SOME INVARIANT $\sigma$-ALGEBRAS FOR MEASURE-PRESERVING TRANSFORMATIONS

BY

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Abstract. For an invertible measure-preserving transformation $T$ of a Lebesgue measure space $(X, \mathcal{B}, m)$ and a sequence $N$ of integers, a $T$-invariant partition $\mathcal{A}_N(T)$ of $(X, \mathcal{B}, m)$ is defined. The relationship of these partitions to spectral properties of $T$ and entropy theory is discussed and the behaviour of the partitions $\mathcal{A}_N(T)$ under group extensions is investigated. Several examples are discussed.

0. Introduction. For an invertible measure-preserving transformation $T$ of a Lebesgue space $(X, \mathcal{B}, m)$ and a sequence $N=\{n_i\}_{i=1}^\infty$ of integers we define a $\sigma$-algebra by $\mathcal{A}_N(T) = \{A \in \mathcal{B} \mid m(T^{n_i}A \Delta A) \to 0\}$. Our aim is to study these $\sigma$-algebras. In §1 we evaluate those elements of $L^2(X, \mathcal{B}, m)$ which are measurable with respect to $\mathcal{A}_N(T)$. The connections the algebras $\mathcal{A}_N(T)$ have with discrete spectrum and entropy theory are discussed in §2. Every ergodic $T$ with discrete spectrum has $\mathcal{A}_N(T) = \mathcal{B}$ for some sequence $N$ and every $T$ with $\mathcal{A}_N(T) = \mathcal{B}$ for some sequence $N$ has zero entropy. It turns out that the algebras $\mathcal{A}_N(T)$ have properties in common with the $\sigma$-algebra generated by the eigenfunctions of $T$ and also properties in common with the $\sigma$-algebra generated by all the finite algebras having zero entropy relative to $T$. Some of these properties are noted in §3 which also contains remarks on the relationship of the $\sigma$-algebras $\mathcal{A}_N(T)$ to mixing properties of $T$. The behaviour of the $\sigma$-algebras $\mathcal{A}_N(T)$ under group extensions is discussed in §4 and in §5 we use Gaussian processes to give examples of weak mixing transformations with $\mathcal{A}_N = \mathcal{B}$ for some sequence $N$. §6 is devoted to a discussion of further properties of the algebras $\mathcal{A}_N(T)$.

Throughout $T$ will denote an invertible measure-preserving transformation of a Lebesgue space $(X, \mathcal{B}, m)$ [14]. $\mathcal{N}$ will denote the trivial $\sigma$-algebra consisting of those members of $\mathcal{B}$ with measure 0 or 1. $\nu$ will denote the trivial partition and $\varepsilon$ will denote the partition into points of any Lebesgue space. Greek letters $\xi, \eta, \zeta$ etc. will be used to denote measurable partitions. We shall use partitions and their associated $\sigma$-algebras interchangeably. The factor space of $X$ by $\xi$ will be denoted by $X/\xi$ and if $T\xi = \zeta$ the factor transformation induced by $T$ on $X/\xi$ will be denoted by $T\xi$. If $\mathcal{A}$ denotes the $\sigma$-algebra generated by the members of $\xi$ then $L^2(\xi)$ and $L^2(\mathcal{A})$ will both denote the collection of all elements of $L^2(X, \mathcal{B}, m)$ measurable
with respect to $\mathcal{A}$. In particular $L^2(e)$ and $L^2(\mathcal{B})$ will stand for $L^2(X, \mathcal{B}, m)$. $U_T$ will denote the unitary operator of $L^2(\mathcal{B})$ defined by $f \mapsto f \circ T$ and $\| \cdot \|_2$ will denote the norm on an $L^2$-space. We shall repeatedly use the spectral theorem which implies that for each $f \in L^2(\mathcal{B})$ there is a Borel measure $\sigma_f$ on the unit circle $K$ with $(U_T f, f) = \int_K \lambda^n \, d\sigma_f(\lambda) \forall n \in \mathbb{Z}$. $K$ will always denote the unit circle. $\sigma_f$ is called the spectral measure of $f$.

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1. The $\sigma$-algebras $\mathcal{A}_N(T)$. For a sequence $N = \{n_i\}$ of integers let $\mathcal{A}_N(T) = \{A \in \mathcal{B} \mid m(T^{n_i} A \Delta A) \to 0\}$. We show below that $\mathcal{A}_N(T)$ is a $\sigma$-algebra. The corresponding partition will be denoted by $\alpha_N(T)$. Let $\mathcal{A}(T) = \bigvee_N \mathcal{A}_N(T)$ (the refinement is taken over all sequences $N$ of integers) and let $\alpha(T)$ denote the corresponding partition. We have $T\mathcal{A}_N(T) = \mathcal{A}_N(T)$, $T\alpha_N(T) = \alpha_N(T)$, $T\mathcal{A}(T) = \mathcal{A}(T)$ and $T\alpha(T) = \alpha(T)$. When $T$ is understood we shall write $\mathcal{A}_N$, $\alpha_N$, $\mathcal{A}$ and $\alpha$.

**Theorem 1.** $\mathcal{A}_N(T)$ is a $\sigma$-algebra.

**Proof.** Clearly $\mathcal{A}_N(T)$ is closed under complementation. $\mathcal{A}_N$ is finitely additive since if $A_1, A_2 \in \mathcal{A}_N$ then

\[
T^n(A_1 \cup A_2) \Delta (A_1 \cup A_2) \subset (T^nA_1 \Delta A_1) \cup (T^nA_2 \Delta A_2)
\]

implies $A_1 \cup A_2 \in \mathcal{A}_N$. It remains to show that if $A_j \in \mathcal{A}_N$ ($j \geq 1$) and $A_1 \subset A_2 \subset A_3 \subset \cdots$ then $A = \bigcup_{j=1}^\infty A_j \in \mathcal{A}_N$. Let $\epsilon > 0$ be given. Choose $j_0$ so that $m(A \setminus A_{j_0}) < \epsilon$. Choose $I$ so that $i > I \Rightarrow m(T^n A_{j_0} \Delta A_{j_0}) < \epsilon$. Then

\[
i > I \Rightarrow m(T^nA \Delta A) \leq m(T^nA \Delta T^nA_{j_0}) + m(T^nA_{j_0} \Delta A_{j_0}) + m(A_{j_0} \Delta A) < 3\epsilon.
\]

Therefore $A \in \mathcal{A}_N$ and $\mathcal{A}_N$ is countably additive.

