A CHARACTERIZATION OF ODD ORDER EXTENSIONS OF THE FINITE PROJECTIVE SYMPLECTIC GROUPS
PSp(4, q)

BY
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Abstract. In a recent paper, W. J. Wong characterized the finite projective symplectic groups PSp(4, q) where q is a power of an odd prime integer by the structure of the centralizer of an involution in the center of a Sylow 2-subgroup of PSp (4, q). In the present paper, finite groups which contain an involution in the center of a Sylow 2-subgroup whose centralizer has a more general structure than in the PSp(4, q) case are classified by showing them to be odd ordered extensions of PSp(4, q).

Introduction and statement. Let p denote an odd prime integer and let q = p^f where f is a positive integer. Let PSp(4, q) denote the projective symplectic group in dimension 4 over a field F_q of q elements. The group PSp(4, q) is described in [8, §1], is simple of order ½q4(q^2 + 1)(q^2 - 1)^2 and has a Sylow 2-subgroup with center of order 2 so that involutions which lie in the centers of Sylow 2-subgroups form a single conjugacy class.

Let a denote an automorphism of F_q. Then a induces, in the natural way, an automorphism of PSp(4, q) (cf. [8, §1]) and centralizes an involution in the center of a Sylow 2-subgroup of PSp (4, q). In fact <a>, the cyclic subgroup of Aut(F_q) generated by a, acts faithfully on PSp(4, q) and one may form the natural semidirect product <a> PSp(4, q). If a is an odd ordered automorphism of F_q, then <a> PSp(4, q) is an odd ordered extension of PSp(4, q) with trivial 2-core. In fact, any odd ordered extension of PSp(4, q) with trivial 2-core is of this form (cf. [1]).

Let t be an involution in the center of a Sylow 2-subgroup of PSp(4, q) such that t is centralized by a. Then the centralizer C(t) of t in <a> PSp(4, q) is a semidirect product <a> C(t) where C(t) denotes the centralizer of t in PSp(4, q) and C(t) has trivial 2-core.

Finite groups G containing an involution t such that C_G(t) ≅ C(t) have been classified in [8]. However, for example in classifying groups by their Sylow 2-subgroups, one may arrive at a situation in which the centralizer C_G(t) of an involution t in a group

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$G$ is an odd ordered extension of $\mathcal{G}$ with trivial 2-core. This is the case in the group $\langle \sigma \rangle \text{PSp}(4, q)$ of the above paragraph.

To handle this situation, we prove the following more general result:

**Theorem.** Let $G$ be a finite group with an involution $t$ such that
(a) $O(C_0(t)) = \{1\}$ and
(b) $C_0(t)$ contains a normal subgroup $\mathcal{S}$ of index $2^\rho$ with $\rho$ odd and an involution $u \in C_0(t) - \mathcal{S}$ such that $\mathcal{S}$ contains a subgroup $L_1 \cong \text{SL}(2, q)$ where $q = p^f$, $p$ is an odd prime integer and $f$ a positive integer such that if $L_2 = L_1^2$, then $\mathcal{S} = L_1L_2$, $[L_1, L_2] = \{1\}$ and $L_1 \cap L_2 = \langle t \rangle$.

Then, either (i) $G = C_0(t)O(G)$ or (ii) $G \cong \langle \sigma \rangle \text{PSp}(4, q)$ where $\sigma \in \text{Aut}(F_q)$ and is of order $p$.

Note that when $\rho = 1$, then $C_0(t)$ is isomorphic to the centralizer of an involution in the center of a Sylow 2-subgroup in $\text{PSp}(4, q)$ and the theorem follows from [8].

Thus for the rest of the paper, we assume that $G$ is a finite group with an involution $t$ such that $C_0(t)$ satisfies the hypotheses of the theorem with $\rho > 1$. Let $q \equiv \delta \pmod{4}$ where $\delta = \pm 1$ and let $q = 2^e$ with $e$ an odd integer and $e \geq 2$.

The paper is organized as follows. In §1, we study the structure of $C_0(t)$ and in §2, we show that if case (i) of the conclusion of the theorem does not hold, then $G$ has exactly two conjugacy classes of involutions. In §3, we determine the structure of the centralizer of an involution in the second conjugacy class, and we show that the situation is exactly as in the group $\langle \sigma \rangle \text{PSp}(4, q)$. Moreover, we show, using a lemma of Thompson, that $|G| = \rho |\text{PSp}(4, q)|$. Then in §4, we find the structure of the normalizer of a Sylow $\rho$-subgroup of $G$. In §5, we construct a normal subgroup $G_0$ of $G$ such that $|G:G_0| = \rho$ and then we show that $G_0$ is isomorphic to $\text{PSp}(4, q)$ by applying [8]. The rest of the theorem now follows easily.

Our notation is fairly standard and tends to follow that of [5]. We use $O(X)$ to denote the 2-core of $X$, i.e., the largest normal subgroup of odd order in the finite group $X$. We write $[x, y] = x^{-1}x^y = x^{-1}y^{-1}xy$ and, if $x^y = z$, we write $y : x \leftrightarrow z$. If $y : x \leftrightarrow x^{-1}$, we say that $y$ inverts $x$ and, if $y : x \leftrightarrow z$ and $y : z \leftrightarrow x$, we write $y : x \leftrightarrow z$.

1. The structure of $C_0(t)$. Since $C_0(t)/\mathcal{S}$ is a group of order $2^\rho$ we have

**Lemma 1.1.** $C_0(t)$ contains a subgroup $\mathcal{A}$ of index 2 such that $\mathcal{A} \supseteq \mathcal{S}$.

**Lemma 1.2.** $\{L_1, L_2\}$ is invariant in $C_0(t)$.

**Proof.** This follows easily from the Krull-Schmidt Theorem applied to the group $\mathcal{S}/\langle t \rangle \cong \text{PSL}(2, q) \times \text{PSL}(2, q)$.

**Corollary 1.2.1.** $N_0(L_1) = N_0(L_2) = \mathcal{A}$.

**Proof.** Because $Z(L_1) = \langle t \rangle$, we have $N_0(L_1) \subseteq C_0(t)$. Also $\mathcal{S} \subseteq N_0(L_1) \cap N_0(L_2)$ and $|\mathcal{A}/\mathcal{S}| = \rho$ is odd. Now we can conclude that $\mathcal{A} \subseteq N_0(L_1)$ from Lemma 1.2. Hence $N_0(L_1) = \mathcal{A}$, Similarly $N_0(L_2) = \mathcal{A}$.
Lemma 1.3. \( C_\odot(\odot) = \langle t \rangle \).

Proof. Since \( C_\odot(\odot) \leq \mathcal{A} \) and \( Z(\odot) = \langle t \rangle \), \( C_\odot(\odot) \) has order \( 2 \cdot r \) where \( r \mid \rho \). But \( C_\odot(\odot) \leq C_\odot(t) \) and \( O(C_\odot(t)) = \{1\} \); hence \( r = 1 \).

Lemma 1.4. There exists a subgroup \( A \) of \( \mathcal{A} \) of order \( \rho \) such that \( \mathcal{A} = \odot A \) and \( A \cap \odot = \{1\} \) and such that \( A \) centralizes a Sylow 2-subgroup of \( \odot \). Moreover if, \( f = 2^a, a \geq 0, \) then \( \rho = 1 \) and the result follows from [8].

