EXTREME POINTS IN A CLASS OF POLYNOMIALS HAVING UNIVALENT SEQUENTIAL LIMITS

BY

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Abstract. This paper concerns a class \( \mathcal{P}_n \) (defined below) of polynomials of degree less than or equal to \( n \) having the properties: each polynomial which is univalent in the unit disk and of degree \( n \) or less is in \( \mathcal{P}_n \) and if \( \{P_n_k\}_{k=1}^\infty \) is a sequence of polynomials such that \( P_n_k \in \mathcal{P}_n \) and \( \lim_{k \to \infty} P_n_k = f \) (uniformly on compact subsets of the unit disk) then \( f \) is univalent. The approach is to study the extreme points in \( \mathcal{P}_n \) (\( P \in \mathcal{P}_n \) is extreme if \( P \) is not a proper convex combination of two distinct elements of \( \mathcal{P}_n \)). Theorem 3 shows that if \( P \in \mathcal{P}_n \) is extreme then \(((n+1)/n)P(z) - (1/n)zP'(z)\) is univalent and Theorem 6 gives a geometric condition on the image of the boundary of the disk under this mapping in order that \( P \) be extreme. Theorem 10 states that the collection of polynomials univalent in the unit disk and having the property \( P(z)=z+a_nz^n+\cdots+a_2z^2+a_0z \), \( a_0 = 1/n \), are dense in the class \( \mathcal{S} \) of normalized univalent functions. These polynomials have the very striking geometric property that the tangent line to the curve \( P(e^{i\theta}), 0 \leq \theta \leq 2\pi \), turns at a constant rate (between cusps) as \( \theta \) varies.

For \( n \geq 1 \), let \( \mathcal{P}_n \) be the collection of polynomials of degree less than or equal to \( n \) of the form \( P(z)=z+a_nz^n+\cdots+a_2z^2+a_0z \) such that the equations

\[
\frac{\Delta_k P(z)}{z} = \frac{P(ze^{ik\pi/(n+1)}) - P(ze^{-ik\pi/(n+1)})}{z(e^{ik\pi/(n+1)} - e^{-ik\pi/(n+1)})} = 1 + \sum_{j=2}^n a_j \frac{\sin k\pi/(n+1)}{\sin k\pi/(n+1)} z^{j-1} = 0, \quad k = 1, 2, \ldots, n,
\]

have no roots in \( |z| < 1 \). Since \( P \) is univalent in \( |z| < 1 \) if and only if for \( 0 < \theta < \pi/2 \) the equation

\[
O = 1 + \sum_{j=2}^n a_j \frac{\sin j\theta}{\sin \theta} z^{j-1}
\]

has no roots in \( |z| < 1 \), [3], \( \mathcal{P}_n \) contains the collection \( U_n \) of all univalent polynomials of degree \( n \) or less which are appropriately normalized.

We say that \( P \in \mathcal{P}_n \) is an extreme point of \( \mathcal{P}_n \) if there do not exist \( P_1 \) and \( P_2 \) in \( \mathcal{P}_n \), \( P_1 \neq P_2 \), such that \( P = tP_1 + (1-t)P_2 \) where \( 0 < t < 1 \). We will show below that \( \bigcup_{n=1}^\infty \mathcal{P}_n \) is a normal family and it is then easy to see that, for each \( n \), \( \mathcal{P}_n \) is a compact

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subset of a locally convex linear topological space. As in [2], the Kreĭn-Milman theorem [4] then applies and \( P_n \) is contained in the closure of the convex hull of its extreme points. Further, any continuous linear functional on \( P_n \) assumes its maximum real part and maximum modulus on the set of extreme points.

It is clear that \( P \in P_n \) is an extreme point if and only if for each real \( \alpha \), \( e^{-\alpha}P(ze^{\alpha}) \) is extreme. Hence in attempting to characterize the extreme points of \( P_n \) we may assume \( P(z) = z + a_2z^2 + \cdots + a_nz^n \) where \( a_n \geq 0 \).

**Theorem 1.** If \( P(z) = z + \sum_{j=2}^{n} a_jz^j \) is an extreme point of \( P_n \) such that \( a_n \geq 0 \) then \( a_{j+1} = \bar{a}_{n-j}, 0 \leq j \leq n-1 \).

**Proof.** Note that
\[
\sin k(j+1)\pi/(n+1) = (-1)^{k-1} \sin [k\pi - k(j+1)\pi/(n+1)] = (-1)^{k-1} \sin k(n-j)\pi/(n+1)
\]
so that \( a_n \leq 1 \) with equality only if all the roots of the equations \( \Delta_k P(z)/z = 0 \) lie on \( |z| = 1 \) and in this case \( a_{j+1} = \bar{a}_{n-j} \) [1]. Thus we need only show \( a_n = 1 \).

Let
\[
\hat{P}(z) = z^{n+1}P(1/z) = \sum_{j=1}^{n} \bar{a}_{n-j+1}z^j
\]
and observe that \( \Delta_k(\hat{P}) = (-1)^{k-1}(\Delta_k P)^\sim \). Assume \( 1 > a_n \) and define \( Q(z) = (1 + a_n)^{-1}[P(z) + \hat{P}(z)], R(z) = (1 - a_n)^{-1}[P(z) - \hat{P}(z)] \). We now show \( Q, R \in P_n \). For any polynomial \( S \) of degree less than or equal to \( n \),
\[
S(e^{i\theta}) = e^{i(n+1)\theta}S(e^{i\theta})
\]
so \( \Delta_k \hat{P}/\Delta_k P = (-1)^{k-1}(\Delta_k P)^\sim/\Delta_k P \) is analytic in a neighborhood of the closed disk. But
\[
|\Delta_k \hat{P}(z)/\Delta_k P(z)| = 1 \quad \text{on} \quad |z| = 1,
\]
\[
= a_n \quad \text{at} \quad z = 0,
\]
so \( |\Delta_k \hat{P}/\Delta_k P| < 1 \) in \( |z| < 1 \). This means \( \Delta_k Q \neq 0 \neq \Delta_k R \) in \( 0 < |z| < 1 \) so \( Q, R \in P_n \). However \( P = (1 + a_n)/2 Q + (1 - a_n)/2 R \) and \( P \) is extreme so \( Q = R \). This implies \( \hat{P} = a_n P \) and equating \( n \)th coefficients, \( a_n^2 = 1, a_n = 1 \) which is a contradiction. This completes the proof of Theorem 1.

