APPROXIMATION IN THE MEAN BY ANALYTIC FUNCTIONS

BY

LARS INGE HEDBERG

Abstract. Let E be a compact set in the plane, let $L^p(E)$ have its usual meaning, and let $L^p_a(E)$ be the subspace of functions analytic in the interior of E. The problem studied in this paper is whether or not rational functions with poles off E are dense in $L^p_a(E)$ (or in $L^p(E)$ in the case when E has no interior). For $1 \le p \le 2$ the problem has been settled by Bers and Havin. By a method which applies for $1 \le p < \infty$ we give new results for $p > 2$ which improve earlier results by Sinanjan. The results are given in terms of capacities.

1. **Introduction.** In recent years the problem of uniform approximation by rational functions in the complex plane has received great attention (see e.g., [26], [28], [11]), but comparatively little work has been devoted to the corresponding problem of approximation in the areal mean. Carleman proved in [5] that if D is a Jordan region, then every function in $L^2(D)$ which is analytic in D can be approximated in $L^2(D)$ by polynomials, and this result was later extended by Farrell and Markuševič to Carathéodory regions (see [21]). Only much later was the corresponding problem for approximation by rational functions studied. See Bers [2], Sinanjan [22], [23], Havin [15]. In particular, Havin discovered that necessary and sufficient conditions for approximation in L^2 can be given in terms of logarithmic capacity. He remarked, however, that his method of proof does not easily extend to L^p, $p > 2$.

In this paper we employ a method of proof which applies to all p, $1 \le p < \infty$, and get new approximation theorems for $p > 2$ which improve those of Sinanjan [23]. We also get new proofs of some of the results of Havin [15]. The method of proof goes back to Bers [2], and is similar to that used by the author in [18].

The main results are stated in §3 of the paper. They are given in terms of analytic p-capacity, or equivalently, q-capacity, which are defined in §2 below, where we also study the relations of these capacities to potential theoretic capacity and "Hausdorff content".

For simplicity we will only consider approximation on bounded sets.

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The following notation will be used. If E is a measurable set, $\bar{E}$ will be its closure, E° its interior, ∂E its topological boundary, and $\mathscr{C}E$ its complement. $L_a^p(E)$ is the closed subspace of $L^p(E)$ which consists of functions analytic in E°. $R^p(E)$ is the closure in $L^p(E)$ of rational functions with poles in $\mathscr{C}E$.

For $f \in L^p(E)$, $\|f\|_p = \{\int_E |f(z)|^p \, dA\}^{1/p}$, where dA is Lebesgue measure.

q will always mean $p/(p-1)$. The disk $\{\zeta; |\zeta - z| < \delta\}$ is denoted by $K_z(\delta)$. Various constants will be denoted by K.

2. **Capacities.** Let E be a compact set, and let $1 < q \leq 2$. By Ω we mean the complex plane when $q < 2$, and a fixed bounded region containing E in its interior in the case $q = 2$. The q-capacity $\Gamma_q(E)$ is defined in the following way.

DEFINITION. $\Gamma_q(E) = \inf_\varphi \int_\Omega |\mathrm{grad}\ \varphi|^q \, dA$, where the infimum is taken over all real-valued Lipschitz continuous functions φ with support in Ω such that $\varphi(z) \geq 1$ on E.

If E is arbitrary we define $\Gamma_q(E) = \sup_F \Gamma_q(F)$, $F \subset E$, F compact.

REMARK. This set function has been studied e.g. by Wallin [27] and Ziemer [29]. Ziemer proved, among other things, that Γ_q satisfies the conditions in Choquet's theorem, and therefore e.g. all Borel sets are capacitable.

We can also define an analytic p-capacity $\gamma_p(E)$, $2 \leq p < \infty$, in the following way.

DEFINITION. $\gamma_p(E) = \sup |(1/2\pi i) \int_c f(z) \, dz|$, where the supremum is taken over all functions f in $L_a^p(\Omega \backslash E)$ with $\|f\|_p \leq 1$, and c is a Jordan curve enclosing E.

If E is arbitrary we define $\gamma_p(E) = \sup_F \gamma_p(F)$, $F \subset E$, F compact.

REMARK. A somewhat different definition of analytic p-capacity was given by Sinanjan in [23]. This restricts his results unnecessarily, as will be seen below.

LEMMA 1. *Let E be compact. Then*

$$\gamma_2(E) = \frac{1}{2\pi} \Gamma_2(E)^{1/2}.$$

For $1 < q < 2$ there are positive constants K_1 and K_2 (only depending on q) such that

$$K_1 \Gamma_q(E)^{1/q} \leq \gamma_p(E) \leq K_2 \Gamma_q(E)^{1/q}.$$

Proof. First assume that E (and Ω) has smooth boundary.

Let $p = 2$, and let $\omega(z)$ be the harmonic measure of E with respect to Ω. Then, for any $f \in L_a^2(\Omega \backslash E)$ with continuous boundary values (I owe this idea to John Wermer),

$$\left| \int_{\partial E} f(z) \, dz \right| = \left| \int_{\partial E} f(z)\omega(z) \, dz \right|$$

$$= \left| \int_{\Omega \backslash E} \frac{\partial}{\partial \bar{z}} (f\omega) \, d\bar{z} \wedge dz \right| = \left| 2i \int_{\Omega \backslash E} f\left(\frac{\overline{\partial \omega}}{\partial z}\right) dA \right|.$$

Thus

$$\left| \int_{\partial E} f(z) \, dz \right|^2 \leq 4 \int_{\Omega \backslash E} |f|^2 \, dA \int_{\Omega \backslash E} \left|\frac{\partial \omega}{\partial z}\right|^2 dA.$$

Equality holds for $f = \partial \omega / \partial z$, which is a single-valued analytic function. It follows that

(1)
$$\left| \frac{1}{2\pi} \int_{\partial E} \frac{\partial \omega}{\partial z} \, dz \right| = \frac{1}{4\pi} \|\text{grad } \omega\|_2^2$$

and since functions with continuous boundary values are dense it follows that $\gamma_2(E) = (1/2\pi)\Gamma_2(E)^{1/2}$.

Now let $p > 2$, and let $\omega(z)$ be any complex-valued Lipschitz function with compact support such that $\omega(z) = 1$ on E. We obtain in the same way

$$\left| \int_{\partial E} f(z) \, dz \right| = \left| 2i \int_{\mathscr{C}E} f \frac{\partial \omega}{\partial \bar{z}} \, dA \right| \leq 2\|f\|_p \left\| \frac{\partial \omega}{\partial \bar{z}} \right\|_q.$$

It follows that $\gamma_p(E) \leq K\Gamma_q(E)^{1/q}$. The general case follows easily.