Proof. Since \( C_\odot(L_1) = L_2 \triangleleft C_\mathcal{A}(L_1) \triangleleft \mathcal{A} \) and \( |\mathcal{A} : L_1 C_\mathcal{A}(L_1)| \) divides \( \rho \), it follows from the structure of \( \text{Aut}(L_1) \) that there exists a subgroup \( A_1 \) of \( \mathcal{A} \) such that \( A_1 \geq C_\mathcal{A}(L_1) \), \( \mathcal{A} = L_1 A_1 \), and \( L_1 \cap A_1 \leq C_\mathcal{A}(L_1) \). Since \( |A_1 : C_\mathcal{A}(L_1)| \) is odd, \( A_1 \) centralizes a Sylow 2-subgroup \( S_1 \) of \( L_1 \). Thus \( \langle t \rangle \leq L_1 \cap A_1 \leq C_\mathcal{A}(L_1) \cap L_1 = \langle t \rangle \) and \( \odot \cap A_1 = L_2 \triangleleft A_1 \). Hence \( |A_1/L_2| \) is odd. Again, it follows that there exists a subgroup \( A_2 \) of \( A_1 \) such that \( A_2 \geq C_{A_1}(L_2) \), \( A_1 = L_2 A_2 \) and \( L_2 \cap A_2 \leq C_{A_1}(L_2) \) and such that \( A_2 \) centralizes a Sylow 2-subgroup \( S_2 \) of \( L_2 \). Thus \( \langle t \rangle \leq A_2 \cap \odot = A_2 \cap \odot \leq L_2 \cap C_{A_1}(L_2) = \langle t \rangle \). Hence \( \langle t \rangle \) is the Sylow 2-subgroup of \( A_2 \) and \( A_2 \) has a subgroup \( A \) of index 2. Then \( \odot A = L_1 L_2 A = L_1 L_2 A_2 = L_1 A_1 = \mathcal{A} \), \( \odot \cap A = A \cap \odot \cap A_2 = A \cap \langle t \rangle = \{1\} \), \( |A| = \rho \) and \( A \) centralizes \( S_1 S_2 \) which is a Sylow 2-subgroup of \( \odot \). Finally, if \( f = 2^a, a \geq 0 \) then

\[
|\mathcal{A} : L_1 C_\mathcal{A}(L_1)| \quad \text{and} \quad |A_1/L_2 C_{A_1}(L_2)|
\]

are powers of 2. Thus \( A_1 = C_\mathcal{A}(L_1), A_2 = C_{A_1}(L_2) \) and \( A_2 \leq C_\odot(\odot) = \langle t \rangle \) and, therefore, \( |A| = 1 = \rho \).

Note that we may now assume that \( q > 3 \) and, hence, that \( L_i/\langle t \rangle \) is simple for \( i = 1, 2 \). We may also assume that \( e > 1 \), for if \( e = 1 \) then \( q - 8 = 2^n \) and [6, Lemma 19.3] yields that either \( q \) is a prime or \( q = 3^\gamma \) both of which have been excluded.

Lemma 1.5. \( A \) is Abelian with at most 2 generators.

Proof. Consider the natural homomorphism

\[
\theta : A \to \big( \text{Aut}(L_1)/\text{Inn}(L_1) \big) \times \big( \text{Aut}(L_2)/\text{Inn}(L_2) \big).
\]

Suppose that \( a \in \text{Ker}(\theta) \). Then there is an \( l_i \in L_i \) such that \( a l_i = C_\odot(l_i) \) for \( i = 1, 2 \). Hence \( a l_1 l_2 \in C_\odot(\odot) = \langle t \rangle \) and \( a \in A \cap \odot = \{1\} \). Thus \( \theta \) is one-to-one and since \( |A| = \rho \) is odd, the lemma follows from the structures of \( \text{Aut}(L_1) \) and \( \text{Aut}(L_2) \).

Lemma 1.6. The subgroup \( A \) may be chosen so that it is normalized by \( u \). If \( A \) is so chosen, there exists a Sylow 2-subgroup \( S_1 \) of \( L_1 \) centralized by \( A \) and hence \( S_1 \) is a Sylow 2-subgroup of \( L_2 \) centralized by \( A \).

Proof. Let \( S_1 \) be a Sylow 2-subgroup of \( L_1 \) centralized by \( A \). Since \( A \) centralizes a Sylow 2-subgroup of \( L_2 \), by conjugating \( A \) by an element of \( L_2 \), we may assume that \( A \) centralizes \( S_1 S_2 \). Since \( C_\odot(S_1 S_2) = \langle t \rangle \times A \) and \( u \in N_\odot(S_1 S_2) \), \( u \) normalizes \( O(C_\odot(S_1 S_2)) = A \).

Henceforth, we assume that \( A \) is normalized by \( u \).
Corollary 1.6.1. \( A = C_A(u) \times [A, u] \) where \([A, u]\) is the subgroup of \( A \) formed by the elements of \( A \) inverted by \( u \) together with the identity element.

Let
\[
B = \{ x_1 x_1^u \mid x_1 \in L_1 \}.
\]

Then \( B \) is a subgroup of \( C_G(t, u) \) and \( B \) is isomorphic to \( \text{PSL}(2, q) \) where the isomorphism is induced by the epimorphism \( \theta: L_1 \to B \) defined by \( \theta(x_1) = x_1 x_1^u \) for \( x_1 \in L_1 \) since \( \text{Ker}(\theta) = \langle t \rangle \).

An easy calculation gives

\[
\text{Lemma 1.7.} \quad C_G(u, t) = (B \times \langle t, u \rangle) C_A(u) \text{ where } \xi \cap C_G(u, t) = (B \times \langle t \rangle) \langle C_A(u, t) \rangle.
\]

Corollary 1.7.1. \( B \) is characteristic in \( C_G(u, t) \).

Proof. Since \( q > 3 \), \( B \) is simple and non-Abelian, thus \( C_G(u, t) = B \).

2. Classes of involutions in \( G \). From the structure of \( \xi \), we have

\[
\text{Lemma 2.1.} \quad \text{All involutions of } \xi - \langle t \rangle = L_1 L_2 - \langle t \rangle \text{ are conjugate in } \xi = L_1 L_2.
\]

An easy consequence of the fact that \( u \) normalizes \( A \) is

\[
\text{Lemma 2.2.} \quad \text{All involutions of } C_G(t) - \xi \text{ are conjugate in } C_G(t) \text{ to } u \text{ or } tu.
\]

The proof of [2, (2B)] yields

\[
\text{Lemma 2.3.} \quad \text{If } H \text{ is a subgroup of } G \text{ and if } T \text{ is a Sylow 2-subgroup of } H \cap C_G(t) \text{ such that } \langle t \rangle \text{ is characteristic in } T, \text{ then } T \text{ is a Sylow 2-subgroup of } H. \text{ In particular, a Sylow 2-subgroup of } C_G(t) \text{ is a Sylow 2-subgroup of } G.
\]

Let \( S_1 = \langle a_1, b_1 \rangle \) where \( a_1^2 = b_1^2 = t \) and \( a_1 b_1 = b_1 a_1^{-1} \) be a Sylow 2-subgroup of \( L_1 \) centralized by \( A \). Set \( b_2 = b_1 \), \( a_2 = a_1 \). Then \( v = (a_1 a_2)^{2n-2} \) and \( w = b_1 b_2 \) are two involutions in \( \xi - \langle t \rangle \) centralized by \( u \). Set \( S = S_1 S_1^u \langle u \rangle = \langle a_1, b_1, a_2, b_2, u \rangle \). Then \( S \) is a Sylow 2-subgroup of \( G \). Note that \( Z(S) = \langle t \rangle \) so that the conjugacy class of \( t \) in \( G \) consists exactly of those involutions which are in the center of some Sylow 2-subgroup of \( G \).

\[
\text{Lemma 2.4.} \quad \text{The involutions of } \xi - \langle t \rangle \text{ are not conjugate in } G \text{ to } t.
\]

Proof. It suffices to show that \( v \) is not conjugate in \( G \) to \( t \). Since \( C_L(a_1^{2n-2}) = \langle d_1 \rangle \) which is cyclic of order \( q - \delta \), we may assume that \( d_1 = a_1 \). Setting \( d_2 = d_1^u \), we have

\[
C_G(t, v) = \langle d_1, d_2, w \rangle A \cdot \langle u \rangle
\]

which is of order \( 2(q - \delta)^2 \). Also \( A \) normalizes \( \langle d_1 \rangle \) and \( \langle d_2 \rangle \) and \( d_1^w = d_1^{-1} \) for \( i = 1, 2 \). A Sylow 2-subgroup of \( C_G(t, v) \) is \( T = \langle a_1, a_2, w, u \rangle \) which is of order \( 2^{2n+1} \). The argument of [8, Lemma 2.2] applies with only a slight modification required if \( n = 2 \).
We now assume

(3) \( G \neq C_\theta(t)O(G) \).

The proof of [8, Lemma 2.3] yields

**Lemma 2.5.** Either \( t \) or \( tu \) is conjugate in \( G \) to \( t \).

Since \( C_\theta(t) \) has an automorphism which interchanges \( u \) and \( tu \) and fixes \( \theta A \), we may assume

(4) \( tu \) is conjugate in \( G \) to \( t \).

**Lemma 2.6.** \( G \) has exactly two conjugacy classes of involutions \( K_1 \) and \( K_2 \) such that \( K_1 \cap C_\theta(t) \) consists of the conjugacy classes in \( C_\theta(t) \) represented by \( t \) and \( tu \) and \( K_2 \cap C_\theta(t) \) consists of the conjugacy classes in \( C_\theta(t) \) represented by \( v \) and \( u \). There exists an element \( z \in N_\theta(C_\theta(tu)) \) such that \( z^2 \in C_\theta(tu) \) and

(5) \( z: t \to uv, \quad u \to tv, \quad v \to v. \)

Also \( z \) normalizes \( C_\theta(u) \).