Now consider the polynomials
\[
Q_p(z; n) = \sum_{j=1}^{n} \frac{\sin j\pi/(n+1)}{\sin p\pi/(n+1)} z^j = \frac{z(1 - (-1)^p z^{n+1})}{1 - 2z \cos p\pi/(n+1) + z^2}, \quad 1 \leq p \leq n.
\]
Since
\[
\Delta_k Q_p(z; n) = \frac{z(1 - (-1)^p z^{n+1})(1 - z^2)}{(1 - 2ze^{ik\pi/(n+1)} \cos p\pi/(n+1) + z^2e^{2ik\pi/(n+1)})(1 - 2ze^{-ik\pi/(n+1)} \cos p\pi/(n+1) + z^2e^{-2ik\pi/(n+1)})},
\]
1 \leq k \leq n, 1 \leq p \leq n, each have \( n - 1 \) zeros on \( |z| = 1 \), we conclude \( Q_p(z; n) \in \mathcal{P}_n \).

Also, we see that

\[
\Delta_k Q_p(1; n) = \begin{cases} 0 & \text{if } 1 \leq k \leq n, k \neq p, \\ \frac{n+1}{(n+1)(2 \sin^2 p\pi/n + 1)} & \text{if } k = p, \end{cases}
\]

so the polynomials \( Q_p(z; n) \) are linearly independent.

**Theorem 2.** If \( P(z) = z + a_2z^2 + \cdots + a_nz^n \in \mathcal{P}_n \) is such that \( a_n = 1 \), then \( P(z) = \sum_{p=1}^n a_pQ_p(z; n) \) where \( a_p \) is real when \( p \) is odd and pure imaginary when \( p \) is even. Further \( \sum_{p \text{ odd}} a_p = 1 \) and \( \sum_{p \text{ even}} a_p = 0 \).

**Proof.** Since the \( Q_p(z; n) \) are linearly independent, we may write

\[
P(z) = \sum_{p=1}^n a_pQ_p(z; n).\]

Then \( \Delta_q P(1) = \alpha_q \Delta_q Q_q(1; n) \) by (4) and we have \( \alpha_q = \Delta_q P(1)(2 \sin^2 p\pi/(n+1))/(n+1) \).

As remarked before, \( a_n = 1 \) implies the coefficient relation \( a_{j+1} = a_{n-j} \) when \( P \in \mathcal{P}_n \) so

\[
\Delta_q P(1) = 1 + \frac{\sin 2p\pi/(n+1)}{\sin p\pi/(n+1)} a_2 + \cdots + (-1)^{p-1} \frac{\sin 2p\pi/(n+1)}{\sin p\pi/(n+1)} a_2 + (-1)^{p-1}
\]

which is real if \( p \) is odd and pure imaginary if \( p \) is even. The rest of the theorem follows from the normalization of \( P \).

From Theorem 1, we easily obtain the following corollary.

**Corollary 1.** If \( P(z) = \sum_{p=1}^n a_pQ_p(z; n) \) where \( a_p \) is real when \( p \) is odd and pure imaginary when \( p \) is even. Further \( \sum_{p \text{ odd}} a_p = 1 \) and \( \sum_{p \text{ even}} a_p = 0 \).

In [6, p. 496] the polynomials \( P(z; n, j) \) defined by

\[
P(z; n, j) = \sum_{k=1}^n \frac{n-k+1}{n} \frac{\sin kj\pi/(n+1)}{\sin j\pi/(n+1)} z^k
\]

were introduced and shown to be univalent. These polynomials are related to the polynomials \( Q_p(z; n) \) by the equation \( P(z; n, p) = \left((n+1)/n\right)Q_p(z; n) - (1/n)zQ'_p(z; n) \).

If \( P \in \mathcal{P}_n \), let \( P^*(z) = ((n+1)/n)P(z) - (1/n)zP'(z) \). We show below that if \( P \in \mathcal{P}_n \) and \( a_n = 1 \) then \( P^* \) is univalent in the disk. We require the following lemma.

**Lemma 1.** If \( P(z) = \sum_{j=1}^n a_jz^j \in \mathcal{P}_n \) then

\[
P^*(z) = \sum_{j=1}^n \frac{n-j+1}{n} a_jz^j \in \mathcal{P}_n.
\]

**Proof.** Observe that

\[
\Delta_k P^*(z) = \frac{n+1}{n} \Delta_k P(z) - \frac{1}{n} \Delta_k [zP'(z)] = \frac{n+1}{n} \Delta_k P(z) - \frac{1}{n} z(\Delta_k P)'(z).
\]
Since \( \Delta_k P(z) = z \prod_{j=1}^{n} (1 - z/z_j), \) \(|z_j| \geq 1,\) we have

\[
\text{Re} \left( \frac{z(\Delta_k P)(z)}{\Delta_k P(z)} \right) = \text{Re} \left( 1 - \sum_{j=1}^{n-1} \frac{z_j}{1 - z/z_j} \right) \leq 1 + \frac{n-1}{2} = \frac{n+1}{2}.
\]

Hence

\[
\left| \frac{\Delta_k P^*(z)}{z} \right| = \left| \frac{\Delta_k P(z)}{nz} \right| \left| n+1 - \frac{z(\Delta_k P)(z)}{\Delta_k P(z)} \right| \geq \frac{|\Delta_k P(z)|}{|nz|} \left( \frac{n+1}{2} \right) \neq 0
\]

when \(|z| < 1.\)

Remark. It is also clear in the above proof that \( \Delta_k P^*(z_0) = 0 \) for some \( z_0 \) on \(|z| = 1\) if and only if \( \Delta_k P(z) \) has a double zero at \( z = z_0.\)

**Theorem 3.** If \( P(z) = \sum_{j=1}^{n} a_j z^j \in \mathcal{P}_n \) and \(|a_n| = 1,\) then \( P^* \) is univalent in \(|z| < 1.\)

**Proof.** We may assume \( a_n = 1.\) Using the coefficient relation \( a_{j+1} = a_{n-j} \) and proceeding as in [6, pp. 497–498] we find \( e^{i\theta} P^*(e^{i\theta}) = e^{i(n+1)\theta/2} R(\theta) \) where \( R \) is real valued and \( \text{Re} \left[ e^{i\theta} P^*(e^{i\theta}) P^*(e^{i\theta}) + 1 \right] = (n+1)/2 \) when \( P^*(e^{i\theta}) \neq 0.\) That is, the tangent line to the curve \( P^*(e^{i\theta}), \) \( 0 \leq \theta \leq \pi, \) turns at a constant rate in a counterclockwise direction as \( \theta \) increases except at the cusps where it reverses direction.