To get the other inequality we will use a standard Hahn-Banach argument (see e.g. Havinson [17]). We note that by the above $\alpha(f) = (1/2\pi i) \int_{\partial E} f(z) \, dz$ is a bounded linear functional on $L_a^p(\mathscr{C}E)$ represented by

$$\frac{1}{\pi} \left(\overline{\frac{\partial \omega}{\partial \bar{z}}} \right) = \frac{1}{\pi} \frac{\partial \bar{\omega}}{\partial z}.$$

On the other hand, it is known (see e.g. [16, Lemma 1]) that functions of the form $\partial \varphi / \partial z$, with $\varphi \in C_0^\infty(\mathscr{C}E)$ form a dense subset of the annihilating space of $L_a^p(\mathscr{C}E)$. Thus, by the Hahn-Banach theorem, there are Lipschitz functions ω with compact support, with $\omega = 1$ on E, such that $(1/\pi)\|\partial \omega / \partial z\|_q$ is arbitrarily close to the norm of the functional α, i.e. to $\gamma_p(E)$. It follows that $\gamma_p(E) = (1/\pi) \inf \|\partial \omega / \partial z\|_q$, for such functions ω. The lemma follows since $\|\text{grad } |\omega|\|_q \leq K\|\partial \omega / \partial z\|_q$ for functions with compact support and $q > 1$, by the Calderón-Zygmund theory [4]. □

We will also examine the relationship between $\gamma_p(E)$ (and $\Gamma_q(E)$) and potential theoretic capacity $C_q(E)$ and "Hausdorff content" $M_q(E)$. If μ is a measure on E we denote the potential of μ with respect to Green's function for Ω by U_2^μ, and the potential with respect to a kernel r^{q-2}, $1 < q < 2$, is denoted by U_q^μ. Then for $1 < q \leq 2$, $C_q(E) = \sup \mu(E)$, the supremum being taken over positive measures on E with $\sup_z U_q^\mu(z) \leq 1$.

For an increasing function $h(r)$, $h(0) = 0$, we define the "Hausdorff content" $M_h(E) = \inf \sum_i h(r_i)$, the infimum being taken over all coverings of E by disks with radii r_i. For $h(r) = r^{2-q}$, $1 < q < 2$, we write $M_q(E)$. For the properties of M_h and its relation to capacities we refer to [7].

The following lemma is essentially contained in [7, Theorem VI:1].

LEMMA 2. *Let E be compact with connected complement. Then*

$$\gamma_2(E) = (2\pi)^{-1/2} C_2(E)^{1/2}.$$

For $1 < q < 2$ there are positive constants K_1 and K_2 (only depending on q), such that

$$K_1 C_q(E)^{1/q} \leq \gamma_p(E) \leq K_2 M_q(E)^{1/q}.$$

Proof. First let $p=2$, and let ω be the harmonic measure for E with respect to Ω. Then ω can be represented as a potential, $\omega(z)=\int_E G(z, \zeta)\, d\mu(\zeta)$, where G is Green's function for Ω, and μ is the corresponding equilibrium measure for E with total mass $C_2(E)$. It follows that, for any Jordan curve c enclosing E,

$$\int_c \frac{\partial \omega}{\partial z}\, dz = \int_c \left\{ \int_E \frac{\partial G(z, \zeta)}{\partial z}\, d\mu(\zeta) \right\} dz = \int_E \left\{ \int_c \frac{\partial G}{\partial z}\, dz \right\} d\mu(\zeta)$$

$$= \int_E \left\{ \int_c \frac{\partial}{\partial z} \log \frac{1}{|z-\zeta|}\, dz \right\} d\mu(\zeta) = \frac{1}{2} \int_E \left\{ \int_c \frac{dz}{\zeta-z} \right\} d\mu(\zeta) = -\pi i \mu(E) = -\pi i C_2(E).$$

Thus by (1)

$$\Gamma_2(E) = \int_\Omega |\text{grad } \omega|^2\, dA = 2 \left| \int \frac{\partial \omega}{\partial z}\, dz \right| = 2\pi C_2(E)$$

and $\gamma_2(E) = \{C_2(E)/2\pi\}^{1/2}$ by Lemma 1.

For $p>2$ we prove $\Gamma_q(E) \leq K M_q(E)$. Cover E with finitely many disks σ_i with radius r_i so that $\sum r_i^{2-q} < M_q(E) + \varepsilon$. Let σ_i' be the expanded disks with radius $2r_i$. Let $\omega_i(z)$ be continuous, $\omega_i(z)=1$ on σ_i, $\omega_i(z)=0$ outside σ_i', and a linear function of the distance to the center of σ_i for z in $\sigma_i'\backslash\sigma_i$. Set $\omega(z)=\text{Max}_i\, \omega_i(z)$. Then

$$\int |\text{grad } \omega|^q\, dA \leq \sum_i \int_{\sigma_i'} |\text{grad } \omega_i|^q\, dA$$

$$= \sum_i 2\pi \int_{r_i}^{2r_i} r_i^{-q} r\, dr = 3\pi \sum r_i^{2-q},$$

which proves the inequality.

For the proof in the other direction we assume that $C_q(E)>0$. Then there is a unit measure μ on E such that $\sup_z U_q^\mu(z) \leq 2/C_q(E)$. The function $\hat\mu(z)=\int d\mu(\zeta)/(\zeta-z)$ is analytic off E, $\lim_{z\to\infty} |z\hat\mu(z)| = 1$, and $\{\int |\hat\mu(z)|^p\, dA\}^{1/p} \leq K/C_q(E)^{1/q}$ by an inequality due to du Plessis [9], and Deny [8]. (A different proof of the inequality is given in [7].) This proves the lemma.

Lemma 3. *If E is a continuum with diameter d there are positive constants K_1 and K_2, only depending on p (and the distance from E to $\partial\Omega$, for $p=2$), such that*

$$K_1/(\log 1/d)^{1/2} \leq \gamma_2(E) \leq K_2/(\log 1/d)^{1/2};$$

$$K_1\, d^{1-2/p} \leq \gamma_p(E) \leq K_2\, d^{1-2/p}, \qquad p > 2.$$

Proof. It is well known that $C_q(E)$ is not increased by a contraction of E. See e.g. Landkof [19, p. 198]. Thus by Lemma 2 it is enough to compute C_q for an interval and M_q for a disk, which is easy by means of a homothety.