**Proof.** Let \( K_1 \) be the conjugacy class of \( t \) in \( G \) and let \( K_2 \) be the conjugacy class of \( v \) in \( G \); clearly \( K_1 \neq K_2 \) by Lemma 2.4. Set

(6) \( E = C_\theta(tu) = \langle a_1a_2, w \rangle \times \langle t, u \rangle. \)

Then \( C_\theta(E) = Z(E) \times C_\theta(u) \) by Lemma 1.7 and the argument of [8, Lemma 2.4] applies with only a slight modification required if \( n = 2 \).

The discussion at the end of [8, §2] yields

**Lemma 2.7.** The focal subgroup of \( S \) in \( G \) is \( S \) and hence \( G \) has no subgroup of index 2.

3. Centralizers of involutions in \( K_2 \). Clearly

(7) \( C_\theta(t, v) = \langle d_1, d_2, w \rangle A \langle tu \rangle. \)

Transforming by the element \( z \) of Lemma 2.6, we obtain

(8) \( C_\theta(u, v) = \langle d_1^z, d_2^z, w^z \rangle A^z \langle tu \rangle. \)

**Lemma 3.1.** \( \langle d_1d_2 \rangle^z = \langle d_1d_2 \rangle, \langle a_1a_2 \rangle^z = \langle a_1a_2 \rangle \) and

(9) \( w = (w(d_1d_2)^m tu)^z \) for some integer \( m \).

**Proof.** Since \( C_\theta(t, u, v) = \langle \langle t, u \rangle \times \langle d_1d_2, w \rangle \rangle C_\theta(u) \) by Lemma 1.7 and since \( \langle d_1d_2, w \rangle \) is dihedral of order \( q - 8 \), we have

(10) \( O^{2\theta}(C_\theta(t, u, v)) = \langle t, u \rangle \times \langle d_1d_2, w \rangle. \)

Since \( z \in N_\theta(\langle t, u, v \rangle) \), we obtain

(11) \( O^{2\theta}(C_\theta(t, u, v)) = \langle t, u \rangle \times \langle (d_1d_2)^z, w^z \rangle. \)
Clearly the proof of [8, Lemma 3.1] applies with $C_G(t, u, v)$ replaced by 

$$O^2(C_G(t, u, v)).$$

Since $C_G(u, v) = \langle d_t^a, d_t^b, w^a \rangle A^2 \langle tu \rangle$ by (8), $\langle tu, a_1^a, a_2^b \rangle$ is a subgroup of $C_G(u, v)$ of order $2^{2n}$. Also $w$ centralizes $tu$ and, by (9), we have $(a_1^a)^w = (a_2^b)^w = (a_1^a)^w = (a_2^b)^w = (a_1^a)^{-1}$. Thus $C_G(u, v)$ contains the subgroup $T = \langle tu, a_1^a, a_2^b, w \rangle$ of order $2^{2n+1}$. Since $u$ does not lie in the center of a Sylow 2-subgroup of $G$, $T$ is a Sylow 2-subgroup of $C_G(u), C_G(v)$ and $C_G(u, v)$.

**Lemma 3.2.** $C_G(u)$ has a normal subgroup $K$ of index 2 with Sylow 2-subgroup $M = \langle a_1^a, a_2^b, w \rangle$. Also $C_G(u) = K \langle t \rangle$ where $K \cap \langle t \rangle = \{1\}$.

**Proof.** The proof of [8, Lemma 3.2] applies to demonstrate the existence of $K$. Since $u = (tu)^s = (a_1^{2n-1})^s(a(a_1a_2)^{2n-2})^s \in M$ and $t \notin K$, the result follows.

**Lemma 3.3.** $K$ has a normal subgroup $L$ of index 2 with Sylow 2-subgroup $J = \langle a_1a_2, w \rangle$.

**Proof.** Clearly $M' = \langle a_1a_2 \rangle^s = \langle a_1a_2 \rangle$ and so $N_K(M) \leq C_K(t) \leq C_G(u, v) = \langle d_t^a, d_t^b, w^a \rangle A^2 \langle tu \rangle$ which has $\langle (d_t^a)^2 \rangle, (d_t^b)^2 \rangle A^2$ as a normal 2-complement. Hence $N_K(M)$ has a normal 2-complement so that $N_K(M)' \cap M = M'$. With only a slight modification, the proof of [8, Lemma 3.4] yields the result.

Since $u \notin L$, we have

**Lemma 3.4.** If $k$ is any integer and $s = a_1^a w (d_1 d_2)^k$, then $s^{2n-1} = u$, the order of $s$ is $2^n$, $s^t = s^{-1}$ and hence $\langle t, s \rangle$ is dihedral of order $2^{n+1}$. Also $K = L \langle s \rangle$ and $L \cap L = \{1\}$.

**Lemma 3.5.** (a) For $i = 1$ or $2$, if $x \in N_G(L_i) = \mathcal{A}$ and $x$ acts trivially on $L_i/\langle t \rangle$ then $x$ centralizes $L_i$.

(b) If $x \in C_G(B) \cap \mathcal{A}$, then $x \in \langle t \rangle$.

**Proof.** Let $x$ satisfy the hypotheses of (a) and let $y \in L_1 \subseteq SL(2, q)$, $y$ of odd order. Clearly $x$ centralizes $y^2$ and hence $[x, y] = 1$. But $L_1 = SL(2, q)$ is generated by its elements of odd order and (a) follows. Let $x$ satisfy the hypothesis of (b) and let $y_1 \in L_1$. Then $(y_1 y_1^x)^s = y_1 y_1^x = y_1^x y_1^x$, so that $y_1^x y_1^x \in L_1 \cap L_2 = \langle t \rangle$. Hence $x$ acts trivially on $L_1/\langle t \rangle$ and we conclude from (a) that $x$ centralizes $L_1$. Similarly $x$ centralizes $L_2$. Thus $x \in \langle t \rangle$ by Lemma 1.3.

We shall need the following well-known result:

**Lemma 3.6.** Let $H$ be a group with $O(H) = \{1\}$ and such that $H$ contains a normal subgroup $M \cong PSL(2, r)$ with $r$ a prime power. Suppose also that $4 \nmid |H:M|$. Then $H/M$ is Abelian.

**Lemma 3.7.** (a) $u$ centralizes $A$ and hence $A \leq L$.

(b) $B$ is characteristic in $L$.

(c) $L = (B \times O(L))A$ and $(B \times O(L)) \cap A = \{1\}$.
Proof. \(A^z \leq L\) and \(B \leq L\) since \(B\) is simple. Also \(B \cap O(L) = \{1\}\) and \(z \in N_C(A(u))\). Moreover, \(C_0(t,u) = \langle B \times \langle t, u \rangle \rangle C_A(u)\) and \(\langle t \rangle \cap C_0(t,u) = B \times \langle t \rangle\) so that \(C_A(u)\) normalizes \(B\). If \(a \in C_A(u)\) centralizes \(B\), then \(a \in \langle t \rangle\) by Lemma 3.5(b). Hence \(a = 1\) and \(C_A(u)\) acts faithfully on \(B\). But \([O(L) \cap C_A(u), B] \leq B \cap O(L) = \{1\}\). Thus \(O(L) \cap C_A(u) = \{1\}\). By a theorem of Gorenstein and Walter [4, Theorem 1], since \(J\) is dihedral and \(B \cong BO(L)/O(L)\) is an odd indexed subgroup of \(L/O(L)\), we have two cases:

(a) \(L/O(L)\) is isomorphic to the alternating group \(A_4\). Hence \(B\) is isomorphic to a subgroup of \(A_4\) which implies that \(q = 5, 7\) or \(9\). These possibilities have been excluded.