We wish to show that for each \( \theta, \) \( 0 \leq \theta < \pi/2,\) the polynomial

\[
S(z, \theta) = (P^*(ze^{i\theta}) - P^*(ze^{-i\theta}))/z(e^{i\theta} - e^{-i\theta}) \quad (= P^*(z) \text{ if } \theta = 0)
\]

has no zeros in \(|z| < 1.\) We first show \( P^*(z) \neq 0 \) in \(|z| < 1\) so suppose \( S(z_0, 0) = 0 \) for some \( z_0, \) \(|z_0| < 1.\) Since for each \( \theta, \) \( S(z, \theta) \) is a polynomial and the zeros of a polynomial vary continuously with the coefficients there is a continuous function \( z(\theta) \) such that \( S(z(\theta), \theta) = 0, \) \( 0 \leq \theta \leq \pi/n + 1, \) \( z(0) = z_0.\) By Lemma 1, \( S(z, \pi/(n+1)) \neq 0 \) in \(|z| < 1 \text{ so } |z(\pi/(n+1))| \geq 1.\) Therefore \(|z(\phi)| = 1 \) for some \( \phi, \) \( 0 < \phi \leq \pi/(n+1).\) For each \( \theta, \) \( 0 < \theta < \phi,\) one can find tangent lines \( L(\theta), M(\theta) \) to the closed curve \( \gamma(\theta) = \{P(z(\theta) e^{i\theta}) : (\theta - \psi) \leq \theta \leq \theta + \phi \} \) such that \( \gamma(\theta) \) is contained between \( L(\theta) \) and \( M(\theta) \) and so that \( L \) and \( M \) vary continuously with \( \theta \) (for example choose \( L \) and \( M \) parallel to the tangent to \( P(z(\theta) e^{i\theta}) \) at \( |\theta - \psi| = \epsilon \) where \( \epsilon \) is small). Hence one obtains parallel tangents \( L(\phi), M(\phi) \) to the closed curve \( \gamma(\phi) \) at the points \( P(z(\phi) \exp(i\psi_1)) \) and \( P(z(\phi) \exp(i\psi_2)) \) where \( 0 < \psi_2 - \psi_1 < 2\phi \leq 2\pi/(n+1). \) But the tangent line turns at the constant rate \((n+1)/2\) on \(|z| = 1\) so \( L \) and \( M \) parallel implies \( \psi_2 - \psi_1 = 2\pi/(n+1) \geq 2\pi/(n+1) \) which is a contradiction. We remark that it seems necessary to find \( L(\phi) \) and \( M(\phi) \) as above to avoid the problem of cusps on the image of \(|z| = 1.\)

Now suppose \( S(z, \theta) = 0 \) for some \( \theta \) and \( z, \) \( 0 < \theta < \pi/2, \) \(|z| < 1.\) Let \( r \) be a minimum such that for some \( z_0 \) and \( \theta_0, \) \( r = |z_0| \) and \( S(z_0, \theta_0) = 0.\) As before, there is a continuous function \( z(\theta), 0 \leq |\theta - \theta_0| \leq \pi/(n+1), \) such that \( S(z(\theta), \theta) = 0 \) and \( z(\theta_0) = z_0.\) Again using Lemma 1, we conclude there are \( \phi_1 \) and \( \phi_2 \) such that \(-\pi/(n+1) < \phi_2 \)
\[ -\theta_0 < \phi_1 - \theta_0 < \pi/(n+1) \text{ and } |z(\phi_1)| = |z(\phi_2)| = 1. \] If for some continuous branch of the argument, we have

\[ \psi_1 = \arg (z(\phi_1) \exp (i\phi_1)) > \arg (z(\phi_2) \exp (i\phi_2)) = \psi_2 \]

and

\[ \psi_3 = \arg (z(\phi_3) \exp (-i\phi_3)) > \arg (z(\phi_1) \exp (-i\phi_1)) = \psi_4 \]

we may proceed as in the proof that \( P^*(z) \neq 0 \) in \(|z| < 1\) to show that there exist \( \psi_1 \) and \( \psi_2 \) satisfying \( \psi_1 > \theta_1 > \psi_2, \psi_3 > \theta_2 > \psi_4 \) and \( \theta_1 - \theta_2 = 2\pi/(n+1) > 0 \). But \( \psi_1 - \psi_4 = 2\phi_1 > \theta_1 - \theta_2 = 2\phi_2 > \psi_2 \) so \( \phi_1 > j\pi/(n+1) > \phi_2 \) and \( S(z(j\pi/(n+1)), j\pi/(n+1)) = 0 \) contradicting Lemma 1.

Let \( \gamma \) be the curve \( P^*(z_0 e^{i\theta}), 0 \leq \theta \leq 2\pi \). The curve \( \gamma \) has a common tangent line where \( \theta = \theta_0 \) and \( \theta = -\theta_0 \) by the way in which \( z_0 \) and \( \theta_0 \) were chosen. Since \( zP^*(z) \) is in the direction of the outward normal when \( P^* \neq 0 \) we must have

\[ \frac{z_0 \exp (i\theta_0)P^*(z_0 \exp (i\theta_0))}{z_0 \exp (-i\theta_0)P^*(z_0 \exp (-i\theta_0))} < 0. \]

Using the fact that \( P^*(z(\theta) e^{i\theta}) - P^*(z(\theta) e^{-i\theta}) = 0 \), we find

\[ \left| \frac{d \arg z(\theta)}{d\theta} \right| = \left| \text{Im} \frac{d \log z(\theta)}{d\theta} \right| = \left| \left( 1 + \frac{z(\theta) e^{i\theta} P^*(z(\theta) e^{i\theta})}{z(\theta) e^{-i\theta} P^*(z(\theta) e^{-i\theta})} \right) / \left( 1 - \frac{z(\theta) e^{i\theta} P^*(z(\theta) e^{i\theta})}{z(\theta) e^{-i\theta} P^*(z(\theta) e^{-i\theta})} \right) \right| < 1 \]

when \( \theta = \theta_0 \). This means that if \( \phi_1 \) and \( \phi_2 \) are sufficiently near \( \theta_0 \) and such that \( \phi_2 < \theta_0 < \phi_1 \) and \(|z(\phi_1)| = |z(\phi_2)|\) then \( \arg (z(\phi_2) \exp (i\phi_2)) < \arg (z(\phi_1) \exp (i\phi_1)) \) and \( \arg (z(\phi_2) \exp (-i\phi_2)) > \arg (z(\phi_1) \exp (-i\phi_1)) \). Therefore to complete the proof of the theorem we need only show that if \( 1 \geq |z(\theta_0)| = |z(\phi_1)| \) and either

\[ \arg (z(\phi_2) \exp (i\phi_2)) = \arg (z(\phi_1) \exp (i\phi_1)) \]

or

\[ \arg (z(\phi_2) \exp (-i\phi_2)) = \arg (z(\phi_1) \exp (-i\phi_1)) \]

we obtain a contradiction to \( P^* \in \mathcal{P}_n \). We have \( 0 < \phi_1 - \phi_2 < \pi/(n+1) \) and by changing notation there are \( z \) and \( \theta = \phi_1 - \phi_2 \) such that \( P^*(z e^{i\theta}) = P^*(z e^{-i\theta}) \), \( 0 < \theta < \pi/(n+1) \). The proof now proceeds as in the proof that \( P^* \neq 0 \) in \(|z| < 1\).

