ENTROPY-EXPANSIVE MAPS

BY

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Abstract. Let \( f: X \to X \) be a uniformly continuous map of a metric space. \( f \) is called \( h \)-expansive if there is an \( \varepsilon > 0 \) so that the set \( \Phi_{\varepsilon}(x) = \{ y : d(f^n(x), f^n(y)) \leq \varepsilon \} \) for all \( n \geq 0 \) has zero topological entropy for each \( x \in X \). For \( X \) compact, the topological entropy of such an \( f \) is equal to its estimate using \( \varepsilon : h(f) = h(f, \varepsilon) \). If \( X \) is compact finite dimensional and \( \mu \) an invariant Borel measure, then \( h_\mu(f) = h_\mu(f, A) \) for any finite measurable partition \( A \) of \( X \) into sets of diameter at most \( \varepsilon \). A number of examples are given. No diffeomorphism of a compact manifold is known to be not \( h \)-expansive.

Let \( f: X \to X \) be a homeomorphism of a metric space. For \( \varepsilon > 0 \) and \( x \in X \) define

\[ \Gamma_{\varepsilon}(x) = \{ y \in X : d(f^n(y), f^n(x)) \leq \varepsilon \} \text{ for all } n \in \mathbb{Z}. \]

\( f \) is called expansive if for some \( \varepsilon \) these sets are as small as possible, i.e. if \( \Gamma_{\varepsilon}(x) = \{ x \} \) for all \( x \). We are concerned with entropy and shall call \( f \) \( h \)-expansive provided that for some \( \varepsilon > 0 \) the \( \Gamma_{\varepsilon}(x) \) are negligible in terms of entropy, i.e. if the topological entropy \( h(f, \Gamma_{\varepsilon}(x)) = 0 \) for all \( x \).

We have two main results for \( h \)-expansive maps with \( X \) compact. First, the topological entropy satisfies \( h(f) = h(f, \varepsilon) \). Second, assuming \( X \) is finite dimensional, \( h_\mu(f) = h_\mu(f, A) \) when \( \mu \) is an \( f \)-invariant normalized Borel measure on \( X \) and \( A \) is a finite measurable partition of \( X \) into sets of diameter at most \( \varepsilon \). Both these results are well known in case \( f \) is expansive (see [11] and [14] respectively). Arov [2] noted that the second statement was true for \( f \) an endomorphism of a torus and \( \mu \) Haar measure when he calculated \( h_\mu(f) \) for this case (see Example 1.2).

1. Definitions and examples. We now review the definition of topological entropy given in [4]. For \( X \) compact this definition was given independently by Dinaburg [7]; is related to the \( \varepsilon \)-entropy of Kolmogorov [12]. Topological entropy was defined first in [1].

Let \( f: X \to X \) be uniformly continuous on the metric space \( X \). For \( E, F \subseteq X \) we say that \( E(n, \delta) \)-spans \( F \) (with respect to \( f \)), if for each \( y \in F \) there is an \( x \in E \) so that \( d(f^n(x), f^n(y)) \leq \delta \) for all \( 0 \leq k < n \). We let \( r_n(F, \delta) = r_n(F, \delta, f) \) denote the minimum cardinality of a set which \((n, \delta)\)-spans \( F \). If \( K \) is compact, then the continuity of \( f \) guarantees \( r_n(K, \delta) < \infty \). For compact \( K \) we define

\[ \tilde{r}_f(K, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(K, \delta) \]

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and
\[ h(f, K) = \lim_{\delta \to 0} \tilde{r}_f(K, \delta) \]
(notice that \( \tilde{r}_f(K, \delta) \) increases as \( \delta \) decreases). Finally let \( h(f) = \sup_K h(f, K) \) where \( K \) varies over all compact subsets of \( X \). If \( X \) is compact, then \( h(f) = h(f, X) \) and we write \( h(f, \delta) = \tilde{r}_f(X, \delta) \).