(b) \(L/O(L)\) is isomorphic to a subgroup of \(\text{PGL}(2, r)\) containing \(\text{PSL}(2, r)\) with \(r\) odd. Clearly \(B\) must be isomorphic to a subgroup of \(\text{PSL}(2, r)\) so that \(q = 5\) (which is excluded) or \(r\) is a power of \(q\). Let \(M\) be the subgroup of \(L\) such that \(L/M > O(L)\) and \(M/O(L) \leq \text{PSL}(2, r)\). Thus \(BO(L) \leq M\) and \(L/M\) is odd. Now \(C_0(u,v) = \langle d_1, d_2, x \rangle A^x \leq \langle s, t, u \rangle\). But \(C_0(u,v) = L(s,t)\); hence \(C_0(u,v) = C_0(v) \leq \langle s, t \rangle\). Clearly \(C_0(v) \leq \langle d_1^{2n_2}, d_1d_2, w \rangle \cdot A^z\) which is a product of two groups. But \(|C_0(t,v)| = |C_0(u,v)| = 2(q - 1)^2 \cdot p\) so that \(|C_0(v)| = 2^n e^2 p\) (for \(L \cap \langle s, t \rangle = \{1\}\)). Now \(\langle d_1^{2n_2}, d_1d_2, d_2 \rangle\) has a normal 2-complement, namely, \(\langle d_1^{2n_2}, (d_1d_2)^{2n_2} \rangle \cap A^z = \langle (d_1d_2)^{2n_2} \rangle \cap A^z\)

Thus \(|d_1^{2n_2}, d_1d_2, w \rangle \cdot A^z| = 2^n e^2 p = |C_0(v)|\) and so \(C_0(v) = \langle d_1^{2n_2}, d_1d_2, w \rangle \cdot A^z\). Clearly \(L = O^2(C_0(u))\).

Now \(M/O(L) = (L/O(L))'\) implies that \(M = L'O(L) \leq C_0(u)\). Moreover \(L \cap \langle t \rangle = \{1\}\) and \(L \langle t \rangle\) is a subgroup of \(C_0(u)\) such that \(O(L \langle t \rangle) = O(L)\) and \(L \langle t \rangle / O(L)\) contains a normal subgroup \(M/O(L)\) of index twice an odd integer. Now Lemma 3.6 applies, and we conclude that \(L \langle t \rangle/M\) is Abelian. Hence \([d_1^{2n_2}, w] = [d_1^{2n_2}, t] = d_1^{-2n_2}d_1^{2n_2} = (d_1^{-1}d_2)^{2n_2} \in M\); but \((d_1^{-1}d_2)^{2n_2 - 2}\) has order \(e\), therefore \((d_1^{-1}d_2)^{2n_2 - 2} \in M\).

Suppose that \(w = w^e(d_1d_2)^{2n_2} \in S^z A^z\). Then \(w \in S^z\), \(tu \in S^z\) and \(tu \in S^z\) which is impossible. Thus \(C_0(v) \cap S^z A^z = \langle d_1^{2n_2}, d_1d_2 \rangle A^z\) and \(C_0(v) \cap S^z = \langle d_1^{2n_2}d_1d_2 \rangle = \langle (d_1d_2)^{2n_2 - 1}, d_1d_2 \rangle\) is normalized by \(A^z\). Also \(C_M(v) = \langle (d_1d_2)^{2n_2 - 1}, d_1d_2, w \rangle \cdot (A^z \cap M)\) is normalized by \(A^z\) and \(C_M(v) \leq \langle (d_1d_2)^{2n_2 - 1} \rangle \times \langle v, w \rangle\). However, \(C_{M/O(L)}(v, w) = (v, w) O(L)/O(L)\) since \(M/O(L) = \text{PSL}(2, r)\). Thus \((d_1d_2)^{2n_2 - 1} \in O(L)\) and \(A^e\) normalizes \(C_0(v) \cap S^z \cap O(L) = \langle (d_1d_2)^{2n_2 - 1} \rangle\). Thus \(A\) normalizes \(\langle (d_1d_2)^{2n_2} \rangle = \langle (d_1d_2)^{2n_2 - 1} \rangle\) which is a group of order \(e > 1\). In the proof of Lemma 2.4, we saw that \(A\) normalizes both \(\langle d_1 \rangle\) and \(\langle d_2 \rangle\). Hence \(A\) normalizes \(\langle d_1^{2n_2} \rangle\) and \(\langle d_2^{2n_2} \rangle\) which are groups of order \(e > 1\). Let \(a \in [A, u]\) and let \(\langle (d_1d_2)^{2n_2} \rangle = \langle (d_1d_2)^{2n_2} \rangle\) where \(\gamma\) is an integer relatively prime to \(e\). Let \(d_1^{2n_2} = d_1^{2n_2}\) where \(a\) is an integer relatively prime to \(e\).
Then $d_2^{a_2u} = d_1^{a_2u} = d_1^{a_2u+1} = d_2^{a_2u} = d_2^{a_2u}$ where $\beta$ is an integer relatively prime to $e$ such that $a\beta \equiv 1 \pmod{e}$ (since $u$ inverts $a$). Hence $(d_1d_2^{-1})^{a_2} = d_1^{a_2}d_2^{-a_2} = d_2^{a_2}d_2^{-a_2}$. Thus $\alpha \equiv \beta \pmod{e}$ and $a^2 \equiv 1 \pmod{e}$. Since $a$ is of odd order, $a$ centralizes $d_1^{a_2}$ and $d_2^{a_2}$. Consequently $a$ centralizes $\langle d_1 \rangle \times \langle d_2 \rangle$ and similarly for $\langle d_2 \rangle$. But a subgroup of order $q - \delta$ contained in $SL(2, q)$ is self-centralizing in $\text{Aut}(SL(2, q))$; thus there exist elements $d_1' \in \langle d_1 \rangle$ and $d_2' \in \langle d_2 \rangle$ such that $d_1'd_2a$ centralizes $L_1L_2 = \hat{S}$. Hence $d_1'd_2a \in C_\hat{S}(\hat{S}) = \langle t \rangle$ and $a \in \hat{S} \cap A = \{1\}$. We have proved that $[A, u] = 1$ and thus $u$ centralizes $A$ and $z$ normalizes $A = C_A(u)$. Hence $C_A(v, w) = \langle \langle d_1d_2^{-1} \rangle^{a_2} \rangle \times \langle v, w \rangle \rangle$ has a normal 2-complement $\langle \langle d_1d_2^{-1} \rangle^{a_2} \rangle A$ and $C_M(v, w) = \langle \langle d_1d_2^{-1} \rangle^{a_2} \rangle \times \langle v, w \rangle \rangle (M \cap A)$. Again $C_{M, O(L)}(\langle v, w \rangle O(L)) = \langle v, w \rangle O(L)$ and $M \cap A \leq O(L)$ so that $M \cap A = \{1\}$. Consequently $C_M(v) = \langle \langle d_1d_2^{-1} \rangle^{a_2} \rangle$, $d_1d_2w$ by $O(L)$, taking orders, we conclude that $r + 1 = q - \delta$ (because $O(L) \cap \langle \langle d_1d_2^{-1} \rangle^{a_2} \rangle$, $d_1d_2$, $w) = \langle \langle d_1d_2^{-1} \rangle^{a_2} \rangle$). But $r$ is a power of $q$; thus $q = r$ and $M = B O(L)$. Since $|C_M(v)| = e(q - \delta) = d_2^{a_2}$, $|C_M(v) \cap O(L)| = e$ and $w, tw$ are conjugate to $v$ in $B$, we also have $|C_{O(L)}(w)| = |C_{O(L)}(w)| = e$. But $|O(L) \cap C_M(v, w)| = e$, and we may apply [3, Lemma 3] to conclude that $|O(L)| = e$. Thus $O(L) = \langle \langle d_1d_2^{-1} \rangle^{a_2} \rangle$. Since $C_M(O(L)) \leq M$, $C_M(O(L))$ contains $\langle O(L), v \rangle$ and $M/O(L)$ is simple, we conclude that $C_M(O(L)) = M$; thus $M = B \times O(L)$. We have $B = M'$ so that (b) follows. Also all of the involutions of $B$ are conjugate in $B$ and $|L/B|$ is odd. Hence $|C_2(v) : C_M(v)| = |L : M| = |A|$, and $L = (B \times O(L))A$ where $(B \times O(L)) \cap A = M \cap A = \{1\}$. This finishes the proof of the lemma.

**Corollary 3.7.1.** $O^2(C_G(t)) = \hat{S}\langle u \rangle$.

Lemma 3.5(b) yields

**Lemma 3.8.** $C_{AB}(B) = \{1\}$.

Consequently $AB$ acts faithfully by conjugation on $B$, and we have an injection $\theta : AB = BA \to \text{Aut}(B)$ such that $\theta$ maps $B$ onto $\text{Inn}(B)$. Thus

$$|AB| = \rho|B| \left| \left| \text{Aut}(B) \right| \right| = 2|B|f$$

so that $\rho$ divides $f$. Moreover, the image of $\theta$, $\text{Im}(\theta)$, is the unique subgroup of $\text{Aut}(B)$ which contains $\text{Inn}(B)$ as a subgroup of index $\rho$. (This follows from the structure of $\text{Aut}(B)$ and the fact that $\rho$ is odd.) Also $A \simeq AB/B$ must be cyclic.