Our next aim is to show $L^2(\mathcal{A}_N(T)) = \{f \in L^2(\mathcal{B}) \mid \| U^n f - f \|_2 \to 0\}$.

**Lemma 1.** Let $f \in L^2(\mathcal{B})$ be real valued and nonconstant. Let $N = \{n_i\}$ be a sequence of integers. If $\| U^n f - f \|_2 \to 0$ then $f^{-1}(\mathcal{C}) \subset \mathcal{A}_N(T)$, where $\mathcal{C}$ denotes the $\sigma$-algebra of Borel subsets of $R$.

**Proof.** Let $b \in R$. Put $B = \{x \mid f(x) \leq b\}$ and $B_\epsilon = \{x \mid f(x) \leq b + \epsilon\}$. Let $\delta > 0$ be given. On $T^{-n} B \setminus B_\epsilon$ we have $|f(T^n x) - f(x)| \geq \epsilon$ and therefore $m(T^{-n} B \setminus B_\epsilon) \to 0$ as $i \to \infty$. Since $B_\epsilon \setminus B$ decreases with $\epsilon$ and $\bigcap_{\epsilon > 0} (B_\epsilon \setminus B) = \emptyset$, choose $\epsilon_0$ so that $m(B_\epsilon \setminus B) < \delta$. Choose $i_0$ so that $i > i_0$ implies $m(T^{-n} B \setminus B_{\epsilon_0}) < \delta$. Then $m(T^{-n} B \setminus B) \leq m(T^{-n} B \setminus B_{\epsilon_0}) + m(B_{\epsilon_0} \setminus B) < 2\delta$ if $i > i_0$. Therefore $m(T^{-n} B \Delta B) \to 0$. We have shown $f^{-1}(\mathcal{C}) \subset \mathcal{A}_N(T)$ and by Theorem 1 $f^{-1}(\mathcal{C}) \subset \mathcal{A}_N(T)$.}

**Theorem 2.** $L^2(\mathcal{A}_N(T)) = \{f \in L^2(\mathcal{B}) \mid \| U^n f - f \|_2 \to 0\}$.

**Proof.** Let $\mathcal{H}$ denote the right-hand side. Certainly $L^2(\mathcal{A}_N(T)) \subset \mathcal{H}$. Suppose $f \in \mathcal{H} \setminus L^2(\mathcal{A}_N(T))$. We can assume $f$ is real valued since either the real or imaginary part of $f$ does not belong to $L^2(\mathcal{A}_N(T))$ but belongs to $\mathcal{H}$. By Lemma 1 $f^{-1}(\mathcal{C}) \subset \mathcal{A}_N(T)$ (where $\mathcal{C}$ = Borel subsets of $R$) and hence $f \in L^2(\mathcal{A}_N(T))$, a contradiction.
Our aim is to study the algebras \( \mathcal{A}_h(T) \). Of particular interest are those transformations with \( \mathcal{A}_h(T) = \mathcal{B} \) (\( a_h(T) = \epsilon \)) for some sequence \( N \), those with \( \mathcal{A}(T) = \mathcal{B} \) (\( a(T) = \epsilon \)) and those with \( \mathcal{A}(T) = \mathcal{N} \) (\( a(T) = \nu \)). The condition \( \mathcal{A}_h(T) = \mathcal{B} \) means \( T^n \) converges to the identity in the space of invertible measure-preserving transformations of \( (X, \mathcal{B}, m) \) with the weak topology [6] or equivalently \( U_{\mathcal{B}}^T \) converges to \( I \) in the space of unitary operators of \( L^2(\mathcal{B}) \) with the weak (or strong) topology. The following result relates the property \( \mathcal{A}_h(T) = \mathcal{B} \) to the maximal spectral type of \( T \). For the theory of spectral measures and types see [13].

**Theorem 3.** \( \mathcal{A}_h(T) = \mathcal{B} \iff \int_K |\lambda^n - 1|^2 \sigma(\lambda) \to 0 \) where \( \sigma \) denotes a finite measure on \( K = \{z \mid |z| = 1\} \) whose type is the maximal spectral type of \( T \).

**Proof.** Suppose the right-hand side holds and \( h \in L^1(\sigma) \). We shall show \( \int_K |\lambda^n - 1|^2 h(\lambda) \sigma(\lambda) \to 0 \). Let \( \delta > 0 \) be given and choose \( h_1, h_2 \) so that \( h = h_1 + h_2, \) \( h_1 \) is bounded (\( |h_1(\lambda)| \leq c_0 \) say) and \( \int |h_2(\lambda)| \sigma(\lambda) < \delta \).

\[
\left| \int |\lambda^n - 1|^2 h(\lambda) \sigma(\lambda) \right| 
\leq \int |\lambda^n - 1|^2 |h_1(\lambda)| \sigma(\lambda) + \int |\lambda^n - 1|^2 |h_2(\lambda)| \sigma(\lambda) 
< c_0 \int |\lambda^n - 1|^2 \sigma(\lambda) + 4\delta < 5\delta
\]

if \( i > i_0 \) and \( i_0 \) is chosen so that \( i > i_0 \) implies \( \int |\lambda^n - 1|^2 \sigma(\lambda) < \delta/c_0 \). Hence \( \int |\lambda^n - 1|^2 h(\lambda) \sigma(\lambda) \to 0 \). If \( f \in L^2(\mathcal{B}) \) the spectral measure \( \sigma_f \) of \( f \) is absolutely continuous with respect to \( \sigma \) and by the above \( \|U^n f - f\|_2 = \int |\lambda^n - 1|^2 \sigma_f(\lambda) \to 0 \). Hence \( f \in L^2(\mathcal{A}_h(T)) \) and \( \mathcal{A}_H(T) = \mathcal{B} \).