Let \( \Phi_t(x) = \bigcap_{n \geq 0} f^{-n}B_\varepsilon(f^n(x)) = \{ y : d(f^n(x), f^n(y)) \leq \varepsilon \text{ for } n \geq 0 \} \) and \( h_\varepsilon^*(\varepsilon) = \sup_{x \in X} h(f, \Phi_t(x)) \). \( f \) is called \( h \)-expansive if \( h_\varepsilon^*(\varepsilon) = 0 \) for some \( \varepsilon > 0 \). In case \( f \) is a homeomorphism we set
\[ \Gamma_\varepsilon(x) = \bigcap_{n \in \mathbb{Z}} f^{-n}B_\varepsilon(f^n(x)) \]
and
\[ h_{\text{homeo}}^*(\varepsilon) = \sup_{x \in X} h(f, \Gamma_\varepsilon(x)) \].

**Remark.** For \( f \) a homeomorphism, \( \Gamma_\varepsilon(x) \subseteq \Phi_\varepsilon(x) \) and so \( h_{\text{homeo}}^*(\varepsilon) \leq h_\varepsilon^*(\varepsilon) \). The definition of \( h \)-expansiveness for homeomorphisms mentioned in the introduction, namely \( h_{\text{homeo}}^*(\varepsilon) = 0 \), is actually equivalent to the above one in case \( X \) is compact.

For in 2.3 we prove \( h_\varepsilon^*(\varepsilon) = h_{\text{homeo}}^*(\varepsilon) \) when \( X \) compact.

**Example 1.0.** Expansive maps.

**Example 1.1.** If \( f: \mathbb{R}^n \to \mathbb{R}^n \) is linear and \( d \) comes from a norm, then \( h_\varepsilon^*(\varepsilon) = 0 \) for every \( \varepsilon \).

**Proof.** \( f \) decomposes into a direct sum of linear maps \( f = f_1 \oplus f_2: E_1 \oplus E_2 \to E_1 \oplus E_2 \) where \( f_1 \)’s eigenvalues have norm at most 1 and \( f_2 \)’s have norm greater than 1. If \( u \in E_2, u \neq 0 \), then \( d(f_2(u), 0) \to \infty \text{ as } n \to \infty \). It follows that \( \Phi_t(0) \subseteq E_1 \). But \( h(f|E_1) = h(f_1) = 0 \) by Theorem 15 of \([4]\). So \( h(f, \Phi_t(0)) = 0 \). But \( \Phi_t(x) = \Phi_t(0) + x \) and \( h(f, K + x) = h(f, K) \) for any compact set \( K \).

**Example 1.2.** An endomorphism \( f \) of a Lie group \( G \) is \( h \)-expansive.

**Proof.** Here we use a right invariant metric \( d \). Then one checks \( \Phi_t(x) = \Phi_t(e) x \) and \( h(f, Kx) = h(f, K) \) for compact \( K \). So it is enough to see \( h(f, \Phi_t(e)) = 0 \) for some \( \varepsilon \).

Now
\[
\begin{array}{ccc}
T_\varepsilon G & \xrightarrow{df} & T_\varepsilon G \\
\downarrow \exp & & \downarrow \exp \\
G & \xrightarrow{f} & G
\end{array}
\]
commutes and \( \exp \) is a homeomorphism of a small neighborhood \( B_\varepsilon(0) \subseteq T_\varepsilon G \) onto a neighborhood of some \( B_\varepsilon(e) \). Then \( \Phi_t(e, f) \subseteq \exp \Phi_\varepsilon(0, df) \) and since \( f|\exp \Phi_\varepsilon(0, df) \) is a quotient of \( df|\Phi_\varepsilon(0, df) \) one has
\[ h(f, \Phi_t(e, f)) \leq h(df, \Phi_\varepsilon(0, df)) = 0. \]

**Example 1.3.** Suppose \( f \) is \( h \)-expansive and \( T \) a uniformly continuous map so that \( (T \cdot f)^n = T_n \cdot f^n \) for \( n \geq 0 \) where the \( T_n \) are isometries. Then \( T \cdot f \) is \( h \)-expansive.
Proof. One checks easily that $0 \leq (x, T^n) = \Phi(x, f^n)$ and that a set which $(n, \delta)$-spans some $F \subseteq X$ with respect to $f$ also $(n, \delta)$-spans $F$ with respect to $T \cdot f$. It follows that

$$h(T \cdot f, \Phi(x, T \cdot f)) \leq h(f, \Phi(x, f)) = 0.$$ 

Example 1.3*. Let $G$ be a Lie group and for $g, u \in G$ define $L_g(u) = gu$ and $R_g(u) = ug$. If $f$ is an endomorphism of $G$ and $g \in G$, then the affine maps $R_g \cdot f$, $f \cdot R_g$, $L_g \cdot f$ and $f \cdot L_g$ are all $h$-expansive.