Let $F_q$ denote the field of $q = p^f$ elements and let $SL(2, F_q)$ denote the multiplicative group of $2 \times 2$ matrices with coefficients in $F_q$ of determinant $1$. Fix an isomorphism

$$\phi_1 : SL(2, F_q) \to L_1.$$ 

Let

$$\phi_2 = I_u \circ \phi_1 : SL(2, F_q) \to L_2.$$
where $I_u$ denotes conjugation by $u$. Clearly

$$
\phi_1\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) = t = \phi_2\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right).
$$

Thus we have the obvious isomorphism

(14) $$
\phi: \text{PSL}(2, F_q) = \text{SL}(2, F_q) / \left\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle \to B
$$

defined by:

if $x \in \text{SL}(2, F_q)$, then $\phi\left(x\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right) = \phi_1(x)\phi_2(x)$.

Since $\rho|/ f$, we may let $f = \rho f_i$ where $f_i$ is a positive integer. Let $\tilde{\sigma}$ denote the automorphism of $F_q$ of order $\rho$ defined by

$$
\tilde{\sigma}(x) = x^{\rho f_i} \text{ for } x \in F_q.
$$

Then $\tilde{\sigma}$ induces an automorphism of order $\rho$ of $\text{SL}(2, F_q)$ and of $\text{PSL}(2, F_q)$

$$
= \text{SL}(2, F_q) / \left\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle.
$$

Since the natural semidirect product $\langle \tilde{\sigma} \rangle \text{ Inn}(\text{PSL}(2, F_q))$ is the unique subgroup of $\text{Aut}(\text{PSL}(2, F_q))$ which contains $\text{Inn}(\text{PSL}(2, F_q))$ of index $\rho$, it follows from the above discussion of the subgroup $AB$ that there exists an element $\sigma \in AB$ of order $\rho$ such that $AB = \langle \sigma \rangle B, B \cap \langle \sigma \rangle = \{1\}$ and

(15) $$
\phi\left(x\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right)^\sigma = \phi\left(x^{\sigma}\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right) \text{ for all } x \in \text{SL}(2, F_q).
$$

But $AB \leq N_{\sigma}(L_1) \cap N_{\sigma}(L_2)$ so that $\phi_1(x)^{\sigma}\phi_2(x)^{\sigma} = \phi_1(x^{\sigma})\phi_2(x^{\sigma})$. Thus $\phi_1(x)^{-\sigma}\phi_1(x^{\sigma}) 
\in L_1 \cap L_2 = \langle t \rangle$ for all $x \in \text{SL}(2, F_q)$. It follows as in the proof of Lemma 3.5(a) that

(16) $$
\phi_i(x)^{\sigma} = \phi_i(x^{\sigma}) \text{ for } i = 1, 2 \text{ and all } x \in \text{SL}(2, F_q).
$$

Moreover, $\Phi \cap (AB) = B = \Phi \cap \langle \langle \sigma \rangle B \rangle$ implies that $\Phi \cap \langle \sigma \rangle = \{1\}$ and

$$(B \times O(L)) \cap (AB) = B = (B \times O(L)) \cap \langle \langle \sigma \rangle B \rangle$$

implies that $(B \times O(L)) \cap \langle \sigma \rangle = \{1\}$. It follows that we may replace $A$ by $\langle \sigma \rangle$, consequently from now on, we will assume

(17) $$
A = \langle \sigma \rangle.
$$

If $\delta = 1$, let $\epsilon$ denote a generator of the multiplicative group $F_q^\times$ of $F_q$ and set

(18) $$
d = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
$$

If $\delta = -1$, note that $f$ is odd and $p \equiv -1 \pmod{4}$ also. Let $\epsilon$ denote a generator of the multiplicative group $F_p^\times$ of $F_p$. Since $F_p^\times$ has no element of order 4, $\sqrt{\epsilon} \notin F_q$ so that, using the norm from $F_p^\times \rightarrow F_p^\times$, there exist elements $\lambda, \mu \in F_p$ such that
\[ \lambda^2 - \epsilon \mu^2 = -1 \] and, using the norm from \( F_{q^2} \to F_q \), there exist elements \( \alpha, \beta \in F_q \) such that \( \alpha + \beta \sqrt{\epsilon} \) is a generator of the subgroup of elements of \( F_{q^2} \) of norm 1. Set

\[
d = \begin{bmatrix} \alpha & \beta \\ \epsilon \beta & \alpha \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \lambda & \mu \\ -\epsilon \mu & -\lambda \end{bmatrix}.
\]

Thus \( b, d \in SL(2, F_q) \) and it is easy to see that \( \sigma \) centralizes \( b \) in both cases and \( d^0 = d^{p_1} \) if \( \delta = 1 \) and \( d^0 = d^{-p_1} \) if \( \delta = -1 \). Moreover, \( b^2 = -I \) and \( b \) inverts \( d \).

By (16), we have \( \phi_i(b)^e = \phi_i(b) \) for \( i = 1, 2 \) and \( \phi_i(d)^e = \phi_i(d)^{e \delta p_1} \) for \( i = 1, 2 \). Hence \( \phi_i(d)^e = \phi_i(d)^{e \delta p_1} \) and since \( p_1 \equiv \delta (\text{mod } 2^n) \) we have \( \phi_i(d)^e = \phi_i(d)^e \).

Now note that we may assume

\[
b_i = \phi_i(b), \quad a_i = \phi_i(d)^e, \quad d_i = \phi_i(d) \quad \text{for } i = 1, 2.
\]

**Lemma 3.9.** For a suitable integer \( k \) and \( s = \alpha_2 \nu(d_1 d_2)^k \), we have

\[
C_0(u) = \langle s, t \rangle = ((B \times O(L)) \langle \sigma \rangle) \langle s, t \rangle
\]

where \( B \cong PSL(2, q) \), \( O(L) \) is cyclic of order \( e \), and \( \langle t, s \rangle \) is dihedral of order \( 2^{n+1} \) where \( t^2 = s^{2^n} = 1, s^4 = s^{-1} \) and \( s^{2^n-1} = u \). Also \( B, O(L), L = (B \times O(L)) \langle \sigma \rangle \) and \( (B \times O(L)) \langle s, t \rangle \) are normal in \( C_0(u) \) and \( O^{2e}(C_0(u)) = (B \times O(L)) \langle s, t \rangle \). The involution \( t \) centralizes \( B \) and inverts \( O(L) \); the element \( s \) centralizes \( O(L) \) and for any

\[
x \in SL(2, F_q)
\]

we have

\[
\phi(x \left[ \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right])^e = \phi(x \left[ \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right])
\]

where \( \delta = [1 \ 0 \ 0 \ 1] (\delta = 1) \) or \([0 \ 1 \ 1 \ 0] (\delta = -1) \). (Here \( e \) denotes a generator of the multiplicative group of \( F_q \).) Also, if \( \delta = -1 \), then \( s \) and \( \sigma \) commute, and if \( \delta = 1 \), then \( (sw)^e \) and \( \sigma \) commute and \( \sigma \) commutes with \( \langle \alpha \rangle \) and \( \langle \alpha \rangle \) centralizes \( \alpha \), \( \sigma \) inverts \( (d_1 d_2)^{k2^n} \) and \( \sigma^2 = \sigma \) since \( s \) inverts \( (d_1 d_2) \). Thus \( (B \times O(L)) \langle s, t \rangle \langle C_0(u) \rangle \). But \( t \) inverts \( O(L) \) and \( B \) is simple, hence, \( (B \times O(L)) \langle s, t \rangle \leq O^{2e}(C_0(u)) \); therefore,

\[
O^{2e}(C_0(u)) = (B \times O(L)) \langle s, t \rangle.
\]

Moreover, when \( \delta = -1 \), \( [s, \alpha] \in C_0(B) \cap B = \{ 1 \} \) so that \( [s, \alpha] = 1 \). When \( \delta = 1 \), then \( \sigma^e \sigma = \sigma(d_1 d_2)^{m \kappa 2^n} \) so that \( \sigma^{e \kappa 2^n} = \sigma(d_1 d_2)^{m \kappa 2^n} = \sigma \), finishing the proof.

A well-known lemma of John Thompson is
Lemma 3.10. Let $G$ be a finite group with more than one class of involutions. Let $\alpha, \beta$ be two nonconjugate involutions and let $R$ consist of a set of representatives for the conjugacy classes of involutions of $G$. For $t \in R$, let $\alpha(\alpha, \beta, t)$ denote the number of ordered pairs of involutions $(\alpha, \lambda)$ where $\alpha, \lambda \in C_G(t)$ such that $\alpha$ is conjugate in $G$ to $\alpha$, $\lambda$ is conjugate in $G$ to $\beta$, and such that $t \in \langle (\alpha \lambda) \rangle$. Then

$$|G| = |C_G(\alpha)| \cdot |C_G(\beta)| \sum_{t \in R} \alpha(\alpha, \beta, t)|C_G(t)|.$$  

We can now prove

Lemma 3.11. $|G| = |\text{PSp} (4, q)| \rho = \frac{1}{4} q^4(q^2 + 1)(q^2 - 1)\rho$.