3. **Main results.** Theorem 1 is known, and Theorems 2 and 4 are only slight improvements of known theorems, but we include them here in the interest of completeness. See the historical remarks below.

We prefer to give the results in terms of γ_p rather than the equivalent Γ_q, since γ_p is what is needed in most of the proofs.

THEOREM 1. *Let E be a bounded measurable set. Then $R^p(E)=L_a^p(E)$, $1 \leqq p < 2$.*

THEOREM 2. *Let E be compact without interior. Then the following statements are equivalent:*

   (i) $R^2(E)=L^2(E)$.

   (ii) *For almost all $z \in E$ (with respect to plane Lebesgue measure)*

(2)
$$\limsup_{\delta \to 0} C_2(K_z(\delta) \backslash E) \delta^{-2} > 0.$$

   (iii) *For all open sets U in Ω, $C_2(U \backslash E) = C_2(U)$. In particular, $C_2(K_z(\delta) \backslash E) = C_2(K_z(\delta)) \approx K/(\log 1/\delta)$ for all z and $\delta > 0$.*

THEOREM 3. *Let E be compact without interior, and let $2 < p < \infty$. Then the following statements are equivalent:*

   (i) $R^p(E)=L^p(E)$.

   (ii) *For almost all $z \in E$ (with respect to plane Lebesgue measure)*

(3)
$$\limsup_{\delta \to 0} \gamma_p(K_z(\delta) \backslash E) \delta^{-2/q} = \infty.$$

   (iii) *For all bounded open sets U, $\gamma_p(U \backslash E) = \gamma_p(U)$. In particular, $\gamma_p(K_z(\delta) \backslash E) = \gamma_p(K_z(\delta)) = K\delta^{1-2/p} = K\delta^{2/q-1}$ for all z and $\delta > 0$.*

THEOREM 4. *Let E be compact. Then the following statements are equivalent:*

   (i) $R^2(E)=L_a^2(E)$.

   (ii) *For almost all $z \in \partial E \backslash \partial E^{\circ}$ (with respect to plane Lebesgue measure)*

$$\limsup_{\delta \to 0} C_2(K_z(\delta) \backslash E) \delta^{-2} > 0,$$

and for almost all $z \in \partial E^{\circ}$ (with respect to logarithmic capacity)

(4)
$$\int_0 C_2(K_z(\delta) \backslash E) \delta^{-1} \, d\delta = \infty.$$

   (iii) *For all open sets U in Ω, $C_2(U \backslash E) = C_2(U \backslash E^{\circ})$.*

THEOREM 5. *Let E be compact, $2 < p < \infty$. Then $R^p(E)=L_a^p(E)$ if the following conditions are satisfied:*

   (i) *For almost all $z \in \partial E \backslash \partial E^{\circ}$ (with respect to plane Lebesgue measure)*

$$\limsup_{\delta \to 0} \gamma_p(K_z(\delta) \backslash E) \delta^{-2/q} = \infty.$$

   (ii) *There is a denumerable sequence of nondecreasing functions $\{h_i\}$ such that for almost all $z \in \partial E^{\circ}$ (with respect to γ_p)*

(5)
$$\liminf_{\delta \to 0} \frac{\gamma_p(K_z(\delta) \backslash E)}{h_i(\delta)} > 0, \quad \text{for some } i,$$

$h_i^p(\delta)\delta^{1-p}$ *are nonincreasing, and*

(6)
$$\int_0 \text{Min} \left(\frac{1}{\delta \log 1/\delta}, \frac{h_i^p(\delta)}{\delta^{p-1}} \right) d\delta = \infty.$$

Lemma 3 gives the following corollary to Theorems 4 and 5.

COROLLARY. *Let E be compact, $2 \leqq p < \infty$. Then $R^p(E) = L^p_a(E)$ if the inner boundary $\partial' E$ (i.e. the set of points in ∂E which are not on the boundary of any component of $\mathscr{C}E$) satisfies $\gamma_p(\partial' E) = 0$.*

In fact, we can take $h_i(\delta) = \delta^{1-2/p}$ when $p > 2$.

THEOREM 6. *Let E be compact, $2 < p < \infty$. If $R^p(E) = L^p_a(E)$, then*
   (i) $\gamma_p(U \setminus E) = \gamma_p(U \setminus E^\circ)$ *for all bounded open sets U;*
   (ii) *for almost all $z \in \partial E^\circ$ (with respect to C_q)*

$$(7) \qquad \int_0 \frac{\gamma_p^q(K_z(\delta) \setminus E)}{\delta^{3-q}} \, d\delta = \infty;$$

   (iii) *for any nondecreasing function h such that*

$$(8) \qquad \int_0 \frac{h^q(\delta)}{\delta^{3-q}} \, d\delta < \infty$$

the set $F \subset \partial E^\circ$ where $\lim_{\delta \to 0} \gamma_p(K_z(\delta) \setminus E)/h(\delta) = 0$ satisfies $\gamma_p(F) = 0$.

REMARK 1. Lemma 2 shows that Theorems 3 and 5 give sufficient conditions for approximation in terms of potential theoretic capacity. It will be seen from the proof that (3) can then be replaced by the somewhat weaker sufficient condition that $\lim \sup_{\delta \to 0} C_q(K_z(\delta) \setminus E)\delta^{-2} > 0$. If γ_p^q is replaced by C_q, (4) and (7) are of course the integral form of Wiener's condition for regular points. See e.g. [19, p. 356].

REMARK 2. Theorem 1 is due to Sinanjan [22] in the case of no interior points, to Bers [2] for $p = 1$ (and open sets), and to Havin [15] for $1 < p < 2$. For $p = 2$ a somewhat weaker necessary and sufficient condition than the equivalence of (i) and (ii) in Theorems 2 and 4 is due to Havin [15]. A necessary and sufficient condition in the case of no interior points (and $p \geqq 2$) had been given earlier by Sinanjan [23], who also gave a sufficient condition for approximation in the general case and $p \geqq 2$. Sinanjan's results are included in ours. Related problems were also studied in Brennan [3]. Bagby [1], [1a] has obtained results related to ours.

The equivalence of (ii) and (iii) in Theorem 2 (and uniform approximation of all continuous functions by harmonic functions) was proved by Gončar [12], [13], [14], and Lysenko and Pisarevskiĭ [20]. This phenomenon, the instability of capacity, was discovered by Vituškin for analytic capacity. See [26, Chapter VI:1].