**Corollary 2.** If \( P \in \mathcal{P}_n \) is an extreme point then \(((n+1)/n)P(z) - (1/n)zP'(z)\) is univalent in \(|z| < 1\).

We have the following converse to Theorem 3.

**Theorem 4.** If

\[ Q(z) = \sum_{j=1}^{n} \frac{n-j+1}{n} a_j z^j, \quad a_1 = 1 = a_n, \]

and \( Q(z) \) is univalent in \(|z| < 1\) then \( P(z) = \sum_{j=1}^{n} a_j z^j \in \mathcal{P}_n \).
Proof. As shown in [1], we must have $a_{j+1} = \bar{a}_{n-j}$, and it then follows that

$$\text{Re } [\Delta_k z P'(z)/\Delta_k P(z)] = \text{Re } [z(\Delta_k P)'(z)/\Delta_k P(z)] = (n+1)/2$$

when $|z| = 1$, $\Delta_k P(z) \neq 0$.

Suppose $\Delta_k P(z) = 0$ for some $z$, $|z| < 1$. Then $w = \Delta_k P'(z)/\Delta_k P(z) = n + 1$ for some $z$, $|z| < 1$ for $w$ assumes every value in a neighborhood of $\infty$ and therefore every value not on the line $\text{Re } w = (n+1)/2$. But $\Delta_k Q(z) = ((n+1)/n) \Delta_k P(z) - (1/n) \Delta_k z P'(z) = 0$ when $w = n + 1$ which contradicts the univalence of $Q$. This proves $\Delta_k P(z) \neq 0$ when $|z| < 1$, $1 \leq k \leq n$ so $P \in \mathcal{P}_n$ and the proof is complete.

Now suppose $P(z) = \sum_{j=1}^{n} a_j z^j \in \mathcal{P}_n$. If $a_n = 1$, then by Theorem 3, $P^*$ is univalent in $|z| < 1$ so $|a_j| < n(3j)(n-j+1) < 6nj(n+1) < 6j$ if $j < (n+1)/2$. Using the coefficient relation, $|a_j| < 6j$ for all $j$. If $0 < a_n < 1$ then as shown in the proof of Theorem 1, $P$ is a convex combination of members of $\mathcal{P}_n$ having $n$th coefficient $\pm 1$. Therefore, in any case $|a_j| < 6j$. Hence for $P \in \mathcal{P}_n$, $|P(z)| < 6 \sum_{j=1}^{n} |jz|^j < 6|z|/(1 - |z|)^2$ so the family $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ is locally uniformly bounded and is therefore a normal family.

Theorem 5. Suppose $P_{n_k} \in \mathcal{P}_{n_k}$ and that $P_{n_k} \rightarrow f$ as $k \rightarrow \infty$. Then $f$ is univalent in $|z| < 1$.

Proof. Suppose $f$ is not univalent in $|z| < 1$. Then there exist $\theta$, $z$ such that $0 < |z| < 1$, $0 < \theta < \pi/2$ and $f\left(z e^{i\theta}\right) = f\left(z e^{-i\theta}\right)$. In fact there exists $r < 1$ and a closed interval $I = [\theta_1, \theta_2]$ such that the equation $f\left(z e^{i\theta}\right) = f\left(z e^{-i\theta}\right)$ has a solution in $D_r = \{z : 0 < |z| < r\}$ for each $\theta \in I$. For fixed $\theta \in I$, there exists $k$ such that if $l > k$ then $P_{n_l}(z e^{i\theta}) = P_{n_l}(z e^{-i\theta})$ has a solution in $D_{(1+r)/2}$. Let $I_k = \{\theta \in I : P_{n_l}(z e^{i\theta}) = P_{n_l}(z e^{-i\theta})\}$ has a solution in $D_{(1+r)/2}$ for all $l > k$. Then $\bigcup_{k=0}^{\infty} I_k = I$ so by Baire's theorem [7, p. 76] some $I_k$ contains an interval. That is $l > k_0$ implies $P_{n_l}(z e^{i\theta}) = P_{n_l}(z e^{-i\theta})$ has a solution in $D_{(1+r)/2}$ for all $\theta$ in some fixed interval. This contradicts the definition of $\mathcal{P}_n$ and completes the proof.

We now wish to obtain some geometric properties of the univalent polynomial $P^*$ associated with extreme points $P \in \mathcal{P}_n$. Note that a double zero of $\Delta_k P(z)$ is a zero of $\Delta_k P^*$. Letting $\gamma = \{P^*(e^{i\theta}) : 0 \leq \theta < 2\pi\}$ we see that a double root of $\Delta_k P$ on $|z| = 1$ corresponds to a point of self-tangency of $\gamma$ and conversely. Further, we have the following lemma.

Lemma 2. If $P \in \mathcal{P}_n$ satisfies $a_n = 1$ then $\Delta_k P$ cannot have a zero of multiplicity greater than 2.

Proof. Assume $\Delta_k P$ has a zero of multiplicity greater than 2. Then $P^*(z e^{i\pi n/(n+1)}) = P^*(z e^{-i\pi n/(n+1)})$ and $z e^{i\pi n/(n+1)} P^*(z e^{i\pi n/(n+1)}) = z e^{-i\pi n/(n+1)} P^*(z e^{-i\pi n/(n+1)})$ for some $z$, $|z| = 1$. If $P^*(z e^{i\pi n/(n+1)}) = 0 = P^*(z e^{-i\pi n/(n+1)})$ then it is clear that the images under the mapping $P^*$ of small sectors of sufficiently large opening inside the unit circle with vertices at $z e^{i\pi n/(n+1)}$ and $z e^{-i\pi n/(n+1)}$ will overlap contradicting
the univalence of \( Q \). Hence the image of \( |z| = 1 \) has a common tangent at the two points under consideration and as seen previously this implies

\[
\frac{ze^{i\kappa(n+1)}P^*(ze^{i\kappa(n+1)})}{ze^{-i\kappa(n+1)}P^*(ze^{-i\kappa(n+1)})} < 0.
\]

This is a contradiction which completes the proof.