Proof. If we set $g_1 = g$ and $g_{n+1} = f(g_n)g$, one sees that $(R_g \cdot f)^n = R_{g_n} \cdot f^n$. As we use a right invariant metric, $R_{g_n}$ is an isometry and 1.3 applies. Now $(L_g \cdot f)(u) = gf(u)(g^{-1})g = (R_g \cdot f^*)(u)$ where $f^*(u) = gf(u)g^{-1}$ is an endomorphism. We leave $f \cdot R_g$ and $f \cdot L_g$ to the reader.

Example 1.4. Suppose $H$ is a uniformly discrete subgroup of the Lie group $G$, i.e. $G/H$ is compact and $\pi: G \rightarrow G/H$ given by $\pi(x) = xH$ is a covering. For an endomorphism of $G$ with $f(H) \subseteq H$ and $g \in G$ define $f^*$ on $G/H$ by $f^*(uH) = gf(u)H$. Then $f^*$ is $h$-expansive.

Proof. For small enough $\pi$ maps $B_\delta(x)$ isometrically onto $B_\delta(xH)$ for every $x$ and $\pi \Phi(x, L_g \cdot f) = \Phi(xH, f^*)$. Then

$$h(f^*, \Phi(x, L_g \cdot f^*)) \leq h(L_g \cdot f, \Phi(x, L_g \cdot f)) = 0.$$ 

So $f^*$ is $h$-expansive (see [4] for some more details).

Example 1.5. For the case of $X$ compact define the nonwandering set

$$\Omega(f) = \left\{ x \in X : \text{for every neighborhood } U \text{ of } x, U \cap \bigcup_{n \geq 0} f^n(U) \neq \emptyset \right\}.$$ 

Then $f|\Omega(f) = \Omega(f)$. If $f|\Omega(f)$ is $h$-expansive, then so is $f$. An example of this is one of Smale’s Axiom A diffeomorphisms [15], where $f|\Omega(f)$ is expansive.

Proof. Splice together the proof of Theorem 2.4 in [5] and that of 2.2 below for $f|\Omega$, $a = 0$ and $x$ staying in a neighborhood of $\Omega$ up to time $n$.

Example 1.6. Suppose $\Phi = \{\varphi_t: X \rightarrow X\}_{t \in \mathbb{R}}$ is a continuous flow on a compact metric space $X$. Suppose also that there are $\varepsilon > 0$ and $\delta > 0$ so that

$$\Gamma_{\varepsilon}(x, \Phi) = \{y \in X : d(\varphi_t(y), \varphi_t(x)) \leq \varepsilon \text{ for all } t \in \mathbb{R}\} \subseteq \varphi_{t-s} \cdot \Phi \subseteq \varphi_{t+s} \cdot \Phi \text{ for all } |r| \leq \delta.$$ 

Then each $\varphi_t$ is $h$-expansive.

Proof. For any $t \in \mathbb{R}$ there is a $\delta$ so that $d(x, y) \leq \delta$ implies $d(\varphi_t(x), \varphi_t(y)) \leq \varepsilon$ for all $|r| \leq \delta$. Then

$$\Gamma_{\varepsilon}(x, \varphi_t) \subseteq \Gamma_{\varepsilon}(x, \Phi) \subseteq \varphi_{t-s} \cdot \Phi \subseteq \varphi_{t+s} \cdot \Phi.$$ 

For $\beta > 0$ choose $\alpha > 0$ such that for all $x \in X$ and all $|r| \leq \alpha$ we have $d(x, \varphi_r(x)) \leq \delta$ (here we use $X$ compact). Let $K$ be a set of numbers so that every point in $[-s, s]$ is...
within $\alpha$ of one of them. Then \{$\varphi_u(x) : u \in K$\} $(n, \delta)$-spans $\varphi_{t-s}(x)$ with respect to $\varphi_t$. Hence
\[ r_n(\varphi_{t-s}(x), \delta, \varphi_t) \leq \text{card } K \]
and
\[ h(\varphi_t, r_n(x, \varphi_t)) \leq h(\varphi_t, \varphi_{t-s}(x)) = 0. \]

**Example 1.6**. Let $\Phi = \{\varphi_t\}$ be one of Smale’s Axiom A flows [15]. Then $\Phi|\Omega(\Phi)$ satisfies the condition of 1.6 [9]. By 1.5 and 1.6, each $\varphi_t$ is $h$-expansive.