We will treat all p, $1 \leqq p < \infty$, simultaneously. The bulk of the paper will be devoted to the proof of Theorem 5. In §4 we prove that functions in $L^p(\partial E)$ can be approximated by rational functions with poles off E, and in §5 we use this result to prove the general result. In §6, finally, we prove Theorem 6.

4. Approximation on nowhere dense sets.

THEOREM 7. *Let E be compact. Then rational functions with poles off E are dense in $L^p(\partial E)$, $1 \leqq p < 2$.*

THEOREM 8. *Let E be compact. Then rational functions with poles off E are dense in $L^p(\partial E)$, $2 \leqq p < \infty$, if for almost all $z \in \partial E$ (with respect to Lebesgue measure)*
$$\lim \sup_{\delta \to 0} \gamma_p^q(K_z(\delta) \backslash E)\delta^{-2} = \infty \ (> 0 \text{ for } p = 2).$$

Theorem 7 is easy. In fact, it is enough to prove that if $g \in L^q(\partial E)$ and the Cauchy transform $\hat{g}(z) = \int_{\partial E} g(\zeta)/(\zeta - z) \, dA = 0$ for all $z \notin E$, then $g \equiv 0$. But $\hat{g}$ is continuous, being the convolution of $g \in L^q$ and $z^{-1} \in L^p_{loc}$. It follows that $\hat{g}(z) = 0$ on ∂E, and since $\hat{g}$ is analytic everywhere off ∂E and $\hat{g}(\infty) = 0$, the maximum principle implies that $\hat{g} \equiv 0$. It is well known that this implies $g \equiv 0$. (See e.g. [6, Lemma 5].)

In order to prove Theorem 8 we will estimate $\hat{g}(z)$ when $g \in L^q(\partial E)$ and $\hat{g}(z) = 0$ off E. We need a few lemmas.

LEMMA 4. *Let $2 < p < \infty$. Let F be a compact set with connected complement and assume $\gamma_p(F) > 0$. Let f be analytic outside F, $f(\infty) = 0$, and $\|f\|_p^p = \int_{\mathscr{C}F} |f|^p \, dA < \infty$. Suppose $F \subset \{|z| \leqq \delta\}$.*
Then for $|z| > r\delta > \delta$
$$|f(z)| \leqq K \frac{\gamma_p(F)}{|z| - r\delta} \|f\|_p,$$

where K only depends on p and r.

Proof. Cf. Sinanjan [23, Lemma 2.1]. Fix z, $|z| > r\delta > \delta$, and consider $\Phi(\zeta) = (f(\zeta) - f(z))/(\zeta - z)$. Then Φ is analytic outside F, $\Phi(\infty) = 0$, and $\lim_{\zeta \to \infty} |\zeta \Phi(\zeta)| = |f(z)|$. Therefore, by definition, $|f(z)| \leqq \gamma_p(F) \|\Phi\|_p$.

We assume that $\|f\|_p = 1$ and claim that $\|\Phi\|_p \leqq K/(|z| - r\delta)$.

For $|\zeta| \geqq r\delta$ we have
$$|f(\zeta)|^p \leqq 1/\pi(r-1)^2\delta^2$$
by the mean value property. Thus, for $|\zeta| = r\delta$
$$|\Phi(\zeta)| \leqq \frac{|f(\zeta)| + |f(z)|}{|z| - r\delta} \leqq \frac{2}{\pi^{1/p}(r-1)^{2/p}\delta^{2/p}} \cdot \frac{1}{|z| - r\delta}.$$

By Schwarz's lemma for $|\zeta| \geqq r\delta$
$$|\Phi(\zeta)| \leqq K \cdot (\delta^{1-2/p}/|\zeta|) \cdot 1/(|z| - r\delta).$$
Thus
$$\int_{|\zeta| \geqq r\delta} |\Phi(\zeta)|^p \, dA \leqq K \frac{\delta^{p-2}}{(|z| - r\delta)^p} \int_{|\zeta| \geqq r\delta} \frac{dA}{|\zeta|^p} = \frac{K}{(|z| - r\delta)^p}.$$

On the other hand
$$\int_{|\zeta| \leqq r\delta} |\Phi(\zeta)|^p \, dA \leqq \frac{K}{(|z| - r\delta)^p} (\|f\|_p^p + |f(z)|^p(r\delta)^2)$$
$$\leqq \frac{K}{(|z| - r\delta)^p} \left(1 + \left(\frac{r}{r-1}\right)^2\right),$$

which proves the lemma.

The above lemma implies the following variant of a lemma of Mergeljan and Vituškin. A weaker form of it, using a smaller analytic p-capacity, was given by Sinanjan [23, Lemma 2.1].

LEMMA 5. *Let* $2 \leqq p < \infty$, *and let* $F \subset \{|z| \leqq \delta\}$ *be compact with connected complement. If* $C_q(F) > 0$ *or* $\gamma_p(F) > 0$ *there exists a function* h, *analytic outside* F *with* $h(\infty) = 0$, *and constants* K *depending only on* p, *such that* $\|h\|_p \leqq K/C_q(F)^{1/q}$ *or* $\|h\|_p \leqq K/\gamma_p(F)$, *respectively, and* $|\zeta^{-1} - h(\zeta)| \leqq K\delta/|\zeta|^2$ *for all* ζ, $|\zeta| \geqq 2\delta$.

Proof. First assume $C_q(F) > 0$, $1 < q \leqq 2$, and let μ be the corresponding unit equilibrium measure for F. Choose $h(\zeta) = \int d\mu(s)/(s - \zeta)$. By the inequalities of du Plessis [9] and Deny [8] referred to earlier, $\|h\|_p \leqq K/C_q(F)^{1/q}$. Moreover, $\lim_{\zeta \to \infty} \zeta h(\zeta) = 1$, so $\lim_{\zeta \to \infty} \zeta^2(\zeta^{-1} - h(\zeta))$ exists. For $|\zeta| = 2\delta$ clearly $|\zeta^2(\zeta^{-1} - h(\zeta))| \leqq 6\delta$. The inequality for $|\zeta| \geqq 2\delta$ follows from the maximum principle.

Now assume $\gamma_p(F) > 0$, $p > 2$, and let h be analytic outside F, $h(\infty) = 0$, $\lim_{\zeta \to \infty} \zeta h(\zeta) = 1$, and $\|h\|_p \leqq 2/\gamma_p(F)$. The lemma follows as above if Lemma 4 is applied to h for $|\zeta| = 2\delta$.

The following lemma will be crucial also in the following section.

