ of the maximal (normal) subgroups of G contains the derived group G'. But G is t-semisimple so that its Frattini subgroup Φ consists of the identity only. Thus G' contains the identity only and G is commutative. But this contradicts our assumption so that the result follows.

COROLLARY 3.4. *If S is a t-semisimple inverse H-semigroup, then S is commutative.*

COROLLARY 3.5. *If S is an inverse H-semigroup with a minimum idempotent e, then S is t-semisimple if and only if* G_e *is t-semisimple and, for each group* G_f *in the semilattice with* $f \geqq e$, *the homomorphism* $\varphi_{f,e}$ *on* G_f *into* G_e, *defined by* $a\varphi_{f,e}$, *is a monomorphism.*

Proof. Only the sufficiency requires proof. Let f, $h \in E$ with $f \geqq h$ and assume there exist $a \neq b$ in G_f such that $ah = bh$ in G_h. Then $ae = ahe = bhe = be$ in G_e implies $a = b$, since $\varphi_{f,e}$ is a monomorphism. Contradiction. Hence, for each f, h in E, where $f \geqq h$, $\varphi_{f,h}$ is a monomorphism. Let $f \in E$ and $a \neq f$ in G_f. Assume $a \in \Phi_f$. Since $\varphi_{f,e}$ is injective then $ae \neq e$ in G_e, so that there exists a maximal subgroup H_e in G_e such that $ae \notin H_e$. Thus $a = ae\varphi_{f,e}^{-1} \notin H_e\varphi_{f,e}^{-1} = H_f$ in G_f. But H_f is a maximal subgroup of G_f and $a \notin H_f$ imply $a \notin \Phi_f$. Contradiction. Hence G_f is t-semisimple. It remains to show that for each f in E, for each $a \neq f$ in G_f, there exists a subsemigroup T_p of S such that $a \notin T_p$ and, for each h in E, $T_p \cap G_f = H_h$, where $H_h = G_h$ or H_h is a maximal subgroup of prime index p in G_h. Let $a \neq f$ in G_f. Since G_f is t-semisimple, there exists a maximal subgroup H_f in G_f such that $a \notin H_f$. It follows that G_f/H_f has no nontrivial subgroups and is therefore cyclic of prime order p so that the cosets of H_f may be written as $H_f, H_fa, \ldots, H_fa^{p-1}$. H_fe is a subgroup of G_e which does not contain ea. Let H_e be a subgroup of G_e maximal with respect to not containing ea and such that $H_fe \subseteq H_e$ [8, p. 22]. But then H_e is a maximal subgroup of G_e and G_e/H_e is cyclic of prime order. Since $\varphi_{f,e}$ is a monomorphism from G_f into G_e and H_f is a maximal subgroup of G_f, it follows that $ea^i \notin H_e$, $1 \leqq i \leqq p-1$, and $H_e, H_ea, \ldots, H_ea^{p-1}$ are distinct cosets of H_e. Hence these must be all the cosets of H_e so that H_e is maximal and of index p in G_e. For each h in E, let $H_h = (H_e)\varphi_{h,e}$. It follows that for each H_h either $H_h = G_h$ or H_h is maximal and of index p in G_h. Further, for h, k in E, let $x \in H_h$, $y \in H_k$. Then

$$xe, ye \in H_e \Rightarrow xye \in H_e \Rightarrow xy \in H_{hk}.$$

Thus the union of all H_h, $h \in E$, as defined above, is a subsemigroup T_p with the desired properties and the proof is complete.

Corollary 3.6. *If S is a finite inverse H-semigroup, then S is t-semisimple if and only if G_e is t-semisimple, where e is the minimum idempotent of E, and for each subgroup G_f, $f > e$, the homomorphism $\varphi_{f,e}$ from G_f into G_e, defined by $a\varphi_{f,e} = ae$, is a monomorphism.*

Corollary 3.7. *If S is a t-semisimple inverse H-semigroup with no nontrivial modular congruences, then S is either a cyclic group of prime order or the unique semilattice of two elements.*

Proof. Since S is t-semisimple, it has maximal modular congruences, that is, ι is a maximal modular congruence. Since there is no nontrivial modular (non-cancellative) congruence on S, then S is a group or S is the semilattice of two elements. In the former case, any congruence on S would be modular, so it follows that S has no nontrivial subgroups, hence is cyclic of prime order.

Corollary 3.8. *If S is an inverse H-semigroup with zero, then S is t-semisimple if and only if S is a semilattice.*

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