**Problem.** Find some differentiable maps which are not $h$-expansive.

2. Calculating topological entropy.

**Assumption.** For the remainder of the paper $X$ is compact.

**Lemma 2.1.** Suppose $0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = n$ and $E_i(t_{i+1} - t_i, \alpha)$-spans $f^i(F)$ for $0 \leq i < r$. Then
\[ r_n(F, 2\alpha) \leq \prod_{0 \leq i < r} \text{card } E_i. \]

**Proof.** For $x_i \in E_i$ write
\[ V(x_0, \ldots, x_{r-1}) = \{x \in F : d(f^{t_j}(x), f^i(x_i)) \leq \alpha \text{ for } 0 \leq j < t_{i+1} - t_i, 0 \leq i < r\}. \]
If $x, y \in V(x_0, \ldots, x_{r-1})$, then by the triangle inequality $d(f^{s}(x), f^s(y)) \leq 2\alpha$ for $0 \leq s < n$. Since $F = \bigcup V(x_0, \ldots, x_{r-1})$ we get an $(n, 2\alpha)$-spanning set for $F$ by taking one element from each nonempty $V(x_0, \ldots, x_{r-1})$.

**Proposition 2.2.** Let $a = h^*_f(e)$ or $h^*_f,\text{homeo}(e)$ (in case $f$ is a homeomorphism). Then for every $\delta > 0$ and $\beta > 0$ there is a $c$ such that
\[ r_n\left( \bigcap_{k=0}^{n-1} f^{-k}B_e(f^k(x)), \delta \right) \leq ce^{(a + \beta)n} \]
for all $x \in X$.

**Proof.** We do the case where $f$ is a homeomorphism and $a = h^*_f,\text{homeo}(e)$. The case where $a = h^*_f(e)$ is slightly simpler and we leave the necessary modifications to the reader.

For each $y \in X$ pick $m(y)$ so that $a + \beta \geq (1/m(y)) \log \text{card } E(y)$ where $E(y)$ is a set which $(m(y), \frac{1}{4} \delta)$-spans $\Gamma_y(y)$. Then $U(y) = \{w \in X : \exists z \in E(y) \text{ such that } d(f^k(w), f^k(z)) < \frac{1}{4} \delta \text{ for all } 0 \leq k < m(y)\}$ is an open neighborhood of the compact set $\Gamma_y(y)$. Let $S_M = \bigcap_{1 \leq M} f^{-1}B_e(f^M(y))$. Then $S_0 \supset S_1 \supset \cdots$ is a decreasing chain of compact sets with intersection $\Gamma_y(y)$; hence there is an integer $N(y)$ so that $S_{N(y)} \subset U(y)$. Consider the compact sets $W_y = \bigcap_{1 \leq N(y)} f^{-1}B_e(f^N(y))$. Then $\bigcap_{s > e} W_y = W_{\epsilon} = S_{N(y)} \subset U(y)$; hence, $W_{\epsilon} \subset U(y)$ for some $\gamma > \epsilon$. Let $V(y)$ be a neighborhood of $y$ such that $d(f^j(u), f^j(y)) < \gamma - \epsilon$ for $|j| \leq N(y)$ when $u \in V(y)$. Then $B_e(f^j(u)) \subset B_e(f^j(y))$ and
\[ \bigcap_{|j| \leq N(y)} f^{-j}B_e(f^j(u)) \subset U(y). \]
Let \( V(y_1), \ldots, V(y_s) \) cover the compact space \( X \) and

\[
N = \max \{ N(y_1), \ldots, N(y_s), m(y_1), \ldots, m(y_s) \} + 1.
\]

Consider now any \( x \in X \) and \( F_n = \bigcap_{k=0}^{n-1} f^{-k}B_x(f^k(x)) \). For any \( t \in [N, n-N] \), \( f^t(x) \) is in some \( V(y_i) \) and

\[
f^t(F_n) = \bigcap_{k=-t}^{n-1-t} f^{-k}B_x(f^k(f^t(x))) \subset \bigcap_{k \leq N(y)} f^{-k}B_x(f^k(f^t(x))) \subset U(y_i).
\]

Now \( E(y_i) \) \((m(y_i), \frac{1}{2}\delta)\)-spans \( U(y_i) \), so it does \( f^t(F_n) \) also.