Conversely if \( \mathcal{A}_h(T) = \mathcal{B} \) then choosing \( f \in L^2(\mathcal{B}) \) with spectral measure \( \sigma_f \) of maximal type we have \( \int |\lambda^n - 1|^2 \sigma_f(\lambda) = \|U^n f - f\|_2 \to 0 \). By the above, if \( \sigma \) is any measure whose type is the maximal spectral type of \( T \) then \( \int |\lambda^n - 1|^2 \sigma(\lambda) \to 0 \).

2. **Some properties of \( \mathcal{A}_h(T) \).** The simplest examples of transformations with \( \mathcal{A}_h = \mathcal{B} \) are given by

**Theorem 4.** If \( T \) is ergodic with discrete spectrum there exists a sequence \( N = \{n_i\} \) with \( \mathcal{A}_{\mu}(T) = \mathcal{B} \).

**Proof.** We can suppose \( T \) is an ergodic rotation \( Tx = ax \) on a compact abelian group \( G \) [6, p. 48]. Choose \( N = \{n_i\} \) so that \( a^{n_i} \to e \) the identity element of \( G \). If \( \gamma \) is a character of \( G \) \( \|U^n \gamma - \gamma\|_2 = |\gamma(a^{n_i}) - 1|^2 \to 0 \). Since the characters generate \( L^2(G) \) we have \( \mathcal{A}_{\mu}(T) = \mathcal{B} \).

Later we shall give more examples of transformations with \( \mathcal{A}_h = \mathcal{B} \).

The algebras \( \mathcal{A}_h(T) \) are related to the work of Katok and Stepin [7] and Chacon and Schwartzbauer [1] on approximation by periodic transformations. It is easily checked that if \( T \) admits an approximation of the second kind by periodic transformations (a.p.t.II) with speed \( o(1/n) \) in the sense of Katok and Stepin [7, p. 78] then \( \mathcal{A}_{(p_n)}(T) = \mathcal{B} \). Also if \( T \) admits an approximation by periodic automorphisms in Chacon and Schwartzbauer's sense [1] then \( \mathcal{A}_{(p_n)}(T) = \mathcal{B} \).
Now we discuss the relationship of the partitions $\alpha_N(T)$ to entropy theory. The notations for entropy are from [15]. Let $\mathcal{D}$ denote the set of partitions with finite entropy. Pinsker [12] has defined the maximum partition with zero entropy for $T$ as $\pi(T) = \bigvee \{ \xi \in \mathcal{D} \mid h(T, \xi) = 0 \}$. We have $T\pi(T) = \pi(T)$ and if $\eta \in \mathcal{D}$ then $\eta \leq \pi(T)$ if and only if $h(T, \eta) = 0$. Using the concept of sequence entropy introduced by Kushnirenko [8], one can define the maximum partition with zero $N$-entropy for $T$ (for a sequence of integers $N$) by $\pi_N(T) = \bigvee \{ \xi \in \mathcal{D} \mid h_N(T, \xi) = 0 \}$. It is straightforward to check that $T\pi_N(T) = \pi_N(T)$ and if $\eta \in \mathcal{D}$ then $\eta \leq \pi_N(T)$ if and only if $h_N(T, \eta) = 0$. The main result of Kushnirenko’s paper [8] implies $\pi_N(T) = \bigwedge_N \pi_N(T)$ is the maximum partition for $T$ such that the associated factor transformation has discrete spectrum. In other words $\pi_N(T)$ is the partition generated by the eigenfunctions of $T$.

**Theorem 5.** For every sequence $N$ of integers, $\alpha_N(T) \leq \pi(T)$ and $\alpha_N(T) \leq \pi_N(T)$. Hence $\alpha(T) \leq \pi(T)$.

**Proof.** We first show $\alpha_N(T) \leq \pi_N(T)$. Suppose $\xi$ is a finite partition and $\xi \leq \alpha_N(T)$. We have

$$H(T^{n_1}\xi \vee T^{n_2}\xi \vee \cdots \vee T^{n_k}\xi) \leq H(T^{n_1}\xi) + H(T^{n_2}\xi) + \cdots + H(T^{n_k}\xi)$$

so if $H(T^{n_1}\xi) = \cdots = H(T^{n_k}\xi) = 0$ then $h_N(T, \xi) = 0$ and $\xi \leq \pi_N(T)$. But $\mathcal{A}_N(T) = \mathcal{B}$ implies $\mathcal{A}_{(n_1, \ldots, n_k)}(T) = \mathcal{B}$ and this readily implies $H(T^{n_1}\xi) = 0$. Hence $\alpha_N(T) \leq \pi_N(T)$.

Similarly, $\alpha_N(T) \leq \pi(T)$ since $\xi \leq \alpha_N(T)$ is finite $h(T, \xi) = H(T^{n_1}\xi) = 0$. Hence $\alpha(T) \leq \pi(T)$.

**Corollary 5.1.** $h(T_{\alpha_N(T)}) = 0$, $h_N(T_{\alpha_N(T)}) = 0$ and $h(T_{\alpha(T)}) = 0$.

By Theorem 4 we know there exists a sequence $N$ such that $\pi_N(T) \leq \alpha_N(T)$. This and Theorem 5 indicate that the partitions $\alpha_N(T)$ may inherit some of the properties of $\pi_N(T)$ and some of the properties of $\pi(T)$. This is shown to be so in the later sections.