**Theorem 6.** If \( P \in \mathcal{P}_n (n > 2) \) is an extreme point then the curve \( \gamma = P^*(e^{i\theta}) : 0 \leq \theta \leq 2\pi \) has \( n-2 \) points of self-tangency. Further, if \( P^*(\exp(i\theta_2)) = P^*(\exp(i\theta_1)) \) then \( \theta_2 - \theta_1 = 2k\pi/(n+1) \) for some integer \( k \).

**Proof.** The last assertion in the theorem follows easily from the fact that \( \Re \left[ e^{i\theta}P^*(e^{i\theta})/P^*(e^{i\theta}) + 1 \right] = (n+1)/2 \) when \( P^*(e^{i\theta}) \neq 0 \) so the tangent line to \( \gamma \) turns at a constant rate between cusps as \( \theta \) varies. Since \( P^*(\exp(i\theta_2)) = P^*(\exp(i\theta_1)) \) implies there is a common tangent line to \( \gamma \) at the above points we must have \( ((n+1)/2)(\theta_2 - \theta_1) = k\pi \) for some integer \( k \).

We now proceed to prove the first part of the theorem.

We may assume \( P(z) = \sum_{j=1}^{n} a_jz^j \), \( a_n = 1 \). Note that \( \Delta_{n+1}P(z) = -\Delta_{n-k}P(-z) \) so it is sufficient to consider \( k \leq (n+1)/2 \). Further, if \( n \) is odd, \( \Delta_{(n+1)/2}P(z) \) is an even function and we may assume for this polynomial that \( 0 \leq \arg z < \pi \). Hence we will show that there are \( n-2 \) values of \( z \) such that \( \Delta_kP(z) \) has a double zero on \( |z| = 1 \) for some \( k \) satisfying \( 1 \leq k < (n+1)/2 \) or \( k = (n+1)/2 \) and \( 0 \leq \arg z < \pi \). Suppose this is not the case. We wish to construct

\[
R(z) = \sum_{n \leq k > 1; k \text{ odd}} \alpha_k(Q_k(z; \kappa) - Q_k(z; \kappa)) + \sum_{n \leq k > \frac{n}{2}; k \text{ even}} \beta_k(Q_k(z; \kappa) - Q_k(z; \kappa))
\]

where \( \alpha_k \) and \( \beta_k \) are real and \( Q_k(z; \kappa) \) is given by (2) so that \( P(z) + tR(z) \in \mathcal{P}_n \) and \( P(z) - tR(z) \in \mathcal{P}_n \) for some \( t > 0 \), \( R \neq 0 \). Then \( P = \frac{1}{2}(P + tR) + \frac{1}{2}(P - tR) \) and \( P \) is not extreme (\( R \) must have the above form in order for \( P + tR \) and \( P - tR \) to satisfy the coefficient relation).

We wish to obtain \( n-2 \) real linear equations in the \( n-2 \) unknowns \( \alpha_k, \beta_k \). For each double zero \( e^{i\theta} \) of \( \Delta_kP(z) \) restricted as discussed above, we obtain an equation by setting \( i^{k-1}e^{-i(n+1)/2}\Delta_kR(e^{i\theta}) = 0 \). The coefficient relation in \( Q_k(z; \kappa) \) implies that the \( \alpha_k \) and \( \beta_k \) have real coefficients in these equations. Suppose \( l \) equations are obtained in this way. The remaining equations are obtained by setting \( e^{-i\pi/2}R(e^{i\pi/(n+1)}) = K, \ j = 1, 2, \ldots, n-l-2, \) where \( K \neq 0 \) if the determinant of the coefficients is 0 and \( K = 1 \) otherwise. Thus in any case, there is a choice of the \( \alpha_k \) and \( \beta_k \) such that \( \Delta_kR(z) = 0 \) when \( \Delta_kP(z) \) has a double zero and \( R(z) \neq 0 \).

Let \( k \) be fixed and consider \( S(\theta) = \Delta_kP(e^{i\theta})/\Delta_kR(e^{i\theta}) \). Suppose \( \Delta_kP(e^{i\theta}) \) and \( \Delta_kR(e^{i\theta}) \) have \( p \) common zeros. Then \( S(\theta) \) has \( n-1-p \) simple zeros and therefore changes sign at each of these zeros. Hence there exists \( t_k > 0 \) such that \( S(\theta) \) assumes the values \( t_k \) and \( -t_k, n-1-p \) times. This means that all the zeros of
$\Delta_k P(z) + t\Delta_k R(z)$ and $\Delta_k P(z) - t\Delta_k R(z)$ lie on $|z| = 1$ when $t \leq t_k$. Setting $t = \min_{1 \leq k \leq n} t_k$ the proof is now complete.

**Examples.**

$n = 1$. $\mathcal{P}_1 = \{z\}$.

$n = 2$. Theorem 2 implies that the extreme points of $\mathcal{P}_2$ are rotations of $z + z^2$.

$n = 3$. Theorem 2 implies that the extreme points of $\mathcal{P}_3$ are rotations of polynomials of the form $P(z) = z + a_2 z^2 + z^3$ where $a_2$ is real. Clearly we may assume $a_2 \geq 0$. Theorem 6 implies that one of the polynomials $\Delta_1 P(z) = z + \sqrt{2}a_2 z^2 + z^3$ or $\Delta_2 P(z) = z - z^3$ has a double zero on $|z| = 1$. It follows that $a_2 = \sqrt{2}$ so all extreme points of $\mathcal{P}_3$ are rotations of $z + \sqrt{2}z^2 + z^3$.

$n = 4$. Theorem 2 implies that the extreme points of $\mathcal{P}_4$ are rotations of polynomials of the form $P(z) = z + a_2 z^2 + a_2 z^3 + z^4$. Theorem 6 implies that

\[
1 + 2a_2 \cos \left(\frac{\pi}{5}\right) z + 2a_2 \cos \left(\frac{\pi}{5}\right) z^2 + z^3
\]

and

\[
1 + 2a_2 \cos \left(\frac{2\pi}{5}\right) z - 2a_2 \cos \left(\frac{2\pi}{5}\right) z^2 - z^3
\]

each have a double zero on $|z| = 1$. Applying this to $P^*$, each of the polynomials,

\[
1 + \frac{1}{2}a_2 \cos \left(\frac{\pi}{5}\right) z + a_2 \cos \left(\frac{\pi}{5}\right) z^2 + \frac{1}{4}z^3
\]

and

\[
1 + \frac{1}{2}a_2 \cos \left(\frac{2\pi}{5}\right) z - a_2 \cos \left(\frac{2\pi}{5}\right) z^2 - \frac{1}{4}z^3
\]

has exactly one zero on $|z| = 1$. By Cohn’s rule [1] and [5, p. 149], if $f(z) = c_0 + c_1 z + \cdots + c_k z^k$ satisfies $|c_0| > |c_k|$ then

\[f^*(z) = (c_0 f(z) - c_k z^k) f(1/z)\]

has the same zeros as $f$ on $|z| = 1$ and the same number of zeros as $f$ in $|z| < 1$. Applying Cohn’s rule twice to each of the polynomials (5) leads to the linear polynomials

\[
\left(\frac{12 \cos \frac{2\pi}{5}}{a_2} \frac{2a_2}{a_2} \right) z + \frac{36 \cos^2 \frac{2\pi}{5}}{|a_2|^2} - 1
\]

and

\[
\left(\frac{12 \cos \frac{\pi}{5}}{a_2} + \frac{2a_2}{a_2} \right) z + \frac{36 \cos^2 \frac{\pi}{5}}{|a_2|^2} - 1
\]