We shall define integers \( 0 = t_0 < t_1 < \cdots < t_r = n \). If \( n \leq N \), let \( r = 1 \) and \( t_1 = n \). If \( n > N \), take \( t_1 = N \) and pick \( V(y_i) \) containing \( f^{t_1}(x) \). Suppose we have chosen \( t_1, \ldots, t_k \) and \( y_1, \ldots, y_k \) (with \( t_k < n \)). If \( t_k > n - N \), then set \( r = k + 1 \) and \( t_r = N \). If \( t_k \leq n - N \), then set \( t_{k+1} = t_k + m(y_{ik}) < n \) and choose \( V(y_{ik+1}) \) containing \( f^{t_{k+1}}(x) \). Eventually this process stops.

Let \( K \) be a set which \((N, \frac{1}{2}\delta)\)-spans \( X \). Then \( K \) \((t_1 - t_0, \frac{1}{2}\delta)\)-spans \( F_n \) and also \((t_r - t_{r-1}, \frac{1}{2}\delta)\)-spans \( f^{t_r}(F_n) \). From the way the \( t_k \)'s and \( y_{ik} \)'s were chosen we see that, for \( 0 < k < r - 1 \), \( E(y_{ik}) \) \((t_{k+1} - t_k, \frac{1}{2}\delta)\)-spans \( f^{t_k}(F_n) \). Lemma 2.1 applies to give

\[
r_n(F_n, \delta) \leq \left( \text{card } K \right)^2 \prod_{0 < k < r - 1} \text{card } E(y_{ik}) \leq \left( \text{card } K \right)^2 \exp \left( \left( a + \beta \right) \left( n(y_{ik}) \right) \right) \leq \left( \text{card } K \right)^2 e^{(a + \beta)n}.
\]

**Corollary 2.3.** If \( f \) is a homeomorphism, then \( h_{\text{homeo}}^*(e) = h_f^*(e) \).

**Proof.** Let \( a = h_{\text{homeo}}^*(e) \). Fixing \( \beta, \delta \) the proposition gives us \( r_n(\Phi_e(x), \delta) \leq ce^{(a + \beta)n} \). Hence \( r_f(\Phi_e(x), \delta) \leq a + \beta \) and \( h(f, \Phi_e(x)) \leq a + \beta \). As \( \beta > 0 \) was arbitrary, \( h(f, \Phi_e(x)) \leq a \) and \( h_f^*(e) \leq a = h_{\text{homeo}}^*(e) \). The reverse inequality we noted before.

**Theorem 2.4.** \( h(f) \leq h(f, e) + h_f^*(e) \). In particular, \( h(f) = h(f, e) \) if \( e \) is an \( h \)-expansive constant for \( f \).

**Proof.** Let \( \delta > 0 \) and \( \beta > 0 \). Let \( E_n(n, \epsilon) \)-span \( X \), i.e.

\[
X = \bigcup_{x \in X} \bigcap_{k=0}^{n-1} f^{-k}B_x(f^k(x)).
\]

By Proposition 2.2 there is a constant \( c \) so that each of the sets in the above union can be \((n, \delta)\)-spanned by using at most \( c e^{(a + \beta)n} \) elements (where \( a = h_f^*(e) \)). Hence \( r_n(X, \delta) \leq \text{card } E_n ce^{(a + \beta)n} \leq r_n(X, \epsilon) ce^{(a + \beta)n} \). It follows that \( h(f, \delta) \leq h(f, \epsilon) + a + \beta \). Letting \( \beta \to 0 \), \( h(f, \delta) \leq h(f, \epsilon) + a \). Now letting \( \delta \to 0 \) we get our result.

If \( h_f^*(e) = 0 \), then \( h(f) \leq h(f, e) \). But \( h(f) \geq h(f, e) \) from the definition of \( h(f) \); hence \( h(f) = h(f, e) \).

**Corollary 2.5.** If \( h(f) = h(f, e) + h_f^*(e) \), then \( (1/n) \log r_n(X, \epsilon) \to h(f, e) \). In particular, if \( h_f^*(e) = 0 \), then \( (1/n) \log r_n(f, e) \to h(f) \).