LEMMA 6. *Let* E *be compact,* $2 \leqq p < \infty$. *Let* $z \in E$ *and suppose* $F \subset K_z(\delta) \backslash E$ *is compact,* $\gamma_p(F) > 0$. *Let* $g \in L^q(E)$, *and suppose* $\hat{g}(\zeta) = 0$ *for all* $\zeta \notin E$. *Then*

$$|\hat{g}(z)| \leqq K\left(g^*(z)\delta \log \frac{1}{\delta} + \left\{\int_{|\zeta - z| < 2\delta} |g(\zeta)|^q \, dA\right\}^{1/q} \frac{1}{\gamma_p(F)}\right),$$

where

$$g^*(z) = \sup_r \frac{1}{\pi r^2} \int_{|\zeta - z| < r; \zeta \in E} |g(\zeta)| \, dA,$$

and K *depends only on* q *and the diameter of* E.

If $C_q(F) > 0$ *and* $1 < q \leqq 2$, *then (if* $g(z) = \lim_{r \to 0} (1/\pi r^2) \int_{|\zeta - z| < r; \zeta \in E} g(\zeta) \, dA$)

$$|\hat{g}(z)| \leqq K\left(g^*(z)\delta \log \frac{1}{\delta} + \left\{\int_{|\zeta - z| < 2\delta} |g(\zeta) - g(z)|^q \, dA\right\}^{1/q} \frac{1}{C_q(F)^{1/q}}\right).$$

Proof. Cf. the proof of [6, Lemma 1]. Assume $\gamma_p(F) > 0$, put $z = 0$, and let h be the function given in Lemma 5. We extend the definition of g by setting $g(\zeta) = 0$, $\zeta \notin E$. By Runge's theorem $\int gh \, dA = 0$, so $\hat{g}(0) = \int g(\zeta)(1/\zeta - h(\zeta)) \, dA$. By Lemma 5

$$|\hat{g}(0)| \leqq \int_{|\zeta| \leqq 2\delta} \frac{|g(\zeta)|}{|\zeta|} \, dA + \int_{|\zeta| \leqq 2\delta} |g(\zeta)h(\zeta)| \, dA + K\delta \int_{|\zeta| > 2\delta} \frac{|g(\zeta)|}{|\zeta|^2} \, dA$$

$$= I_1 + I_2 + I_3.$$

Here

$$I_1 = \sum_{n=0}^{\infty} \int_{\delta \cdot 2^{-n} \leqq |\zeta| < \delta \cdot 2^{-n+1}} \frac{|g(\zeta)|}{|\zeta|} \, dA$$

$$\leqq \sum_{n=0}^{\infty} \delta \cdot 2^n \cdot \int_{|\zeta| < \delta \cdot 2^{-n+1}} |g(\zeta)| \, dA \leqq K\delta g^*(0).$$

Similarly

$$I_3 = \sum_{n=1}^{N} \int_{\delta \cdot 2^n \leq |\zeta| \leq \delta \cdot 2^{n+1}} \frac{|g(\zeta)|}{|\zeta|^2} \, dA \leq A g^*(0) \delta N,$$

where N is chosen so that $R < \delta 2^{N+1} \leq 2R$ if E is contained in $\{|\zeta| \leq R\}$.

An application of Hölder's inequality to I_2 completes the proof of the first inequality.

Now assume $C_q(F) > 0$ and choose $h(\zeta) = \int d\mu(s)/(s - \zeta)$ as in Lemma 5. We now write

$$I_2 = \left| \int_{|\zeta| \leq 2\delta} g(\zeta) h(\zeta) \, dA \right|$$

$$\leq \int_{|\zeta| \leq 2\delta} |g(\zeta) - g(0)| \, |h(\zeta)| \, dA + |g(0)| \int_{|\zeta| \leq 2\delta} |h(\zeta)| \, dA$$

$$\leq \left\{ \int_{|\zeta| \leq 2\delta} |g(\zeta) - g(0)|^q \, dA \right\}^{1/q} \frac{K}{C_q(F)^{1/q}} + |g(0)| \int d\mu(s) \int_{|\zeta| \leq 2\delta} \frac{dA_\zeta}{|s - \zeta|}.$$

The last integral is less than $K\delta$, and this proves the lemma.

We now assume that $g \in L^q(\partial E)$, $1 < q \leq 2$, and $\hat{g}(\zeta) = 0$, $\zeta \notin E$. We shall prove that $g = 0$ almost everywhere if E has the property in Theorem 8. We observe that Lemma 6 implies that $\hat{g}(\zeta) = 0$ almost everywhere on ∂E, and this proves the asssertion when E has no interior.

REMARK. The same argument as in Lemma 6 applies to measures and gives a short proof of the following theorem of Vituškin (see [26, Chapter VI: 2], where a slightly stronger result is proved).

$R(E) = C(E)$ if for almost all $z \in E$ (with respect to plane Lebesgue measure) $\lim \sup_{\delta \to 0} \gamma(K_z(\delta) \setminus E)/\delta^2 = \infty$.

Here γ is analytic capacity, and $R(E)$ is the uniform closure of the rational functions.

To prove Theorem 8 in the general case we need the following two lemmas.

LEMMA 7. Let $g \in L^q(\partial E)$, $1 < q \leq 2$, let z be given, and suppose $\hat{g}(\zeta) = 0$ in some set $F \subset K_z(\delta)$ with mes $F > \frac{7}{8}\pi\delta^2$. Then $|\hat{g}(z)| \leq K g^*(z) \delta$.

Proof. Suppose $z = 0$. The intersection of F and a sector

$$S_\alpha = \{re^{i\theta}; 0 \leq r \leq \delta, \alpha \leq 0 \leq \alpha + \pi/2\}$$

has measure at least $\frac{1}{8}\pi\delta^2$. Let χ_1 and χ_2 be characteristic functions of subsets of $S_{\pi/4}$ and $S_{5\pi/4}$ with measure $\frac{1}{8}\pi\delta^2$. It is then easily seen that we can determine $a(\zeta) = a_1\chi_1(\zeta) + a_2\chi_2(\zeta)$ so that $\int a(\zeta) \, dA = 1$, $\int \zeta a(\zeta) \, dA = 0$, and then $|a(\zeta)| \leq K\delta^{-2}$.