**Theorem 6.** (a) $\bigcap_N \mathcal{A}_N(T)$ is the $\sigma$-algebra of $T$-invariant members of $\mathcal{B}$.

(b) The class of invertible measure-preserving transformation with $\mathcal{A}_N = \mathcal{B}$ is closed under (i) factors, (ii) countable direct products, and (iii) inverse limits. In fact (iii) can be strengthened to the property:

if $\xi_n \not\approx \xi$ and $T^{n_k} = \xi_n$, $T^{n_k} = \xi$, then $\alpha_N(T_{\xi_n}) \not\approx \alpha_N(T_{\xi})$.

**Proof.** (a) Let $A \in \bigcap_N \mathcal{A}_N(T)$. Then $m(TA\Delta A) \leq m(T^{n+1}A\Delta A) + m(A\Delta T^nA) \rightarrow 0$.

(b) is trivial.

(ii) Let $T_i$ act on $(X_i, \mathcal{B}_i, \mu_i)$ and let $T_\infty = \bigcap_{i=1}^\infty T_i$ acting on $(X, \mathcal{B}, m) = \bigcap_{i=1}^\infty (X_i, \mathcal{B}_i, \mu_i)$. Assume $\mathcal{A}_N(T_i) = \mathcal{B}$ for each $i$. It is easy to show that each measurable rectangle is in $\mathcal{A}_N(T_\infty)$ and hence $\mathcal{A}_N(T_\infty) = \mathcal{B}$. 

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(iii) It suffices to take $\xi = e$. Let $f \in L^2(\alpha_N(T))$ and put $f_n = E(f/\xi_n)$, where $E(f/\xi_n)$ is the conditional expectation of $f$ relative to the $\sigma$-algebra generated by $\xi_n$. Then $\|f - f_n\|_2 \to 0$ and $\|U_{T_n}f_n - f_n\|_2 \leq \|U_{T_n}f - f\|_2 \to 0$. Hence $f_n \in L^2(\alpha_N(T_{T_n}))$ and $\alpha_N(T_{T_n}) \to \alpha_N(T_e)$.

Let $\mathcal{W}$ denote the class of invertible measure-preserving transformations of $(X, \mathcal{B}, m)$ with the weak topology [6]. $\mathcal{W}$ is a complete metric space and hence has the Baire property that a countable intersection of open dense sets is dense. From Theorem 1.1 of [7] it follows that the collection of all transformations with $\mathcal{A}_N = \mathcal{B}$ for some $N$ contains a dense $G_\delta$ in the space $\mathcal{W}$. Since the weak-mixing transformations form a dense $G_\delta$ in $\mathcal{W}$([6]) it follows that the class of all weak-mixing transformations with $\mathcal{A}_N = \mathcal{B}$ for some $N$ contains a dense $G_\delta$ in $\mathcal{W}$. It follows from this and the next theorem that the class of weak-mixing transformations which are not strong-mixing contains a dense $G_\delta$ in $\mathcal{W}$.

3. Mixing properties. An example of a property of $\alpha_N(T)$ inherited from $\pi_\alpha(T)$ is

**Theorem 7.** There are no nonconstant mixing functions in $L^2(\mathcal{A}_N(T))$ (i.e. $(U_{T_n}f, f) \to (f, 1)(f, f)$ for $f \in L^2(\mathcal{A}_N(T))$ implies $f = \text{constant}$).

**Proof.** If $(U_{T_n}f, f) \to (f, 1)(f, f)$ and $f \in L^2(\mathcal{A}_N(T))$ then $(f, f) = (f, 1)(f, f)$ and $f$ is constant.

**Corollary 7.1.** $T_{\alpha_N(T)}$ has singular spectrum.

**Proof.** If the spectrum is not singular there exists $f \in L^2(\mathcal{A}_N(T))$ with absolutely continuous spectral measure $\sigma$. Then $(U_{T_n}f, f) = \int \lambda^n d\sigma(\lambda) \to 0$ by the Riemann-Lebesgue lemma and since $(U_{T_n}f, f) \to \|f\|^2_2$ we have $f = 0$.

**Corollary 7.2.** If $T$ is totally ergodic with quasi-discrete spectrum then $\alpha_N(T) \leq \pi_\alpha(T)$ for all sequences $N$ and there exists a sequence $N$ with $\alpha_N(T) = \pi_\alpha(T)$. Hence $\alpha(T) = \pi_\alpha(T)$.

**Proof.** $L^2(\mathcal{A}) = L^2(\pi_\alpha(T)) \oplus \mathcal{H}$ where $U_T \mathcal{H} = \mathcal{H}$ and $U_T|\mathcal{H}$ has Lebesgue spectrum. Let $f \in L^2(\alpha_N(T))$ and $f = f_1 + f_2$, $f_1 \in L^2(\pi_\alpha(T))$, $f_2 \in \mathcal{H}$. Then $\|U_{T_n}f - f\|_2^2 = \|U_{T_n}f_1 - f_1\|_2^2 + \|U_{T_n}f_2 - f_2\|_2^2$ implies $f_1 \in L^2(\alpha_N(T))$ and $f_2 \in L^2(\alpha_N(T))$. By Theorem 7 $f_2 = 0$ and hence $\alpha_N(T) \leq \pi_\alpha(T)$. The rest of the corollary follows from Theorem 4.

Later we shall give examples of weak-mixing transformations with $\mathcal{A}_N(T) = \mathcal{B}$ for some $N$. Consideration of $\mathcal{A}(T)$ gives an interesting connection with mixing. We first give a definition.

**Definition 1.** $T$ is intermixing if whenever $m(A) > 0$ and $m(B) > 0$, $A, B \in \mathcal{B}$, we have $\lim \inf_{n \to \infty} m(T^n A \cap B) > 0$.