**Proof.** Otherwise there is an increasing sequence of integers \( \{ n_k \} \) so that \( (1/n_k) \log r_{n_k}(X, \epsilon) \to b < h(f, e) \). Let \( a = h_f^*(e) \). Then \( h(f) > a + b \) and, for \( \gamma > 0 \)
small enough, \( h(f, \varphi) > a + b \). Choose \( \beta > 0 \) so that \( h(f, \varphi) > a + b + \beta \). For some \( c \), as in the proof of the theorem, we have \( r_{n_k}(X, \frac{1}{k}\varphi) \leq r_{n_k}(X, \epsilon)c \exp((a+\beta)n_k) \). So
\[
\limsup_{k \to \infty} \frac{1}{n_k} \log r_{n_k}(X, \frac{1}{k}\varphi) \leq b + a + \beta < h(f, \varphi).
\]
Choose \( R \) so that \((1/R) \log r_R(X, \frac{1}{k}\varphi) = \alpha < h(f, \varphi)\). This means there is an \((R, \frac{1}{k}\varphi)\)-spanning set for \( X \) with \( e^{R_\alpha} \) elements. By Lemma 2.1 (using \( t_k = kR \)) one gets \( r_{R_\alpha}(X, \varphi) \leq (e^{R_\alpha})^p \). For \( 0 \leq q \leq R \),
\[
r_{R_\alpha + q}(X, \varphi) \leq r_{R_\alpha + 1}(X, \varphi) \leq e^{R_\alpha + 1} \leq e^{R_\alpha + 1}.
\]
Hence
\[
h(f, \varphi) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(X, \varphi) \leq \limsup_{p \to \infty} \frac{(p + 1)R_\alpha}{Rp} = \alpha.
\]
But we chose \( \alpha < h(f, \varphi) \), a contradiction.

If \( \alpha^*(\epsilon) = 0 \), then \( h(f, \epsilon) \) by the theorem, and so the first statement applies.

REMARKS. For expansive homeomorphisms the second part of 2.4 was proved in [11] and the second part of 2.5 in [6]. In the original definition of topological entropy using open covers [1] certain limits existed whose analogues might not exist when one uses spanning sets. 2.5 is a technical result giving us conditions which insure that these limits exist. It has an application in counting periodic orbits of the Axiom A diffeomorphisms and flows of Smale (see [6]).

3. Measures. We continue to assume \( f: X \to X \) is continuous and \( X \) a compact metric space. \( \mu \) denotes a Borel measure on \( X \) with \( \mu(X) = 1 \) which is \( f \)-invariant, i.e. \( \mu(f^{-1}(E)) = \mu(E) \) for Borel sets \( E \).

We call \( A = \{A_1, \ldots, A_r\} \) a (finite) Borel partition provided the \( A_i \) are pairwise disjoint Borel sets whose union is \( X \). (Note that any finite \( \mu \)-measurable partition is \( \mu \)-equivalent to a Borel partition.) We write
\[
H_\mu(A) = \sum_{i=1}^{r} -\mu(A_i) \log \mu(A_i).
\]
If \( A, B \) are two Borel partitions, so is \( A \vee B = \{A \cap B : A \in A, B \in B\} \). Setting \( A^n = A \vee f^{-1}A \vee \cdots \vee f^{-(n-1)}A \), one defines the entropies (of Kolmogorov and Sinai, see [3])
\[
h_\mu(f, A) = \lim_{n \to \infty} \frac{1}{n} H_\mu(A^n) \quad \text{and} \quad h_\mu(f) = \sup_A h_\mu(f, A).
\]
An important device for calculating \( h_\mu(f) \) in some examples is Goodwyn's theorem [8]: \( h_\mu(f) \leq h(f) \). We shall use his ideas to prove a stronger statement for the case of \( X \) finite dimensional: \( h_\mu(f) \leq h_\mu(f, A) + h^*(\epsilon) \) where \( A \) is a finite Borel partition with \( \text{diam } A = \max \{\text{diam } A : A \in A\} \leq \epsilon \). This reduces to Goodwyn's theorem when we take \( A = \{X\} \) and \( \epsilon = \text{diam } X \). If \( \epsilon \) is an \( h \)-expansive constant for \( f \), it gives \( h_\mu(f) = h_\mu(f, A) \).
**Lemma 3.1.** If \( a_1, \ldots, a_n \geq 0 \) and \( s = \sum_{i=1}^{n} a_i \leq 1 \), then
\[
-\mu(a_i) \log \mu(a_i) \leq s(\log n - \log s).
\]

**Proof.** This is a well-known case of Jensen’s inequality [13, pp. 11–12].

**Lemma 3.2.** Let \( A_1, A_2, \ldots \) be finite Borel partitions of \( X \) with \( \text{diam } A_m \to 0 \). Then \( h_n(f, A_m) \to h_n(f) \).