Now put $h(\zeta) = \int a(s)\, dA_s/(\zeta - s)$. Then $\sup |h(\zeta)| \leq K\delta^{-1}$, and as in Lemma 5 we find $|1/\zeta - h(\zeta)| \leq K\delta^2/|\zeta|^3$, for $|\zeta| \geq 2\delta$. Now $\int g(\zeta)h(\zeta)\, dA = -\int a(s)\hat{g}(s)\, dA = 0$, so

$$|\hat{g}(0)| = \left| \int g(\zeta)\left(\frac{1}{\zeta} - h(\zeta)\right) dA \right|$$

$$\leq K \int_{|\zeta| \leq 2\delta} \frac{|g(\zeta)|}{|\zeta|}\, dA + K\delta^2 \int_{|\zeta| \geq 2\delta} \frac{|g(\zeta)|}{|\zeta|^3}\, dA \leq Kg^*(0)\delta$$

by estimates similar to those in the proof of Lemma 6.

The following is a stronger version of a well-known lemma (see e.g. [6, Lemma 5]).

LEMMA 8. *If $g \in L^q(\partial E)$, $1 < q \leq 2$, and $\hat{g}(\zeta) = 0$ almost everywhere on ∂E, then $g(\zeta) = 0$ almost everywhere.*

Proof. As usual we put $g(\zeta) = 0$, $\zeta \notin \partial E$. By Fubini's and Cauchy's theorems

$$\frac{1}{2\pi i} \int_{|\zeta - z| = \delta} \hat{g}(\zeta)\, d\zeta = \frac{1}{2\pi i} \int g(s)\, dA \int_{|\zeta - z| = \delta} \frac{d\zeta}{s - \zeta} = -\int_{|s - z| \leq \delta} g(s)\, dA$$

for almost all z and δ. Therefore the lemma will follow if we show that for almost all $z \in \partial E$

$$\liminf_{\delta \to 0} \frac{1}{\delta^2} \int_{|\zeta - z| = \delta} |\hat{g}(\zeta)|\, |d\zeta| = 0.$$

Let z be a density point for ∂E, and assume moreover that

$$\lim_{\delta \to 0} \frac{1}{\pi\delta^2} \int_{|\zeta - z| \leq \delta; \zeta \notin \partial E} g^*(\zeta)\, dA = 0.$$

This is true for almost all $z \in \partial E$, since $g^* \in L^q(\partial E)$. (See e.g. [24, Theorem I.1], for a simple proof.)

By Lemma 7 we get for sufficiently small δ

$$\int_{|\zeta - z| \leq \delta} |\hat{g}(\zeta)|\, dA \leq K \int_{|\zeta - z| \leq \delta; \zeta \notin \partial E} g^*(\zeta)|\zeta - z|\, dA = o(\delta^3)$$

by the choice of z. It follows that

$$\liminf_{\delta \to 0} \frac{1}{\delta^2} \int_{|\zeta - z| = \delta} |\hat{g}(\zeta)|\, |d\zeta| = 0,$$

which proves the lemma and Theorem 8.

5. Approximation on arbitrary compact sets. We shall now apply the results of the preceding section to the general case. Suppose $g \in L^q(E)$ and $\hat{g}(z) = 0$ for $z \in \mathscr{C}E$. We will show that $\int_E fg\, dA = 0$ for $f \in L_a^p(E)$.

We first observe that it suffices to prove that $\int_{E^\circ} fg\, dA = 0$ for $f \in L_a^p(E)$. In fact, this implies that

$$\int_{E^\circ} \frac{g(\zeta)\, dA}{\zeta - z} = 0, \qquad z \notin E,$$

and therefore

$$\int_{\partial E} \frac{g(\zeta)\, dA}{\zeta - z} = 0, \qquad z \notin E.$$

But then $g(\zeta) = 0$ a.e. on ∂E by Theorems 7 and 8, so $\int_E fg\, dA = \int_{E^\circ} fg\, dA = 0$.

For $p = 2$, (4) implies that $\int_{E^\circ} fg\, dA = 0$ as in [15], which proves that (ii) implies (i) in Theorem 4. Now let $p \neq 2$. Following Bers [2], we let $\{\omega_n\}$ be an increasing sequence of functions with compact support in E° such that $0 \leqq \omega_n \leqq 1$ and $\lim_{n \to \infty} \omega_n = 1$ for almost all $z \in E^\circ$. It is sufficient to prove that $\lim_{n \to \infty} \int \omega_n fg\, dA = 0$ for a suitable choice of the sequence. But by Green's theorem and the analyticity of f

$$\int \omega_n fg\, dA = -\frac{1}{\pi} \int \omega_n f \frac{\partial \hat{g}}{\partial \bar{z}}\, dA = \frac{1}{\pi} \int f \frac{\partial \omega_n}{\partial \bar{z}} \hat{g}\, dA,$$

and

$$\left| \int f \frac{\partial \omega_n}{\partial \bar{z}} \hat{g}\, dA \right| \leqq \|f\|_p \|\hat{g} \operatorname{grad} \omega_n\|_q.$$

We shall show that the ω_n can be chosen so that $\|\hat{g} \operatorname{grad} \omega_n\|_q \to 0$, $n \to \infty$.

The proof will follow from the following lemmas. Lemma 9 is a counterpart to Lemma 6 for $q > 2$.

LEMMA 9. *If $g \in L^q(E)$, $2 < q < \infty$, then for $|z_1 - z| \leqq \delta$*

$$|\hat{g}(z) - \hat{g}(z_1)| \leqq K \left(g^*(z)\delta \log \frac{1}{\delta} + \left\{ \int_{|\zeta - z| < 2\delta} |g(\zeta)|^q\, dA \right\}^{1/q} \cdot \delta^{1 - 2/q} \right).$$

For $q = \infty$

$$|\hat{g}(z) - \hat{g}(z_1)| \leqq K \|g\|_\infty \delta \log \frac{1}{\delta}.$$

The proof is similar to that of Lemma 6 and is omitted.

LEMMA 10. *Let F be a compact subset of ∂E° such that the assumptions in Theorem 5 are satisfied for all $z \in F$. ($F = \partial E^\circ$ for $p < 2$.) Then for given $\varepsilon > 0$ there exists a function ω, $0 \leqq \omega(z) \leqq 1$, such that $\omega(z) = 0$ in a neighborhood of F, and $\omega(z) = 1$ outside an ε-neighborhood of F, and $\|\hat{g} \operatorname{grad} \omega\|_q < \varepsilon$.*

Proof. By considering h_i/n, $n = 1, 2, \ldots$, and reenumerating, we can assume that, for each $z \in F$, $\gamma_p(K_z(\delta) \backslash E) \geqq h_m(\delta)$ for all $\delta \leqq 1/m$, for some m. Let F_m be the set of such z, so $F = \bigcup_1^\infty F_m$.