(we have used the fact that $\cos 2\pi/5 = (\sqrt{5} - 1)/4$ and $\cos \pi/5 = (\sqrt{5} + 1)/4$ so $\cos (2\pi/5) \cos \pi/5 = 1/4$ each of which has a zero on $|z| = 1$. This fact yields the equations

\[
|a_2|^4 - 16 \Re a_2^2 \cos 2\pi/5 + 72|a_2|^2 \cos^2 2\pi/5 - 432 \cos^4 2\pi/5 = 0, \tag{6}
\]

\[
|a_2|^4 + 16 \Re a_2^2 \cos \pi/5 + 72|a_2|^2 \cos^2 \pi/5 - 432 \cos^4 \pi/5 = 0.
\]

Eliminating $\Re a_2^2$ from the equations (6) we obtain $|a_2|^4 + 18|a_2|^2 - 54 = 0$ so $|a_2|^2 = 3\sqrt{15} - 9$. Substitution into either equation in (6) then yields

\[
\cos (3 \arg a_2) = \frac{3}{16} \sqrt{(9 + 5\sqrt{15})}.
\]
If we choose a value of arg $a_2$ to satisfy (7) and choose $|a_2|$ so the equations (6) are satisfied then the extreme points in $\mathcal{P}_4$ are rotations of the polynomials $z + a_2z^2 + \bar{a}_2z^2 + z^4$ and $z + \bar{a}_2z^2 + a_2z^2 + z^4$.

$n = 5$. The extreme points of $\mathcal{P}_5$ are rotations of polynomials of the form $P(z) = z + (a + bi)z^2 + cz^3 + (a - bi)z^4 + z^5$ where $a$, $b$, and $c$ are real. By considering $-iP(iz)$, $-P(-z)$ and $(P(z))^-$ we see that we may assume $a$, $b$ and $c$ are non-negative. Theorem 6 implies that among the roots of the equations

$$1 + \sqrt{3}(a + bi)z + 2cz^2 + \sqrt{3}(a - bi)z^3 + z^4 = 0$$

$$1 + (a + bi)z - (a - bi)z^3 - z^4 = 0$$

$$-cz^2 + z^4 = 0$$

there must be three double roots on $|z| = 1$ (only half of the double roots of the third equation are to be counted). Observe that $(1 + e^{iaz})^2(1 + e^{ibz})^2 = 1 + 2(e^{ia} + e^{ib})z + (e^{2ia} + e^{2ib} + 4e^{(a + ib)})z^2 + 2(e^{i(2a + b)} + e^{i(2b + a)})z^3 + e^{2(a + b)}z^4$ so the second equation in (8) cannot have two double roots on $|z| = 1$. Suppose the first equation has two double roots on $|z| = 1$. Then the left hand-side has the form above where $\beta = -\alpha$ or $\beta = \pi - \alpha$. Since $c \geq 0$, we must have $\beta = -\alpha$ so $b = 0$. The second equation in (8) then has roots $\pm 1$ together with the zeros of $1 + az + z^2$. Hence in this case the second equation cannot have a double root (the only possibility is $-1$ and it is either a simple root or a triple root). This means the third equation has a double root so $c = 2$ and $a = \sqrt{8}/3$. But

$$z + \sqrt{8}/3z^2 + 2z^3 + \sqrt{8}/3z^4 + z^5 = ((\sqrt{8} + 3)/6)Q_4(z; 5) + ((3 - \sqrt{8})/6)Q_5(z; 5)$$

and this polynomial is not an extreme point.

Thus we conclude that if $P$ is extreme then each of the equations in (8) has a double root on $|z| = 1$. From the third equation, $c = 2$. Assume that $e^{i\phi}$ and $e^{i\theta}$ are double roots of the first and second equations respectively. We obtain the system

$$\cos 2\phi + \sqrt{3}a \cos \phi + \sqrt{3}b \sin \phi + 2 = 0,$$

$$2 \sin 2\phi + \sqrt{3}a \sin \phi - \sqrt{3}b \cos \phi = 0,$$

$$\sin 2\theta + a \sin \theta - b \cos \theta = 0,$$

$$2 \cos 2\theta + a \cos \theta + b \sin \theta = 0.$$

Solving for $a$ and $b$ in terms of $\theta$ and $\phi$ we find

$$a = -2 \cos^3 \theta = (1/2\sqrt{3})(\cos 3\phi - 7 \cos \phi),$$

$$b = 2 \sin^3 \theta = (1/2\sqrt{3})(\sin 3\phi - \sin \phi).$$

From equations (10),

$$a^2 + b^2 = \frac{1}{3}(8 + 4 \cos 2\phi - 3 \cos^2 2\phi)$$

so $a^2 + b^2 < 28/9$.

Now let $\theta = \theta(\phi)$ satisfy $-2 \cos^3 \theta = (1/2\sqrt{3})(\cos 3\phi - 7 \cos \phi)$ and set $g(\phi) = 2 \sin^3 \theta - (1/2\sqrt{3})(\sin 3\phi - \sin \phi)$. Since $a > 0$ and $b > 0$ we have $\pi > \phi$, $\theta > 3\pi/4$. 

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Also \( g'(\phi) = (10 - 12 \cos^2 \phi) \cos (\phi - \theta)/(2\sqrt{3} \cos \theta) \), \( g(\pi) > 0 \), \( g(\phi_0) < 0 \) and \( g(3\pi/4) > 0 \) where \( \cos^2 \phi_0 = \frac{1}{6} \). Thus we conclude the system (10) has two solutions. Assume the values of \( \phi \) corresponding to these solutions are \( \phi_1 \) and \( \phi_2 \) where \( \pi > \phi_1 > \phi_0 > \phi_2 > 3\pi/4 \). From (11), we find \( a^2 + b^2 = 3 \) when \( \cos 2\phi = 1 \), \( \frac{1}{2} \) while \( g(\phi) < 0 \) when \( \cos 2\phi = \frac{1}{2} \) so \( 3 < a^2 + b^2 < 28/9 \) for the solution corresponding to \( \phi_1 \) and \( a^2 + b^2 < 3 \) for the solution corresponding to \( \phi_2 \). Denote the polynomials \( P \) corresponding to the solutions of (10) for \( \phi = \phi_1, \phi_2 \) by \( P_1 \) and \( P_2 \) respectively and let \( \theta_1 \) and \( \theta_2 \) be the corresponding values of \( \theta \). Clearly \( P_1 \) is an extreme point since it maximizes \( |a_2| \) \( (P_1 \in \mathcal{P}_2 \) since there must be an extreme point satisfying \( \sqrt{3} \leq |a_2| \)).