**Proof.** This is a slight variation of a well-known result of Rohlin. Looking at 6.3, 8.6 and 9.5 of [16], one sees that our lemma is implied by the following statement:

Given a Borel partition \( \beta = \{B_1, \ldots, B_n\} \) and \( \epsilon > 0 \), then for large \( m \) we can find a partition \( \alpha = \{C_1, \ldots, C_n\} \) coarser than \( A_m \) (i.e. each \( C_i \) is the union of members of \( A_m \)) so that \( \mu(B_i \triangle C_i) < \epsilon \) for \( 1 \leq i \leq n \).

We now prove this statement. Since \( \mu \) is a Borel measure, one can choose compact sets \( K_i \subset B_i \) with \( \mu(B_i \setminus K_i) < \epsilon/n \). Choose \( \delta > 0 \) so that \( d(K_i, K_j) > \delta \) for \( i \neq j \) and suppose \( \text{diam } A_m < \delta \). Form \( \alpha = \{C_1, \ldots, C_n\} \) coarser than \( A_m \) by putting \( A \in A \) into

(a) \( C_i \) if \( A \cap K_i \neq \emptyset \) or
(b) \( C_k \) if \( A \cap K_i = \emptyset \) for all \( i \).

This makes sense, for if \( x \in A \cap K_i \) and \( y \in A \cap K_j \), then
\[
d(K_i, K_j) \leq d(x, y) \leq \text{diam } A < \delta
\]
and so \( i = j \).

Clearly \( C_i \supset K_i \). Hence \( \mu(B_i \setminus C_i) \leq \mu(B_i \setminus K_i) < \epsilon/n \). Since \( C_i \setminus B_i \subset \bigcup_{j \neq i} (B_j \setminus C_j) \),
\[
\mu(C_i \setminus B_i) < (n-1)\epsilon/n.
\]
Thus \( \mu(B_i \triangle C_i) < \epsilon \).

**Remark.** We used 3.2 in [4] but stated there (in the introduction) instead a stronger form—which we cannot prove.

Suppose now that \( \mathcal{B} \) is any finite cover of \( X \). For \( E \subset X \) let
\[
F(E, \mathcal{B}) = \{ B \in \mathcal{B} : B \cap E \neq \emptyset \}.
\]

We give a very slight modification of Proposition 2 of [8].

**Lemma 3.3.** Let \( \mathcal{B} \) be a finite cover of \( X \) by closed sets such that each point \( x \in X \) lies in at most \( m \) elements of \( \mathcal{B} \). There is a \( \delta > 0 \) so that \( \text{card } F(E, \mathcal{B}^n) \leq r_n(\delta, E)m^n \) for all \( E \subset X, n \geq 0 \).

**Proof.** For each \( x \in X \) choose a neighborhood \( U_x \) intersecting at most \( m \) elements of \( \mathcal{B} \). Let \( U_{x_1}, \ldots, U_{x_{m}} \) cover \( X \) and \( \delta > 0 \) be a Lebesgue number for this open cover. For each \( n \) let \( K_n \) be a set which \( (n, \delta) \)-spans \( E \) and has \( r_n(E, \delta) \) elements. For each \( \beta \in F(E, \mathcal{B}^n) \) pick \( p(\beta) \in E \cap \beta \) and \( q(\beta) \in K_n \) so that \( d(f^t(p(\beta)), f^t(q(\beta))) \leq \delta \) for \( 0 \leq t \leq n \). If \( \beta = \bigcap_{i=1}^{m} B_i \), \( B_i \in \mathcal{B} \), then \( f^t(p(\beta)) \in B_i(f^t(q(\beta))) \cap B_i \neq \emptyset \). Since \( B_i(f^t(q(\beta))) \) lies inside some \( U_{x_i} \), for a given \( q(\beta) \) there are at most \( m \) possibilities for \( B_i \). It follows that, for \( z \in K_n \), card \( q^{-1}(z) \leq m^n \). Hence card \( F(E, \mathcal{B}^n) \leq (\text{card } K_n)m^n \).
DEFINITION. For $A, B$ two Borel partitions let
\[ b(A, B) = \max_{A \in A} \text{card } F(A, B). \]