We now fix m, and we shall construct a function ω_m which is 0 in a neighborhood of $\bar{F}_m$, 1 outside a neighborhood of $\bar{F}_m$ and is such that $\|\hat{g} \operatorname{grad} \omega_m\|_q < \varepsilon/2^m$.

By Lemma 6 applied to a point z at distance $\delta(z)$ from F_m and to the disk $K_z(2\delta(z))$, we obtain

$$|\hat{g}(z)| \leqq K \left(g^*(z)\delta(z) \log \frac{1}{\delta(z)} + \left\{ \int_{|\zeta - z| < 4\delta(z)} |g(\zeta)|^q\, dA \right\}^{1/q} \frac{1}{h_m(\delta(z))} \right).$$

(Put $h(z) = \delta^{2/q - 1}$ for $p < 2$ by Lemma 9.)

To construct ω_m we now apply an idea of Ahlfors which appears in [2].

Let $\lambda(t)$, $t>0$, be a nonincreasing function such that $\int_r \lambda(t)\,dt = \Lambda(r) \to \infty$ as $r \to +0$. Let $k(t)$, $t>0$, be a continuous piecewise linear function such that $k(t)=0$ for $t \leqq 1$, $k(t)=1$ for $t \geqq 2$, and $|k'(t)| \leqq 1$. Put $K_n(t) = k(n/\Lambda(t))$, $n=1,2,\ldots$. Then $K_n'(t) = k'(n/\Lambda(t)) \cdot (-n/\Lambda^2(t)) \cdot (-\lambda(t))$, so $K_n'(t)=0$ except for $0 < t_n' \leqq t \leqq t_n''$. For $t_n' \leqq t \leqq t_n''$, $\lambda(t)/n \leqq K_n'(t) \leqq 4\lambda(t)/n$, and $\int_{t_n'}^{t_n''} K_n'(t)\,dt = 1$.

We now choose $\lambda(t) = \text{Min}\,(1/(t \log 1/t),\, h_m^p(t)/t^{p-1})$, and put $\omega_m(z) = K_n(\delta(z))$, where n is to be determined. Then $|\text{grad}\,\omega_m(z)| \leqq K_n'(\delta(z))$. A short calculation gives

$$|\hat{g}(z)\,\text{grad}\,\omega_m(z)|^q \leqq \frac{K}{n^{q-1}}\left(g^*(z)^q + \frac{K_n'(\delta(z))}{\delta(z)} \int_{|\zeta-z|<4\delta(z)} |g(\zeta)|^q\,dA\right).$$

By the Hardy maximal theorem (see e.g. [24, Theorem I.1] for a simple proof) we only have to estimate the integral of the second term. We find, since λ is non-increasing,

$$\int \frac{K_n'(\delta(z))}{\delta(z)}\,dA_z \int_{|\zeta-z|<4\delta(z)} |g(\zeta)|^q\,dA_\zeta$$

$$\leqq \frac{K}{n} \int dA_z \int_{|w|<4\delta(z)} \text{Min}\left(\frac{\lambda(t_n')}{t_n'}, \frac{\lambda(|w|/4)}{|w|/4}\right)|g(z+w)|^q\,dA_w$$

$$\leqq \frac{K}{n} \int \text{Min}\left(\frac{\lambda(t_n')}{t_n'}, \frac{\lambda(|w|/4)}{|w|/4}\right) dA_w \int |g(z)|^q\,dA_z$$

$$\leqq \frac{K}{n}\left(\lambda(t_n')\cdot t_n' + \int \lambda(t)\,dt\right)\|g\|_q^q \leqq K\|g\|_q^q,$$

since $\lambda(t) \leqq 1/t$. Thus $\|\hat{g}\,\text{grad}\,\omega_m\|_q < \varepsilon/2^m$ if n is large enough.

Now put $\omega^{(m)} = \prod_1^m \omega_i$. Then $|\text{grad}\,\omega^{(m)}| \leqq \sum_1^m |\text{grad}\,\omega_i|$, so $\|\hat{g}\,\text{grad}\,\omega^{(m)}\|_q < \varepsilon$. Also $\omega^{(m)}=0$ in a neighborhood O_m of $\bar{F}_m$, and the sequence $\{O_m\}$ is increasing. Therefore, by the compactness of F, $F \subset O_N$ for some N. We choose $\omega = \omega^{(N)}$, and this finishes the proof of the lemma and of Theorem 1.

LEMMA 11. *Let $2 \leqq p < \infty$ and let F be a Borel subset of ∂E° with $\gamma_p(F)=0$. Then there exist functions ω_n, $0 \leqq \omega_n(z) \leqq 1$, such that $\lim_{n\to\infty} \|\hat{g}\,\text{grad}\,\omega_n\|_q = 0$, $\lim_{n\to\infty} \omega_n(z) = 1$ in E°, and $\omega_n(z)=0$ in a neighborhood of F.*

Proof. By Lemma 1, and since Borel sets are capacitable for Γ_q (see [29]), there is for every n an open neighborhood O_n of F such that $\Gamma_q(O_n)^{1/q} \leqq \varepsilon_n/n$, and $\lim_{n\to\infty} \varepsilon_n = 0$. Then there is an increasing sequence of compact sets, $K_i \subset O_n$, $O_n = \bigcup_1^\infty K_i$, and functions $\varphi_n^{(i)}$, $\varphi_n^{(i)}(z)=1$ on K_i, such that $\|\text{grad}\,\varphi_n^{(i)}\|_q \leqq \varepsilon_n/n$. We choose a weakly convergent subsequence of $\{\varphi_n^{(i)}\}_i$, and denote the limit function by φ_n. Then $\varphi_n(z)=1$ on O_n, and $\|\text{grad}\,\varphi_n\|_q \leqq \varepsilon_n/n$. Moreover (see [29]), $\Gamma_q^{1/q}\{z;\,\varphi_n(z)>\frac{1}{2}\} \leqq K\varepsilon_n/n$. Set $\varphi_n'(z) = 2\,\text{Max}\,(\varphi_n(z)-\frac{1}{2},\,0)$. It follows that $\lim_{n\to\infty}\varphi_n'(z)=0$ a.e. in E°, at least for a subsequence, which we still denote by $\{\varphi_n'\}$.

Now set $S_n = \{z; |\hat{g}(z)| > n\}$. By the Calderón-Zygmund theory for singular integrals (see [3]) $\|\text{grad } \hat{g}\|_q \leq K\|g\|_q$, and moreover $|\text{grad } |\hat{g}|\,| \leq |\text{grad } \hat{g}|$ almost everywhere, so $\Gamma_q(S_n)^{1/q} \leq (K/n)\|g\|_q$.