Proof. All involutions of $C_G(t)$ and $C_G(u)$ are contained in $O^2'(C_G(t)) = (L_1 L_2)\langle u \rangle$ and $O^2'(C_G(u)) = (B \times O(L))\langle s, t \rangle$ respectively. Moreover, the structures of $O^2'(C_G(t))$ and $O^2'(C_G(u))$ are completely determined.

It is easy to see that the $G$-fusion of involutions in both $C_G(u)$ and $C_G(t)$ is determined. Since the semidirect product $H = \langle \sigma \rangle \text{PSp} (4, q)$ satisfies the hypotheses of the theorem, it follows from the above discussion and Lemmas 2.6 and 3.10 that

$$|G| = |H| = \rho |\text{PSp} (4, q)| = \frac{1}{4} q^4(q^2 + 1)(q^2 - 1)\rho.$$  

4. The $p$-structure of $G$. For $\alpha \in F_q$, set

$$(23) \theta(\alpha) = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}. $$

Then $\{\theta(\alpha) \mid \alpha \in F_q\}$ is a Sylow $p$-subgroup of $\text{SL}(2, F_q)$. Set

$$(24) \theta_i = \phi_i \circ \theta : F_q \to L_i$$

and

$$(25) P_i = \text{Image} (\theta_i) \text{ for } i = 1, 2.$$  

Then $\theta_i : F_q \to P_i$ is an isomorphism of the additive group of $F_q$ onto $P_i$ which is a Sylow $p$-subgroup of $L_i$ for $i = 1, 2$. Thus $P_i$ is elementary Abelian of order $q$ and is normalized by $A = \langle \sigma \rangle$. The subgroup

$$(26) R = P_1 P_2 = P_1 \times P_2$$

is a Sylow $p$-subgroup of $\hat{S}$. The subgroup

$$(27) D_1 = \{\theta_1(\alpha)\theta_2(\alpha) \mid \alpha \in F_q\}$$

is a Sylow subgroup of $B$ and is elementary Abelian of order $q$. Set

$$(28) D_2 = \{\theta_1(\alpha)\theta_2(-\alpha) \mid \alpha \in F_q\}.$$  

Then

$$(29) R = P_1 \times D_1 = D_1 \times D_2.$$
Put
\[(30)\quad h = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} \quad \text{and} \quad h_i = \phi_i(h) \quad \text{for} \quad i = 1, 2,\]

where \(\varepsilon\) is a generator of the multiplicative group of \(F_q\). If \(\delta = 1\), use the same generator \(\varepsilon\) as that used in the element \(\delta\) of Lemma 3.9. Note that \(h = d\) when \(\delta = 1\), but not when \(\delta = -1\). Then

\[(31) \quad H = \langle h_1, h_2 \rangle\]

is an Abelian subgroup of \(\mathfrak{S}\) of order \(\frac{1}{2}(q-1)^2\) and

\[(32) \quad h_1^{q-1/2} = h_2^{q-1/2} = t.\]

Clearly

\[(33) \quad N_G(R) \cap C_G(t) = (RH\langle u \rangle \langle \sigma \rangle).\]

Put
\[(34) \quad y = (sw)^{2^a - 2e} \quad \text{if} \quad \delta = 1, \quad \text{and} \quad y = s^{2^a - 2e} \quad \text{if} \quad \delta = -1,\]

where \(s\) is the element of \(C_G(u)\) referred to in Lemma 3.9. Then

\[(35) \quad y \in N_G(D_1) \quad \text{and} \quad [y, \sigma] = 1.\]

\[(36) \quad (h_1 h_2)^y = h_1 h_2 \quad \text{and} \quad w^y = vw \quad \text{if} \quad \delta = 1 \quad \text{and} \quad w^y = w \quad \text{if} \quad \delta = -1.\]

Also

\[(37) \quad y^2 = uv \quad \text{if} \quad \delta = 1 \quad \text{and} \quad y^2 = u \quad \text{if} \quad \delta = -1\]

and

\[(38) \quad t^y = tu \quad \text{and} \quad (tu)^y = t.\]

Thus \(R^u\) is a Sylow \(p\)-subgroup of \(O^2(C_G(tu))\) by (38).

Now \(D_1 \leq R \cap R^u \leq C_G(t) \cap C_G(tu) = C_G(t, u) = \langle \langle t, u \rangle \times B \rangle \langle \sigma \rangle \). But

\[(39) \quad R \leq O^2(C_G(t)) = \mathfrak{S} \langle u \rangle\]

so that \(R \cap R^u \leq O^2(C_G(t)) \cap C_G(t, u) = (\mathfrak{S} \langle u \rangle) \cap C_G(t, u) = \langle \langle t, u \rangle \times B \rangle.\) Hence

\[(39) \quad D_1 = R \cap R^u.\]

The proof of [8, Lemma 4.1] yields

**LEMMA 4.1.** If \(\{1\} < F \leq O(L)\), then \(N_G(F) \cap R = N_G(F) \cap R^u = D_1.\)

**LEMMA 4.2.** The group \(C_G(D_1)\) has a normal \(2\)-complement \(M\) which is a semi-direct product \(M = O(L)Q, Q \triangleleft M, O(L) \cap Q = \{1\}\) and where \(Q = R^u R\) and \(|Q| = q^3.\)
Proof. It is easy to see that

\[ C_\sigma(D_1) \cap C_\sigma(u) = (D_1 \times O(L))\langle t, s^2 \rangle \]

and

\[ C_\sigma(D_1) \cap C_\sigma(t) = R\langle t, u \rangle. \]

Thus

\[ C_\sigma(D_1) \cap C_\sigma(tu) = R^v\langle t, u \rangle. \]

Now the proof of [8, Lemma 4.2] applies to finish the proof.

Note that \( Q \) is characteristic in \( M \), which is characteristic in \( C_\sigma(D_1) \), which is normal in \( N_\sigma(D_1) \) so that

\[ N_\sigma(D_1) \subseteq N_\sigma(Q). \]

Lemma 4.3. The group \( Q \) is elementary Abelian of order \( q^3 \) and is the normal 2-complement of \( C_\sigma(R) \). Also

\[ C_\sigma(Q) = Q. \]

Proof. From \( N_\sigma(R) \cap C_\sigma(t) = (RH\langle u \rangle)\langle v \rangle, \) we conclude that \( C_\sigma(R) \cap C_\sigma(t) = R\langle t \rangle. \) Hence \( \langle t \rangle \) is a Sylow 2-subgroup of \( C_\sigma(R) \) so that \( C_\sigma(R) \) has a normal 2-complement \( K \). The four group \( \langle t, u \rangle \) normalizes \( R \) and hence acts on \( K \). Also \( C_\sigma(t) = R \) and \( C_\sigma(u) \cap C_\sigma(R) \leq C_\sigma(u) \cap C_\sigma(D_1) = (D_1 \times O(L))\langle t, s^2 \rangle. \) But

\[ C_\sigma(R) \cap O(L) = \{1\} \]

by Lemma 4.1 so that \( C_\sigma(u) = D_1 \); consequently, \( C_\sigma(tu) \geq C_\sigma(t, u) = D_1. \) Assume that \( C_\sigma(tu) \cap R^v = D_1. \) Then

\[
C_\sigma(tu) \cap K = K \cap C_\sigma(tu) \cap C_\sigma(D_1) = C_\sigma(tu) \cap (R^v\langle t, u \rangle) \\
= C_\sigma(tu) \cap R^v = D_1.
\]

Then [8, Lemma 3.6] implies that \( K = R \). The Frattini argument implies that

\[
N_\sigma(R) = C_\sigma(R)(N_\sigma(R) \cap C_\sigma(t)) = (R\langle t \rangle)(N_\sigma(R) \cap C_\sigma(t)) \\
= N_\sigma(R) \cap C_\sigma(t) = (RH\langle u \rangle)\langle v \rangle.
\]

Hence \( R\langle v \rangle \) contains a Sylow \( p \)-subgroup of \( N_\sigma(R) \). However, \( Q \cap \langle v \rangle \leq C_\sigma(D_1) \cap \langle v \rangle = \{1\} \) and \( \langle v \rangle \leq N_\sigma(Q) \) so that \( N_\sigma(R)\langle v \rangle \leq N_\sigma(R). \) Since \( N_\sigma(R) > R, \) \( R\langle v \rangle \) cannot contain a Sylow \( p \)-subgroup of \( N_\sigma(R) \) which is a contradiction. Thus \( C_\sigma(tu) \cap R^v > D_1, \) and the argument of [8, Lemma 4.3] yields \( K = Q, \) \( R^v \leq C_\sigma(R) \) and \( C_\sigma(Q) = Q, \) completing the proof.