Friedman and Ornstein [2] give examples of intermixing transformations which are not strong-mixing.
Theorem 8.

\[ T \text{ strong-mixing } \implies T \text{ intermixing } \Rightarrow \mathcal{A}(T) = \mathcal{N} \overset{\sim}{=} T \text{ weak-mixing}. \]

**Proof.** \( T \text{ strong-mixing } \Rightarrow T \text{ intermixing } \) is clear. The example of Friedman and Ornstein mentioned above shows the converse is false. If \( T \) is intermixing and \( 0 < m(A) < 1, A \in \mathfrak{B} \), then \( \liminf_{n \to \infty} m(T^n A \cap A^c) > 0 \) and so \( A \notin \mathcal{A}_n(T) \) for any sequence \( N \). Therefore \( \mathcal{A}(T) = \mathcal{N} \). Theorem 4 shows \( \mathcal{A}(T) = \mathcal{N} \Rightarrow T \text{ weak-mixing} \), and the converse is false by the examples of §5.

We do not know if \( \mathcal{A}(T) = \mathcal{N} \) implies \( T \) is intermixing. Pinsker [12] has shown that if \( T \xi = \xi \) and \( \pi(T) = \nu \) then \( \pi(T) \) and \( \xi \) are independent partitions. We shall show the corresponding result for the partitions \( \alpha_n(T) \). Two Borel measures (or types of measures) on the unit circle \( K \) will be called singular modulo \( \{1\} \) if their restrictions to \( K \setminus \{1\} \) are singular.

Theorem 9.

If \( \mathcal{H} \) is a \( U_T \)-invariant subspace of \( L^2(\mathfrak{B}) \) with \( L^2(\mathcal{A}_n(T)) \cap \mathcal{H} = \{0\} \) or the constants, then the maximal spectral types of \( U_T | L^2(\mathcal{A}_n(T)) \) and \( U_T | \mathcal{H} \) are singular modulo \( \{1\} \).

**Proof.** Let \( \sigma \) be a measure with type equal to the maximal spectral type of \( T_{\alpha_n(T)} \) and \( \mu \) a measure with type the maximal spectral type of \( U_T | \mathcal{H} \). If \( \sigma \) and \( \mu \) are not singular modulo \( \{1\} \) there exists a measure \( \tau \) not concentrated on \( \{1\} \) with \( \tau \leq \sigma \) and \( \tau \leq \mu \). As in the proof of Theorem 3, \( \int |\lambda^n - 1|^2 d\tau \to 0 \). Let \( g \in \mathcal{H} \) have spectral measure \( \tau \). \( g \) is not constant and \( ||U_T^n g - g||_2 = \int |\lambda^n - 1|^2 d\tau \to 0 \) so \( g \in L^2(\mathcal{A}_n(T)) \), a contradiction.

The next corollary is the analogue of the result of Pinsker mentioned above.

Corollary 9.1. Suppose \( T \xi = \xi \) and \( \alpha_n(T) = \nu \). Then \( \xi \) and \( \alpha_n(T) \) are independent partitions.

**Proof.** By Theorem 9, \( T \xi \) and \( T_{\alpha_n(T)} \) have singular types mod \( \{1\} \). Let \( f \in L^2(\alpha_n(T)) \) and \( g \in L^2(\xi) \) both have integral zero. Then \( f \) and \( g \) have singular spectral types and hence are orthogonal [13, p. 124].

Corollary 9.2. If \( \alpha_n(T) = e \) then \( T \) is disjoint from all strong-mixing transformations. (For the definition of disjointness see [3].)

**Proof.** By Theorem 8 and Corollary 9.1.

This corollary is a strengthening of Theorem 7. The converse to Corollary 9.2 is false since the transformation of the 2-torus \( T(z, w) = (e^{2\pi i a z}, zw) \), where \( a \) is irrational, is disjoint from all strong-mixing transformations [3] and yet \( \alpha(T) \neq e \) by Corollary 7.2 since \( T \) is totally ergodic with quasi-discrete spectrum.

4. Group extensions. We now investigate how the partitions \( \alpha_n(T) \) behave under group extensions.
Theorem 10. Let $G$ be a compact abelian metric group acting as a group of measure-preserving transformations of $(X, \mathcal{B}, m)$ such that $gT = Tg$. Let $\xi(G)$ denote the partition of $X$ into orbits of $G$. If $\alpha_N(T_{\xi(G)}) = \nu$ then $T_{\alpha_N(T)}$ is conjugate to a rotation on a factor group of $G$. (The triviality of this factor group means $\alpha_N(T) = \nu$ and this will occur if $T$ is weak-mixing.)

Proof. $gT = Tg$ implies $g\alpha_N(T) = \alpha_N(T)$ and so $G$ acts on $X/\alpha_N(T)$. We first show that $G$ acts ergodically on $X/\alpha_N(T)$. Let $\xi(G, N)$ denote the partition of $X$ determined by the partition of the space $X/\alpha_N(T)$ into orbits of $G$. Then $\xi(G, N) \leq \xi(G)$ and so $\alpha_N(T_{\xi(G, N)}) = \nu$. But $\xi(G, N) \leq \alpha_N(T)$ and therefore $\xi(G, N) = \nu$. That $T_{\alpha_N(T)}$ is conjugate to a rotation on a factor group of $G$ follows from Lemma 3 of [11].

Results of this nature have been proved about $\pi(T)$ by Parry [11] and Thomas [17].

Corollary 10.1. Suppose $T$ is totally ergodic and $G$ is a finite group acting as measure-preserving transformations of $(X, \mathcal{B}, m)$ so that $gT = Tg$ for each $g \in G$. If $\alpha_N(T_{\xi(G)}) = \nu$ then $\alpha_N(T) = \nu$.