Define $\psi_n(z) = 0$ on S_n, $\psi_n(z) = 1$ outside $S_{n/2}$, and $\psi_n(z) = 2(1 - \hat{g}(z)/n)$ otherwise. It follows that $\|\text{grad } \psi_n\|_q = o(1/n)$, and that $\lim_{n \to \infty} \psi_n(z) = 1$ almost everywhere. We now set $\omega_n = (1 - \varphi_n')^2 \psi_n^2$. Then

$$|\text{grad } \omega_n| \leq 2(1 - \varphi_n)\psi_n^2 |\text{grad } \varphi_n| + 2(1 - \varphi_n)^2 \psi_n |\text{grad } \psi_n|$$
$$\leq 2\psi_n(|\text{grad } \varphi_n| + |\text{grad } \psi_n|).$$

We find $\int |\hat{g}\psi_n \text{ grad } \psi_n|^q \, dA \leq n^q \|\text{grad } \psi_n\|_q^q < \varepsilon$ for n large enough, and similarly $\int |\hat{g}\psi_n \text{ grad } \varphi_n|^q \, dA \leq n^q \|\text{grad } \varphi_n\|_q^q < \varepsilon$. This proves the lemma.

To complete the proof of Theorem 5 we now only have to observe that, by assumption, (5) is satisfied except on a set G with $\gamma_p(G) = 0$. We choose a function ω' as in Lemma 11, with $\omega' = 0$ in a neighborhood O of G. Then $\partial E \backslash O$ satisfies the assumptions of Lemma 10, and we choose $\omega'' = 0$ in a neighborhood of $\partial E \backslash O$ as in Lemma 10. We then set $\omega = \omega'\omega''$, and the result follows.

6. **Necessary conditions for approximation.** We shall first show that if $R^p(E) = L_a^p(E)$ then $\gamma_p(U \backslash E) = \gamma_p(U \backslash E^\circ)$ for all open bounded sets U. The proof is similar to the corresponding proof for uniform approximation as given e.g. in [28, p. 104].

LEMMA 12. *Let U be a bounded open set. Suppose that $f \in R^p(E)$, f is analytic outside a compact subset of U and that $f(\infty) = 0$. Then $f \in R^p(\mathscr{C}U \cup E)$.*

Proof. By Runge's theorem $(z - a)f$ can be uniformly approximated on $\mathscr{C}U$ for all a by rational functions with poles in U. If a is chosen in U it follows that $f \in R^p(\mathscr{C}U)$, $p \geq 2$, where the norm is taken over a fixed disk for $p = 2$. The lemma now follows exactly as in [28, 11.7 and 15.9].

Now assume $R^p(E) = L^p(E)$ and let U be any open set with $\gamma_p(U \backslash E^\circ) > 0$. Then for given $\varepsilon > 0$ there exists a function f which is analytic outside a compact subset of $U \backslash E^\circ$, $\int_{\mathscr{C}(U \backslash E^\circ)} |f|^p \, dA = 1$, and $|f'(\infty)| \geq \gamma_p(U \backslash E^\circ) - \varepsilon$. We put $f(z) = 0$ for $z \in U \backslash E^\circ$.

The restriction of f to E is in $R^p(E)$ by assumption, so by Lemma 12 we can find a sequence of rational functions f_n with poles in $U \backslash E$ tending to f in $L^p(\mathscr{C}U \cup E)$. Clearly $f_n'(\infty) \to f'(\infty)$, and

$$\int_{\mathscr{C}U \cup E} |f_n|^p \, dA \to \int_{\mathscr{C}U \cup E} |f|^p \, dA = \int_{\mathscr{C}U \cup E^\circ} |f|^p \, dA = 1.$$

It follows that $\gamma_p(U \backslash E) \geq \gamma_p(U \backslash E^\circ) - \varepsilon$, which proves Theorems 2(iii), 3(iii), 4(iii), and 6(i).

We shall now prove that these statements imply Theorems 4(ii), and 6(ii) and (iii). (Cf. Vituškin [26, Chapter V: 2].)

Suppose $\gamma_p(U \backslash E) = \gamma_p(U \backslash E^\circ)$ for all bounded open U, and assume that there is a

compact set $F \subset \partial E^\circ$ with $\gamma_p(F) > 0$ such that $\int_0 \gamma_p^q(K_z(\delta) \backslash E)\delta^{q-3} \, d\delta < \infty$ for all $z \in F$. Then, for all $z \in F$

$$\int_0 \gamma_p^q(K_z(\delta) \cap F)\delta^{q-3} \, d\delta \leqq \int_0 \gamma_p^q(K_z(\delta) \backslash E^\circ)\delta^{q-3} \, d\delta$$

$$= \int_0 \gamma_p^q(K_z(\delta) \backslash E)\delta^{q-3} \, d\delta < \infty.$$

By Lemma 2, $\int_0 C_q(K_z(\delta) \cap F)\delta^{q-3} \, d\delta < \infty$, and that this implies $C_q(F) = 0$ is Kellogg's lemma. See e.g. [25, Theorem III:33] or [19, pp. 327 and 353], compare also [26, Chapter III:6].

In order to conclude that the assumptions in Theorem 6(iii) imply that $\gamma_p(F) = 0$ it is clearly enough to prove the following lemma.

LEMMA 13. *Let F be compact, $1 < q \leqq 2$. If for all $z \in F$ and δ, $0 < \delta < \delta_0$, $\Gamma_q(K_z(\delta) \cap F) \leqq h(\delta)$, where $h > 0$ is increasing and $\int_0 h(\delta)\delta^{q-3} \, d\delta < \infty$, then $\Gamma_q(F) = 0$.*

Proof. Suppose $\Gamma_q(F) > 0$. For any $d > 0$ we can then find a subset F' of F with diameter $\leqq d$ and $\Gamma_q(F') > 0$. If F' is covered by disks $K_{z_i}(\delta_i)$ it follows from the assumption and the subadditivity of Γ_q that $\Gamma_q(F') \leqq \sum_i h(\delta_i)$, so $\Gamma_q(F') \leqq M_h(F')$. But on the other hand, it is easily seen from [7, Theorem IV:1] and Lemma 2 above that

$$M_h(F') \leqq C_q(F') \int_0^d h(\delta)\delta^{q-3} \, d\delta \leqq K\Gamma_q(F') \int_0^d h(\delta)\delta^{q-3} \, d\delta$$

which is a contradiction if d is small enough. This proves the lemma.

Theorem 6 follows easily, and this finishes the proof.

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DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, UPPSALA, SWEDEN