We now show \( P_2 \) is also an extreme point of \( \mathcal{P}_2 \). Note that in any of the equations in (8), the roots which do not lie on \( |z|=1 \) must occur as pairs of roots which are inverse points with respect to \( |z|=1 \). Hence if two roots vary continuously beginning on \( |z|=1 \) and ending as inverse points with respect to the circle then they must at some time coincide. The third equation in (8) has its roots on the circle when \( c=2 \). Setting \( a(\theta) = -2 \cos^2 \theta \) and \( b(\theta) = 2 \sin \theta \), \( e^{i\theta} \) is a double root of the second equation and the other roots lie on \( |z|=1 \) for we can never have \( a(\alpha) = a(\theta) \) and \( b(\alpha) = b(\theta) \) when \( \alpha \neq \theta \mod 2\pi \) (and the roots do lie on \( |z|=1 \) when \( \theta=0 \)).

Now set \( a(\phi) = (1/2\sqrt{3})(\cos 3\phi - 7 \cos \phi) \) and \( b(\phi) = (1/2\sqrt{3})(\sin 3\phi - \sin \phi) \). All roots of the first equation in (8) lie on \( |z|=1 \) when \( \phi=\pi \). Further, if \( \pi > \phi > 3\pi/4 \) then \( a(\phi) = a(\alpha) \) and \( b(\phi) = b(\alpha) \) together imply \( \phi = \alpha \). Hence for \( \phi = \phi_2 \), all roots of the first equation in (8) lie on \( |z|=1 \). Hence \( P_2 \in \mathcal{P}_5 \).

Now suppose \( P_2 \) is not an extreme point. Then \( P_2 \) is in the convex hull of the set

\[ \{ e^{-ia}P_1(e^{ia}z) : \alpha \text{ is real} \} \cup \{ e^{i/2}P_1(\bar{z}e^{i/2}) : \beta \text{ is real} \}. \]

Since \( P_2 \) has third coefficient 2 and fifth coefficient 1, \( P_2 \) is a convex combination of

\[ P_1(z), \quad P_1(\bar{z}), \quad -P_1(-z) \quad \text{and} \quad -P_1(-\bar{z}). \]

Therefore, \( \sin^2 \theta_2 \) is a convex combination of \( \sin^3 \theta_1 \) and \( -\sin^3 \theta_1 \). But \( \sin \theta_2 > \sin \theta_1 \) and this is impossible so \( P_2 \) is extreme.

For the next theorem, we restrict ourselves to the subclass \( \mathcal{R}_n \subset \mathcal{R}_n \) having the property \( P \in \mathcal{R}_n \) implies \( P \) has real coefficients.

**Theorem 7.** For each \( p=1, 2, \ldots, n \), \( Q_p(z; n) \) is an extreme point of \( \mathcal{R}_n \). Further \( \mathcal{R}_n \subset \mathcal{C} \{Q_p(z; n)\}_{p=1}^n \) where \( \mathcal{C} \) is the convex hull of \( A \).

**Proof.** Let \( P(z) = z + \sum_{n=2}^p a_n z^n \in \mathcal{R}_n \) be extreme. As in the proof of Theorem 1, we may show that if \( P \) is extreme then \( a_n = \pm 1 \). Using Theorem 3 above, Theorem 2 of [6, p. 500] and the fact mentioned previously that \( Q^*_p(z; n) = P(z; n, p) \) we have

\[ P(z) = \sum_{n=1}^p a_n Q_p(z; n) \]

where \( a_p \geq 0 \) and \( \sum_{n=1}^p a_n = 1 \). This proves

\[ \mathcal{R}_n \subset \mathcal{C} \{Q_p(z; n)\}_{p=1}^n \].

Every \( Q \in \mathcal{R}_n \) can be written uniquely in the form \( Q(z) = \sum_{p=1}^n a_p Q_p(z; n) \), \( a_p \geq 0 \), \( \sum_{p=1}^n a_p = 1 \) and the functional \( J_k \) defined on \( \mathcal{R}_n \) by \( J_k(Q) = a_k \) is a continuous linear
functional. Clearly $a_k = 1$ is its maximum which is assumed only when $Q(z) = Q_k(z; n)$. This proves $Q_k(z; n)$ is extreme in $R_n$.

From Theorems 6 and 7 above, it seems likely that no extreme points of $R_n$ can have all real coefficients when $n \geq 4$. At least we have the following theorem.

**Theorem 8.** If $n > 5$ and $1 < j < n$ then

$$
\max_{P \in R_n} |a_j| > \frac{\sin jn/(n+1)}{\sin \pi/(n+1)} = \max_{P \in R_n} |a_j|
$$

where $P(z) = \sum_{j=1}^n a_j z^j$.

**Proof.** The equality

$$
\max_{P \in R_n} |a_j| = \frac{\sin jn/(n+1)}{\sin \pi/(n+1)}
$$

follows from Theorem 3 and [6, Theorem 3]. From (3) we see that $\Delta_k Q(z; n)/z$ has simple zeros except possibly at $\pm 1$ and $\Delta_k (Q(z; n) - Q_n(z; n))/z$ has a simple zero at $\pm 1$ when $\Delta_k Q(z; n)/z$ has a double zero, $n > 5$. It then follows by the same argument as used in the proof of Theorem 6 that for $t$ sufficiently small, $Q(z) = Q_1(z; n) + it[(Q(z; n) - Q_n(z; n))e^{\in P} and all zeros of $\Delta_k Q$ are simple zeros, $1 \leq k \leq n$. For fixed $j$, choose $p$ odd so that

$$
\left| \frac{\sin jp\pi/(n+1)}{\sin p\pi/(n+1)} \right| < \frac{\sin jn/(n+1)}{\sin \pi/(n+1)}.
$$

Applying the same argument, it follows that, for sufficiently small $s$,

$$
P(z) = Q_1(z; n) - s(Q_n(z; n) - Q_1(z; n)) + it(Q_n(z; n) - Q_2(z; n))
$$

is in $R_n$. Then $|a_j| \geq \Re a_j > (\sin j\pi/(n+1))/(\sin \pi/(n+1))$. Actually the conclusion of the theorem holds for $n > 3$ except for the case $n = 5, j = 3$.