**Lemma 3.4.**
\[ h_{\mu}(f, A \vee B) \leq h_{\mu}(f, A) + \lim_{n \to \infty} \frac{1}{n} \log b(A^n, B^n). \]

**Proof.** Since $(A \vee B^n) = A^n \vee B^n$,
\[ H_{\mu}((A \vee B)^n) = \sum_{\alpha \in A^n} \sum_{\beta \in F(\alpha, B^n)} -\mu(\alpha \cap \beta) \log \mu(\alpha \cap \beta). \]
By Lemma 3.1
\[ \sum_{\beta \in F(\alpha, B^n)} -\mu(\alpha \cap \beta) \log \mu(\alpha \cap \beta) \leq \mu(\alpha) \log b(\alpha, B^n) - \log \mu(\alpha) \]
and so
\[ H_{\mu}((A \vee B)^n) \leq \log b(A^n, B^n) + H_{\mu}(A^n). \]
Divide by $n$ and let $n \to \infty$.

**Theorem 3.5.** Assume $X$ is finite dimensional. Let $A$ be a Borel partition of $X$ with diam $A \leq \varepsilon$. Then $h_{\mu}(f) = h_{\mu}(f, A) + h_{\mu}^f(\varepsilon)$ for any normalized $f$-invariant Borel measure $\mu$. If $e$ is an $h$-expansive constant for $f$, then $h_{\mu}(f) = h_{\mu}(f, A)$.

**Proof.** Say dim $X = m - 1$. Then for each $\gamma > 0$ we can find a finite closed cover $\mathfrak{B} = \mathfrak{B}(\gamma)$ with diameter $< \gamma$ and no point of $X$ in more than $m$ elements of $\mathfrak{B}$ (see [10]). Let $M$ be a fixed positive integer.

Let $B = \{B_1^\gamma, \ldots, B_s^\gamma\}$ be a Borel partition of $B$ where $B_i^\gamma \subset B_i$ and $\mathfrak{B} = \{B_1, \ldots, B_s\}$. We consider $f^M$ with respect to the partition $B \vee A_1^\psi$. If $\alpha \in (A_1^\psi)^{\mathfrak{B}}$ and $x \in \alpha$, then $\alpha \subset \bigcap_{n=0}^{M-1} f^{-n} B_1(f^n(x))$. Let $\delta > 0$ be as in Lemma 3.3 and $\beta > 0$ arbitrary. By 2.2 we have
\[ r_n(\alpha, \delta, f^M) \leq r_{Mn}(\alpha, \delta, f) \leq c \varepsilon (\alpha + \beta)n^M \]
where $a = h_{\mu}^f(\varepsilon)$. Using Lemma 3.3 we get (the first inequality is obvious)
\[ \text{card } F(\alpha, B_i^\gamma) \leq \text{card } F(\alpha, B_i) \leq c \varepsilon (\alpha + \beta)n^M \]
Applying Lemma 3.4,
\[ h(f^M, B \vee A_i^\gamma) \leq h(f^M, A_i^\gamma) + M(\alpha + \beta) + \log m. \]
Letting $\gamma \to 0$, diam $B \vee A_i^\psi$ \leq diam $B$ \leq diam $\mathfrak{B}(\gamma) \to 0$ and so by Lemma 3.2
\[ h_{\mu}(f^M) \leq h_{\mu}(f^M, A_i^\gamma) + M(\alpha + \beta) + \log m. \]
Now $h_{\mu}(f^M) = Mh_{\mu}(f)$ and $h_{\mu}(f^M, A_i^\gamma) = Mh_{\mu}(f, A)$. So
\[ h_{\mu}(f) \leq h_{\mu}(f, A) + a + \beta + \frac{1}{M} \log m. \]
Letting $\beta \to 0$ and then $M \to \infty$, we get our result.
REFERENCES


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