Note that \( C_\sigma(R) = Q\langle t \rangle \) and \( N_\sigma(R) = (Q\langle t \rangle)(N_\sigma(R) \cap C_\sigma(t)) = (QH\langle u \rangle)\langle v \rangle. \) Also \( Q \) is characteristic in \( C_\sigma(R) \) which is normal in \( N_\sigma(R) \). Hence

\[ N_\sigma(R) \leq N_\sigma(Q). \]
Set

\[ P_3 = D_2^y \quad \text{and} \quad \theta_3(a) = (\theta_1(a)\theta_2(-a))^y \quad \text{for} \ a \in F_q. \]

Then, by (29) and (35), we have \( R^y = D_1 \times P_3 \) so that

\[ Q = P_1 \times P_2 \times P_3. \]

Moreover, \( H \leq N_\sigma(R) \leq N_\sigma(Q) \), and the proof of [8, Lemma 4.4] (provided we observe that \([v, \sigma]=1\) so that replacing \( s \) by \( sv \), if necessary, is permitted) yields

**Lemma 4.4.** *The action of \( H \) on \( Q \) is given by*

\[
\begin{align*}
  h_1: \quad & \theta_1(a) \to \theta_1(e^2a), \quad \theta_2(a) \to \theta_2(a), \quad \theta_3(a) \to \theta_3(ea), \\
  h_2: \quad & \theta_1(a) \to \theta_1(a), \quad \theta_2(a) \to (e^2a), \quad \theta_3(a) \to \theta_3(ea).
\end{align*}
\]

**Lemma 4.5.** Let \( V = O(C_\sigma(P_1)) \). Then \( C_\sigma(P_1) = L_2V \), \( L_2 \cap V = \{e\} \) and \( N_\sigma(P_1) = (L_2V\langle h_1 \rangle\langle \sigma \rangle) \). The group \( V/P_1 \) is Abelian, \( V \) is nilpotent of class at most 2 and \( Q \cap V = P_1 \times P_3 \) and \( (L_2V\langle h_1 \rangle) \cap \langle \sigma \rangle = \{e\} \).

**Proof.** Clearly \( N_\sigma(P_1) \cap C_\sigma(t) = \langle P_1\langle h_1 \rangle\rangle \langle \sigma \rangle \) so that \( C_\sigma(P_1) \cap C_\sigma(t) = P_1L_2 \). The proof of [8, Lemma 4.5] applies to give all but the last equation of the lemma.

Finally, by considering the action of an element of \( (C_\sigma(P_1)\langle h_1 \rangle) \cap \langle \sigma \rangle \) on \( P_1 \), we have \( (C_\sigma(P_1)\langle h_1 \rangle) \cap \langle \sigma \rangle \leq C_\sigma(P_1) \cap \langle \sigma \rangle = 1 \). This completes the proof.

**Lemma 4.6.** \( C_\sigma(P_3) \) has a normal 2-complement, and \( O(C_\sigma(P_3)) \leq QH \).

**Proof.** If \( \delta = 1 \), then \( a_2^{2z} = 1 \); \( D_1 \to D_2 \), so that \( C_\sigma(D_2) \) has a normal 2-complement by Lemma 4.2. If \( \delta = -1 \), then \( N_\sigma(D_2) \cap C_\sigma(t) = (R\langle h_1h_2 \rangle\langle t, u \rangle\langle \sigma \rangle) \). Thus \( \langle t \rangle \) is a Sylow 2-subgroup of \( C_\sigma(D_2) \) so that \( C_\sigma(D_2) \) has a normal 2-complement. Since \( P_3 = D_2^y \), \( C_\sigma(P_3) \) always has a normal 2-complement. In either case, \( \langle t, u \rangle \leq N_\sigma(D_2) \) so that \( \langle t, u \rangle \) acts on \( O(C_\sigma(D_2)) \). From the above, we have \( O(C_\sigma(D_2)) \cap C_\sigma(t) = R \). Since \( D_1 \leq B < C_\sigma(u) \) and every pair of distinct Sylow \( p \)-subgroups of \( B \) generates \( B \), every odd ordered subgroup of \( C_\sigma(u) \) containing \( D_1 \) is contained in \( N_\sigma(D_1) \cap C_\sigma(u) \). By Lemma 3.9, this group has a normal 2-complement \( \langle \sigma \rangle FO(L) \) where \( F = D_1\langle (h_1h_2)^{2z} \rangle \) is the normal 2-complement of \( N_\sigma(D_1) \). If \( C_\sigma(D_2) \) contains an element of form \( x^t \) where \( x \in F \) and \( z \in O(L) \), then \( \langle t, x^t \rangle \) is a Sylow 2-subgroup of \( C_\sigma(D_2) \). By Lemma 4.1, \( t^z = 1 \) so that \( z = 1 \). Hence \( O(C_\sigma(D_2)) \cap C_\sigma(u) \leq \langle \sigma \rangle F \). However, no nontrivial element of form \( (h_1h_2)^{2z} \) centralizes \( D_2 \). Hence \( O(C_\sigma(D_2)) \cap C_\sigma(u) = D_1 \). Since \( D_1 \) is normalized by \( \langle \sigma \rangle \), there exists a maximal odd order subgroup \( X \) of \( C_\sigma(t) \) containing \( D_1 \) which is normalized by \( \langle \sigma \rangle \). Thus \( X \leq \widetilde{\Sigma}\langle \sigma \rangle \) and \( X = (X \cap \widetilde{\Sigma})\langle \sigma \rangle \). But \( D_1 \leq X \cap \widetilde{\Sigma} \) and \( RH \) contains a unique largest odd order subgroup of \( \widetilde{\Sigma} \) containing \( D_1 \) and hence \( X \leq RH\langle \sigma \rangle \). Since \( \langle \sigma \rangle \leq N_\sigma(D_2)^{-1} \) and \( D_1 \leq O(C_\sigma(D_2)^{-1}) \cap C_\sigma(t) \), we conclude that \( O(C_\sigma(D_2)^{-1}) \cap C_\sigma(t) \leq RH\langle \sigma \rangle \), so that \( O(C_\sigma(D_2)) \cap C_\sigma(u) \leq R RH\langle \sigma \rangle \). Hence \( O(C_\sigma(D_2)) \leq D_1 RH\langle \sigma \rangle \), so that \( O(C_\sigma(D_2)) \leq QH\langle \sigma \rangle \). Since no element of \( H\langle \sigma \rangle - H \) centralizes \( P_3 \), \( O(C_\sigma(P_3)) \leq QH \).
Lemma 4.7. \( V = O(C_G(P_1)) \) is a p-group and \( V = P_1 P_3 P_4 \) where \( P_4 = P_3^c \) with \( c = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) and \( c_2 = \phi_2(c) \).

Proof. Now
\[
C_V(P_3) \leq V \cap O(C_G(P_3))
\]
\[
= V \cap O(C_G(P_3)) \cap QH \cap C_G(P_1) = V \cap O(C_G(P_3)) \cap (Q\langle h_2 \rangle) \leq V \cap Q
\]
\[
= P_1 \times P_3 \leq C_V(P_3)
\]
so that \( C_V(P_3) = P_1 \times P_3 \) which is a p-group. Since \( V \) is nilpotent, \( V \) is a p-group. Also \( c_2 \in C_G(t) \cap C_G(P_1) \) so that \( P_4 = P_3^c \leq V \). Since \( h_1h_2^{-1} \in C_G(P_3) \), we have \( (h_1h_2^{-1})^2 = h_1h_2 \in C_G(P_4) \). But \( h_1h_2 \) acts fixed point free on \( P_1 \times P_3 \) by Lemma 4.4; hence \( P_1 P_3 \cap P_4 = \{1\} \). We have seen that \( P_1 \cong V \) so \( P_1 P_3 P_4 \) is a subgroup of \( V \) of order \( q^3 \). But \( |P_2 V\langle \sigma \rangle| = q^3 |V| \) so that \( |V| \leq q^3 \) by Lemma 3.11. Thus \( V = P_1 P_3 P_4 \) which completes the proof of the lemma.