Proof. By Theorem 10, $X/\alpha_N(T)$ is a finite space and the total ergodicity of $T$ implies it is one point. Hence $\alpha_N(T) = \nu$.

Let $G$ be a compact connected abelian metric group which acts freely as a group of homeomorphisms on a compact metric space $X$. Let $T: X \to X$ be a homeomorphism with $gT = Tg$ for every $g \in G$. Suppose $T$ and $G$ preserve a measure $m$ defined on the completion of the Borel subsets of $X$. $T$ induces a homeomorphism $T_G: X/G \to X/G$ of the orbit space and every lift of $T_G$ to $X$ is of the form $x \to \phi(x)T(x)$ where $\phi \in C_0(X, G) = \{\phi: X \to G \mid \phi$ is continuous and $\phi(gx) = \phi(x) \forall g \in G, x \in X\}$ [4]. $C_0(X, G)$ becomes a complete metric space when endowed with the metric $D(\phi, \psi) = \sup_{x \in X} d(\phi(x), \psi(x))$ where $d$ is an invariant metric for $G$. $T_G$ preserves the measure on $X/G$ determined by $m$ and the maps $x \to \phi(x)T(x)$ preserve the measure $m$. An (unpublished) result of Jones and Parry announced in [4] states that if $T_G$ is weak-mixing the set of $\phi$ making $x \to \phi(x)T(x)$ weak-mixing contains a dense $G_\delta$ in $C_0(X, G)$. From this, Theorem 8 and Theorem 10 we conclude

Corollary 10.2. (i) If $\alpha_N(T_G) = \nu$ and $T_G$ is weak-mixing the set of $\phi \in C_0(X, G)$ having the property that $x \to \phi(x)T(x)$ has $\alpha_N = \nu$ contains a dense $G_\delta$ in $C_0(X, G)$.

(ii) If $\alpha(T_G) = \nu$ the set of $\phi \in C_0(X, G)$ having the property that $x \to \phi(x)T(x)$ has $\alpha = \nu$ contains a dense $G_\delta$ in $C_0(X, G)$.

We now consider the problem of extending a transformation with $\alpha_N = \nu$ to obtain one with the same property. We shall consider only extensions by $\mathbb{Z}^2 = \{1, -1\}$. The measure on $\mathbb{Z}^2$ is always taken to be the measure giving weight $\frac{1}{2}$ to each point.

Theorem 11. Let $(Y, \mathcal{G}, \mu)$ be a Lebesgue space and let $X = Y \times \mathbb{Z}^2$. Define $T: X \to X$ by $T(y, e) = (Sy, \phi(y)e)$ where $S: Y \to Y$ is measure-preserving and
\( \phi : Y \to \mathbb{Z}^2 \) is measurable. If \( \mathcal{A}_\phi(S) = \mathcal{C} \) then
\[
\mathcal{A}_\phi(T) = \mathcal{B} \Rightarrow \mu(\{y \mid \phi(S^{n-1}y)\phi(S^{n-2}y) \cdots \phi(y) = -1\}) \to 0.
\]

**Proof.** Suppose \( \mathcal{A}_\phi(T) = \mathcal{B} \) and take \( f(y, e) = e \). Then
\[
\int |\phi(S^{n-1}y) \cdots \phi(y) - 1|^2 d\mu(y) = \|U_y^n f - f\|_2^2 \to 0 \quad \text{as} \quad i \to \infty
\]
and hence \( \mu(\{y \mid \phi(S^{n-1}y) \cdots \phi(y) = -1\}) \to 0 \).

Conversely, if this condition holds, the above function \( f \) belongs to \( L^2(\mathcal{A}_\phi(T)) \) and hence \( \mathcal{A}_\phi(T) = \mathcal{B} \).

We use this in the following theorem the proof of which comes from ideas in [7].

**Theorem 12.** Let \( T : K \times \mathbb{Z}^2 \to K \times \mathbb{Z}^2 \) be defined by \( T(z, e) = (e^{2\pi i n_1 z}, \varphi(z)e) \) where
\[
\varphi(z) = \begin{cases} 
-1 & \text{if } \arg z \leq \gamma 2\pi \\
1 & \text{if } \arg z > \gamma 2\pi
\end{cases} \quad (0 < \gamma < 1).
\]

Then \( \mathcal{A}_\phi(T) = \mathcal{B} \) if there exist integers \( p_i \) and even integers \( r_i \) with \( (p_i, n_1) = 1 \), \( |a - p_i|/n_1| = o(1/|n_1^n|) \) and \( |\gamma - r_i/n_1| = o(1/n_1) \).

**Proof.** For the proof we shall consider the circle group \( K \) as the additive group \([0, 1)\) with addition modulo 1. Set
\[
\phi(x) = \phi(x) \cdots \phi((n_1 - 1)a + x)
\]
and
\[
\phi^*_i(x) = \phi(x) \cdots \phi((n_1 - 1)/(n_i)p_i + x);
\]
\( \{x \mid \phi(x) = -1\} \subseteq \{x \mid \phi^*_i(x) = -1\} \cup \{x \mid \phi(x) \neq \phi^*_i(x)\} \).