**Theorem 9.** Every function $f$ in the class $S$ of functions univalent in $|z| < 1$ and normalized by setting $f(0) = 0, f'(0) = 1$, is the limit of polynomials of the form $P(z) = z + \sum_{j=2}^n a_j z^j \in R_n$ which satisfy $a_n = 1$.

**Proof.** Let $f \in S$. One can obtain a sequence of univalent polynomials by taking appropriate partial sums of $r_k f(r_k z)$ where $\{r_k\}_{k=1}^\infty$ is a strictly increasing sequence of real numbers such that $\lim_{k \to \infty} r_k = 1$. Let $\{Q_{n_k}\}$ be such a sequence where $Q_{n_k}$ is of degree $n_k$. Define

$$
P_{n_k}(z) = Q_{n_k}(z) + z^{2n_k + 1} Q_{n_k}(1/z) = \sum_{j=1}^{2n_k} a_j z^j.
$$

By an argument similar to that used to prove Theorem 1, $P_{n_k} \in R_{2n_k}$ and $a_{2n_k} = 1$. Also $\lim_{k \to \infty} P_{n_k}(z) = f(z)$ so $\{P_{n_k}\}_{k=1}^\infty$ is the required sequence.
Theorem 10. The univalent polynomials of the form \( z + \sum_{j=2}^{n} a_j z^j \) which satisfy 
\[(j+1)a_{j+1} = (n-j)a_{n-j} \quad \text{[and thus } a_n = 1/n] \]
are dense in the class \( S \).

Proof. The polynomials \( \{P_n^*\}_{n=1}^{\infty} \) have the same limit as \( \{P_n\}_{n=1}^{\infty} \) above.

We observe that in Theorem 9, if \( f \in S \) has real coefficients then the sequence
\( \{P_k\}_{k=1}^{\infty} \) has real coefficients and we have a new proof of the Bieberbach conjecture
for functions having real coefficients.

Let \( \mathcal{D}_n \) be the class of polynomials of degree \( n \) which are univalent in \( |z| < 1 \) and
of the form \( z + \sum_{j=2}^{n} a_j z^j \), \((j+1)a_{j+1} = (n-j)a_{n-j-1}\). Applying Theorems 2 and 4
above and using the definition of \( P(z; n, j) \) in [6] we can represent any \( P \in \mathcal{D}_n \) in the form

\[
P(z) = P(z; n, 1) + \sum_{j \text{ odd}} a_j [P(z; n, j) - P(z; n, 1)]
\]

(12)

\[+ i \sum_{j \text{ even}} \beta_j [P(z; n, j) - P(z; n, 2)].\]

Since the tangent line to the curve \( P(e^{i\theta}) \) \((0 \leq \theta \leq 2\pi)\) turns at a constant rate
\((\text{Re} \ P'(e^{i\theta})/P'(e^{i\theta}) + 1) = (n+1)/2\) when \( P'(e^{i\theta}) \neq 0\) as \( \theta \) varies the tangent line
to the curve is horizontal when \( \theta \) is an odd multiple of \( \pi/(n+1) \) and vertical when
\( \theta \) is an even multiple of \( \pi/(n+1) \). Recall that among all polynomials in \( \mathcal{D}_n \) having
real coefficients, \( P(z; n, 1) \) maximizes every coefficient. Also \( \Delta_k P(1; n, 1) = 0 \) when \( k \) is odd, \( k > 1 \)
(i.e. \( P(e^{i(k\pi/(n+1))}; n, 1) = P(e^{-i(k\pi/(n+1))}; n, 1) \) when \( k \) is odd, \( k > 1 \)).

Note also that for even \( j > 2 \) and odd \( k > 1 \), \( \Delta_k [P(1; n, j) - P(1; n, 2)] \) is purely imaginary.

Hence the effect of adding \( it \{P(z; n, j) - P(z; n, 2)\} \) (where \( t \) is real and near 0 and \( j \) is even, \( j > 2 \)) to \( P(z; n, 1) \) is to shift the values at \( e^{ik\pi/(n+1)} \) and \( e^{-i(k\pi/(n+1))} \) apart
horizontally. Finally, in the representation (12), for odd \( k > 1 \) we have \( \alpha_k \text{ Im} [\Delta_k P(1; n, k)] = \text{Im} [\Delta_k P(1)] \) so \( \alpha_k \) is negative when \( \text{Im} P(e^{-i(k\pi/(n+1))}) > \text{Im} P(e^{i(k\pi/(n+1))}) \). Since the coefficients in \( P(z; n, k) - P(z; n, 1) \) are all nonpositive,
it would appear that to obtain the maximum modulus for any coefficient, one
should choose the \( \beta_j \) to shift the values \( P(e^{i(k\pi/(n+1))}) \) and \( P(e^{-i(k\pi/(n+1))}) \) apart
and then choose the \( \alpha_j \) negative, the choices being made to satisfy Theorem 6. This
discussion leads to the following conjecture.

Conjecture. Among all polynomials \( P(z) = z + \sum_{j=2}^{n} a_j z^j \in \mathcal{D}_n \), the quantities
\( |a_j|, 2 \leq j \leq n-1 \), are all maximized by a single polynomial having the property that
in the representation (12), \( \alpha_j \leq 0 \) for all odd \( j, n \geq j > 1 \).

This conjecture implies the Bieberbach conjecture as shown by the following argument. Suppose \( P_n(z) \) maximizes \( |a_j|, 2 \leq j \leq n-1 \), in \( \mathcal{D}_n \). Let \( f(z) = \sum_{j=1}^{n} b_j z^j \in S \)
and let \( Q_n \in \mathcal{D}_n \) where \( \{Q_n\}_{n=1}^{\infty} \) is a sequence of polynomials having \( f \) as limit.

Let \( P_n(z) = \sum_{j=1}^{n} a_j z^j, Q_n(z) = \sum_{j=1}^{n} b_j z^j \). Then \( |a_{j,n_k}| \geq |b_{j,n_k}| \) for each \( k \) and \( 2 > |a_{2,n_k}| \geq (n_k - 1)/n_k \) \( \cos (n_k + 1) \). Hence \( \lim_{n_k \to \infty} |a_{2,n_k}| = 2 \) and any convergent
subsequence of \( \{P_n\} \) must converge to a Koebe function. After possibly renaming
we have to obtain a convergent subsequence, we therefore have

\[
j = \lim_{k \to \infty} |a_{j,n_k}| \geq \lim_{k \to \infty} |b_{j,n_k}| = |b_j|.
\]
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