Note that \([a, c_2] = 1\) so that if
\[
\theta_4(\alpha) = \theta_3(\alpha)^{c_2} \quad \text{for} \ \alpha \in F_q,
\]
then
\[
\theta_4(\alpha^e) = (\theta_3(\alpha)^{c_2})^{c_2} = \theta_3(\alpha^{c_2}) = \theta_4(\alpha^e) \quad \text{for} \ \alpha \in F_q.
\]
Also \( c_2 \) centralizes \( h_1 \) and inverts \( h_2 \) so that
\[
\theta_4(h_1) = \theta_4(e) \quad \text{and} \quad \theta_4(h_2) = \theta_4(e^{-1}) \quad \text{for} \ \alpha \in F_q
\]
and
\[
H\langle \sigma \rangle \cong N_0(P_4).
\]

Lemma 4.8. If \( U = P_2 V = P_1 P_2 P_3 P_4 \), then \( N_0(U) \cap N_0(P_1) = UH\langle \sigma \rangle \leq N_0(Q) \). Also \( (UH) \cap \langle \sigma \rangle = \{1\} \), \( P_1 P_3 \unlhd U \) and \( U/P_1 P_3 \) is Abelian.

Proof. Since \( U = QP_4 \), we have \( N_{P_4}(Q) \cap Q \neq \{1\} \). But if \( \alpha \in F_q \), then \( \theta_4(\alpha)^{h_1} = \theta_3(\alpha)^{c_2} \) by Lemma 4.4 and \( h_1 \in N_0(Q) \). Thus \( h_1 \) normalizes \( N_{P_4}(Q) \) and acts irreducibly on \( P_4 \). Consequently, \( N_{P_4}(Q) = P_4 \) so \( Q \triangleleft U \). Then
\[
N_0(U) \cap N_0(P_1) = N_0(U) \cap ((L_2 V\langle h_1 \rangle) \langle \sigma \rangle) = ((N_0(U) \cap L_2 V\langle h_2 \rangle) \langle \sigma \rangle = ((N_0(P_2) \cap L_2 V\langle h_1 \rangle) \langle \sigma \rangle = UH\langle \sigma \rangle \cong N_0(Q).
\]
Clearly \( P_1 \leq Z(U) \) and \( UH\langle \sigma \rangle \leq N_0(P_1) \). By considering the action of \( UH\langle \sigma \rangle \) on \( P_1 \), we conclude that \( UH \cap \langle \sigma \rangle = \{1\} \). Also \( P_1 P_3 = Q \cap \langle \sigma \rangle \leq U \) and \( U = QV \). However, \( \langle Q, V \rangle \leq Q \cap V = P_1 \times P_3 \) and \( Q/P_1 P_3 \) and \( V/P_1 P_3 \) are Abelian, consequently \( U/P_1 P_3 \) is Abelian.

5. The (BN)-pair. The proof of [8, Lemma 5.1] yields

Lemma 5.1.
\[
u: \theta_1(\alpha) \rightarrow \theta_2(\alpha), \ \theta_2(\alpha) \rightarrow \theta_1(\alpha), \ \theta_3(\alpha) \rightarrow \theta_3(-\alpha)
\]
and
\[
c_2: \theta_1(\alpha) \rightarrow \theta_1(\alpha), \ \theta_3(\alpha) \rightarrow \theta_4(\alpha), \ \theta_4(\alpha) \rightarrow \theta_3(-\alpha)
\]
for all \( \alpha \in F_q \).
Since \( \langle t \rangle \) is characteristic in \( H \), \( N_\alpha(H) = N_\alpha(H) \cap C_\alpha(t) = \langle H, u, c_2 \rangle \langle \sigma \rangle \). Set

\[
N = (\langle u \rangle) \cap N_\alpha(H) = \langle H, u, c_2 \rangle.
\]

Clearly \( \langle \sigma \rangle \) normalizes \( N \).

**Lemma 5.2.** The structure of \( N = \langle H, u, c_2 \rangle \) is determined by the relations:

\[
h_1^{-1/2} = h_2^{-1/2} = c_2 = t, \quad t^2 = u^2 = 1, \quad [h_1, h_2] = [h_1, c_2] = 1, \quad h_1^u = h_2, \quad h_2^u = h_2^{-1}, \quad (uc_2)^4 = 1.
\]

The group \( W = N/H \) is dihedral of order 8.

The proof of this lemma is the same as that of [8, Lemma 5.2].

Set \( r_1 = uH \) and \( r_2 = c_2H \); then \( r_1, r_2 \) are involutions which generate \( W = N/H \). The elements of \( W \) written in shortest possible form in terms of \( r_1 \) and \( r_2 \) are as given in [8, p. 30]. For \( \alpha \in W \), let \( \lambda(\alpha) \) be the numbers of factors \( r_i \) when \( \alpha \) is expressed in its shortest form. Also define \( \omega: W \to N \) as in [8, p. 30] so that \( \alpha = \omega(\alpha)H \) for all \( \alpha \in W \). If \( K \) is any subgroup of \( G \) containing \( H \), write \( \alpha K \) and \( K \omega(\alpha) \) for the cosets \( \omega(\alpha)K \) and \( K \omega(\alpha) \) respectively. If \( K \) is any subgroup normalized by \( H \), write \( K^\alpha \) for \( K \omega(\alpha) \). Set

\[
\mathcal{B} = UH.
\]

Clearly \( \mathcal{B} \cap N = H \).

The proofs of the following two lemmas are the same as those of [8, Lemmas 5.3 and 5.4].

**Lemma 5.3.** Let \( G_i = \mathcal{B} \cup \mathcal{B} r_i \mathcal{B} \) for \( i = 1, 2 \). Then \( G_1 \) and \( G_2 \) are subgroups of \( G \).

**Lemma 5.4.** If \( \alpha \in W \) and \( \lambda(r_i \alpha) \geq \lambda(\alpha) \) for \( i = 1 \) or 2, then \( r_1 \mathcal{B} \alpha \subseteq \mathcal{B} r_1 \mathcal{B} \).

**Lemma 5.5.** Set \( G_0 = \mathcal{B} \mathcal{N} \mathcal{B} \). Then \( G_0 \) is a subgroup of \( G \), and \( G_0 \) is the disjoint union of the eight double cosets \( \mathcal{B} \alpha \mathcal{B} \) for \( \alpha \in W \). Also \( \langle \sigma \rangle \trianglelefteq N_\alpha(\mathcal{B}) \cap N_\alpha(N) \trianglelefteq N_\alpha(G_0) \) and \( G_0 \cap \langle \sigma \rangle = \{1\} \).

**Proof.** The first statement follows from Lemmas 5.3 and 5.4 by a theorem of Tits [7]. Since the second statement is already known, it remains to prove that \( G_0 \cap \langle \sigma \rangle = \{1\} \). But \( G_0 = U \mathcal{N} U \) and \( N \cap (U \langle \sigma \rangle) = N \cap C_\alpha(t) \cap (U \langle \sigma \rangle) = N \cap (R \langle \sigma \rangle) = \{1\} \) by the structure of \( C_\alpha(t) \) and the lemma follows.

Using our knowledge of the action of \( H \) on \( P_4 \) as given in (50), the same proof of [8, Lemma 5.6] yields

**Lemma 5.6.** \( U \cap U' t_1 t_2 t_3 t_4 = \{1\} \).

Similarly the arguments of [8, Lemmas 5.7 and 5.8] yield

**Lemma 5.7.** \( |G_0| = |PSp(4, q)| \).

**Lemma 5.8.** \( G_0 \cong PSp(4, q) \), \( G_0 \trianglelefteq G_0 \langle \sigma \rangle \), \( \langle \sigma \rangle \cap G_0 = \{1\} \) and \( O(G) = \{1\} = C_\alpha(G_0) \).

Finally \( G \) is isomorphic to the semidirect product \( \langle \sigma \rangle \bigcirc PSp(4, q) \) described in the introduction.
Proof. By Lemmas 5.5, 5.7 and 3.11, we have that $G = G_0 \langle \sigma \rangle \triangleright G_0$. We now prove that $G_0 \cong \text{PSp}(4, q)$.

Clearly $G_0$ contains $\langle P_2, P_2^2 \rangle = L_2$ and hence $G_0$ contains $L_1 = L_2^k$. Thus $C_0(\tau) \cap G_0 = (L_1L_2)^{\langle \mu \rangle}$ by Lemma 5.5. Since $|G/G_0|$ is odd, $t$ and $uv$ are involutions in $G_0$ which are fused in $G_0$ by Lemma 2.6. Now [8] implies that $G_0 \cong \text{PSp}(4, q)$. The rest follows easily.

References


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