Let \( \mu \) denote Lebesgue measure on \([0, 1)\).
\[
\mu(\{x \mid \phi(x) \neq \phi^*_i(x)\}) \leq \sum_{j=0}^{n_1-1} \mu(\{x \mid \phi(ja + x) \neq \phi((ja/jp_i + x)\})
\leq \sum_{j=0}^{n_1-1} \mu(\{ja - jp_i/n_1\} \leq (n_1(n_1 + 1)/2)o(1/n_1^n) \to 0.
\]

If \( n_1 \gamma - r_i \geq 0 \) then
\[
\phi^*_i(x) = (-1)^{r_i+1} \quad \text{if } \{n_1x\} \leq n_1 \gamma - r_i,
\]
\[
= (-1)^{r_i} \quad \text{if } \{n_1x\} > n_1 \gamma - r_i.
\]

If \( n_1 \gamma - r_i < 0 \) then
\[
\phi^*_i(x) = (-1)^{r_i} \quad \text{if } \{n_1x\} \leq 1 + n_1 \gamma - r_i,
\]
\[
= (-1)^{r_i-1} \quad \text{if } \{n_1x\} > 1 + n_1 \gamma - r_i.
\]

Therefore
\[
\mu(\{x \mid \phi^*_i(x) = -1\}) = \mu(\{x \mid \{n_1x\} \leq n_1 \gamma - r_i\} \cup \{x \mid \{n_1x\} > 1 + n_1 \gamma - r_i\})
\leq 2|n_1 \gamma - r_i| \to 0 \quad \text{as} \quad i \to \infty.
\]

Hence \( \mu(\{x \mid \phi(x) = -1\}) \to 0 \) as \( i \to \infty \) and \( \mathcal{A}_\phi(T) = \mathcal{B} \) by Theorem 11.
5. Further examples. In this section we shall consider some weak-mixing transformations with \( \mathcal{A}_N = \mathcal{B} \) for some \( N \). \( T \) will be the shift generated by a stationary Gaussian process. Let \( X = \prod_{\mathbb{N}} R \), \( \mathcal{C} \) the product \( \sigma \)-algebra generated by the Borel subsets of \( R \) and let \( p_j \) denote the \( j \)th coordinate function. Hence if \( x = \{x_n\} \), then \( p_j(x) = x_j \). One assigns a probability measure to \((X, \mathcal{C})\) by requiring that \( \{p_j\} \) be a stationary Gaussian process with covariance sequence \( R(n) \) where \( R(n) = \int_K \lambda^n \, d\mu(\lambda) \) and \( \mu \) is a finite measure on the unit circle \( K \) symmetric with respect to the real axis. \( \mu \) is called the covariance measure of the process. Let \( \mathcal{B} \) denote the completion of \( \mathcal{C} \) and let the measure on \( \mathcal{B} \) be \( m \). \( T \) is then defined by \( p_{i-1}(x) = p_i(Tx) \), and is an invertible measure-preserving transformation of \((X, \mathcal{B}, m)\). Hence every symmetric finite Borel measure on \( K \) is the covariance measure of a stationary Gaussian process.

**Theorem 13.** Let \( T \) be the shift on a stationary Gaussian process with covariance measure \( \mu \). Then \( \mathcal{A}_N(T) = \mathcal{B} \Leftrightarrow \int |\lambda^n - 1|^2 \, d\mu(\lambda) \to 0 \).

**Proof.** Suppose \( \mathcal{A}_N(T) = \mathcal{B} \). \( \int |\lambda^n - 1|^2 \, d\mu(\lambda) = \| \sum p_k - p_1 \|_2^2 \to 0 \). Conversely \( \int |\lambda^n - 1|^2 \, d\mu(\lambda) \to 0 \) implies \( p_1 \in L^2(\mathcal{A}_N(T)) \) and hence \( p_k \in L^2(\mathcal{A}_N(T)) \) for each \( k \) and hence \( L^2(\mathcal{A}_N(T)) = L^2(\mathcal{B}) \).

**Theorem 14.** Let \( \mu \) be a continuous symmetric finite measure concentrated on \( D \cup D^{-1} \) where \( D \) is a Kronecker subset of \( K \). Let \( T \) be the shift on the Gaussian process determined by \( \mu \). Then \( T \) is weak-mixing and \( \mathcal{A}_N(T) = \mathcal{B} \) for some sequence \( N \) (and hence is not strong-mixing or intermixing).

**Proof.** The conclusion about mixing is in [10]. We have

\[
\int_{D \cup D^{-1}} |\lambda^n - 1|^2 \, d\mu(\lambda) \leq \int_D |\lambda^n - 1|^2 \, d\mu(\lambda) + \int_{D^{-1}} |\lambda^n - 1|^2 \, d\mu(\lambda)
\]

\[
= 2 \int_D |\lambda^n - 1|^2 \, d\mu(\lambda).
\]

Let \( \epsilon_i \to 0 \) and for each \( i \) choose \( n_i \in \mathbb{Z} \) with \( \sup_{x \in D} |1 - x^{n_i}| < \epsilon_i \). This is possible since \( D \) is a Kronecker set. Then \( \int_D |\lambda^n - 1|^2 \, d\mu(\lambda) < e_i^2 \mu(D) \to 0 \) as \( i \to \infty \). Therefore \( \mathcal{A}_N(T) = \mathcal{B} \) by Theorem 13 if \( N = \{n_i\} \).

We also note the following

**Theorem 15.** If \( S \) is an invertible measure-preserving transformation with \( \alpha_S(S) = \epsilon \) and \( S \) does not have discrete spectrum there exists a weak-mixing shift \( T \) of a stationary Gaussian process with \( \alpha_S(T) = \epsilon \).

**Proof.** Let \( \mu_S \) denote a measure having type equal to the maximal spectral type of \( S \). \( \mu_S \) can be chosen symmetric with respect to the real axis. Let \( \mu \) be its continuous part which is nontrivial (by the assumption that \( S \) does not have discrete spectrum) and is symmetric. By the proof of Theorem 3, since \( \mu \ll \mu_S \) we have \( \int_K |\lambda^n - 1|^2 \, d\mu(\lambda) \to 0 \). So letting \( T \) be the shift defined on the Gaussian process with covariance
measure \( \mu \) we obtain a transformation with \( \alpha_\infty(T) = e \) by Theorem 13, and \( T \) is weak-mixing since \( \mu \) is a continuous measure [9].

Other examples of weak-mixing transformations with \( \mathcal{A}_\infty = \mathcal{B} \) for some \( N \) are constructed in [7] by taking transformations induced from rotations of the unit circle.

6. Problems. We now discuss whether properties of ergodic transformations with discrete spectrum carry over to ergodic transformations with \( \alpha_\infty(T) = e \) for some \( N \). Ergodic transformations with discrete spectrum have simple spectrum and this may account for the fact that some properties do not carry over. We first note that if \( T \) is ergodic and \( \alpha_\infty(T) = e \) then \( T \) need not have simple spectrum, for we could choose \( T \) weak-mixing and then \( T \times T \) does not have simple spectrum but is ergodic and \( \alpha_\infty(T \times T) = e \) (Theorem 6).

If \( T \) is ergodic with discrete spectrum then \( T \) is coalescent, i.e. if \( S \) is measure-preserving and \( ST = TS \) then \( S \) is invertible. All transformations with simple spectrum have this property but it does not hold for all ergodic \( T \) with \( \alpha_\infty(T) = e \). Let \( T \) acting as \((X, \mathcal{B}, \lambda)\) be weak-mixing and \( \alpha_\infty(T) = e \) then \( T_\infty = \prod_{i=1}^\infty T \) acting on \( Y = \prod_{i=1}^\infty X \) is ergodic with \( \alpha_\infty(T_\infty) = e \) but commutes with the 1-sided shift with state space \( X \).

The main result of [1] is that if \( T \) admits an approximation by periodic automorphisms (in the sense of Chacon and Schwarzbaucer) and if \( S \) is an invertible measure-preserving transformation commuting with \( T \) there exists a sequence \( \{j_n\} \) of integers such that \( m(T^{j_n}A\Delta SA) \to 0 \) for every \( A \in \mathcal{B} \). This property is, of course, true for an ergodic \( T \) with discrete spectrum since every measure-preserving transformation commuting with an ergodic rotation of a compact abelian group is itself a rotation. It is not true in general for ergodic transformations with \( \alpha_\infty = e \) as the following example shows. Let \( T \) acting on \((X, \mathcal{B}, \lambda)\) be weak-mixing with \( \alpha_\infty(T) = e \). Put \( T_\infty = \prod_{i=1}^\infty T \), \( X_\infty = \prod_{i=1}^\infty X \), \( \mathcal{B}_\infty = \prod_{i=1}^\infty \mathcal{B} \), \( m_\infty = \prod_{i=1}^\infty m \), \( S = \prod_{i=1}^\infty T_i \). \( T_\infty \) and \( S \) both act on \( X_\infty \), \( \alpha_\infty(T_\infty) = e \) and \( T_\infty S = ST_\infty \). However there is no sequence \( \{j_n\} \) with \( m_\infty(T^{j_n}A\Delta SA) \to 0 \) for all \( A \in \mathcal{B}_\infty \). However if \( T \) admits an approximation by periodic automorphisms in Chacon and Schwartzbaucer's sense then \( T \) has simple spectrum and so we could pose the following problem that we have been unable to solve. If \( T \) has simple spectrum and \( \alpha_\infty(T) = e \) for some \( N \) and if \( S \) is an invertible measure-preserving transformation with \( ST = TS \) then does there exist a sequence \( \{j_n\} \) of integers with \( m(T^{j_n}A\Delta SA) \to 0 \) for every \( A \in \mathcal{B}_\infty \)?

Another property enjoyed by an ergodic transformation \( T \) with discrete spectrum is that if \( S \) is measure-preserving and \( ST = TS \) then \( \mathcal{A}_M(S) = \mathcal{B} \) for some sequence \( M \). It is possible that if \( T \) has simple spectrum and \( \mathcal{A}_N(T) = \mathcal{B} \) for some sequence \( N \) then each measure-preserving transformation \( S \) commuting with \( T \) has \( \mathcal{A}_M(S) = \mathcal{B} \) for some sequence \( M \). This is false if the condition of simplicity of the spectrum of \( T \) is replaced by ergodicity since we could take \( T \) to be the 2-sided direct product of a weak-mixing transformation with \( \mathcal{A}_\infty = \mathcal{B} \) and then \( T \) commutes with the 2-sided shift \( S \) which is invertible and \( \mathcal{A}_M(S) = \mathcal{N} \) for every sequence \( M \) (Theorem 8).
Another property of ergodic transformations with discrete spectrum is that \( \{B \in \mathcal{B} \mid (B, X \setminus B) \text{ is a generator} \} \) is dense in the metric space \( \mathcal{B} \text{ (mod 0)} \) with the symmetric difference metric [16]. This is also true for totally ergodic transformations with quasi-discrete spectrum [5]. We have been unable to decide whether it is true for ergodic \( T \) with \( A_f(T) = \mathcal{B} \) for some \( N \).

7. Noninvertible transformations. Suppose now that \( T \) is a noninvertible measure-preserving transformation of a Lebesgue space \( (X, \mathcal{B}, m) \). If we define \( A_n(T) = \{A \in \mathcal{B} \mid m(T^{-n}A \setminus A) \to 0\} \) for a sequence \( N = \{n_i\}_{i=1}^\infty \) of nonnegative integers and let \( \alpha_n(T) \) denote the corresponding partition, then \( T^{-1}\alpha_n(T) \leq \alpha_N(T) \) and one can show, as in the proof of Theorem 5, that \( \alpha_n(T) \leq \pi(T) \) for each sequence \( N \). Since \( T_{\pi(T)} \) is an invertible measure-preserving transformation with zero entropy we have \( T^{-1}\alpha_n(T) = \alpha_n(T) \text{ (mod 0)} \) for each sequence \( N \). Hence to study the algebras \( \alpha_n(T) \) for a noninvertible \( T \) it suffices to study \( \alpha_n(T_{\pi(T)}) \) for the invertible transformation \( T_{\pi(T)} \).

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