ENTROPY-EXPANSIVE MAPS

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Abstract. Let \( f : X \to X \) be a uniformly continuous map of a metric space. \( f \) is called \( h \)-expansive if there is an \( \varepsilon > 0 \) so that the set \( \Phi_{\varepsilon}(x) = \{ y : d(f^n(x), f^n(y)) \leq \varepsilon \} \) for all \( n \geq 0 \) has zero topological entropy for each \( x \in X \). For \( X \) compact, the topological entropy of such an \( f \) is equal to its estimate using \( \varepsilon : h(f) = h(f, \varepsilon) \). If \( X \) is compact finite dimensional and \( \mu \) an invariant Borel measure, then \( h_\mu(f) = h_\mu(f, A) \) for any finite measurable partition \( A \) of \( X \) into sets of diameter at most \( \varepsilon \). A number of examples are given. No diffeomorphism of a compact manifold is known to be not \( h \)-expansive.

Let \( f : X \to X \) be a homeomorphism of a metric space. For \( \varepsilon > 0 \) and \( x \in X \) define

\[ \Gamma_{\varepsilon}(x) = \{ y \in X : d(f^n(y), f^n(x)) \leq \varepsilon \} \]

\( f \) is called expansive if for some \( \varepsilon \) these sets are as small as possible, i.e. if \( \Gamma_{\varepsilon}(x) = \{ x \} \) for all \( x \). We are concerned with entropy and shall call \( f \) \( h \)-expansive provided that for some \( \varepsilon > 0 \) the \( \Gamma_{\varepsilon}(x) \) are negligible in terms of entropy, i.e. if the topological entropy \( h(f, \Gamma_{\varepsilon}(x)) = 0 \) for all \( x \).

We have two main results for \( h \)-expansive maps with \( X \) compact. First, the topological entropy satisfies \( h(f) = h(f, \varepsilon) \). Second, assuming \( X \) is finite dimensional, \( h_\mu(f) = h_\mu(f, A) \) when \( \mu \) is an \( f \)-invariant normalized Borel measure on \( X \) and \( A \) is a finite measurable partition of \( X \) into sets of diameter at most \( \varepsilon \). Both these results are well known in case \( f \) is expansive (see [11] and [14] respectively). Arov [2] noted that the second statement was true for \( f \) an endomorphism of a torus and \( \mu \) Haar measure when he calculated \( h_\mu(f) \) for this case (see Example 1.2).

1. Definitions and examples. We now review the definition of topological entropy given in [4]. For \( X \) compact this definition was given independently by Dinaburg [7]; is related to the \( \varepsilon \)-entropy of Kolmogorov [12]. Topological entropy was defined first in [1].

Let \( f : X \to X \) be uniformly continuous on the metric space \( X \). For \( E, F \subseteq X \) we say that \( E(n, \delta) \)-spans \( F \) (with respect to \( f \)), if for each \( y \in F \) there is an \( x \in E \) so that \( d(f^n(x), f^n(y)) \leq \delta \) for all \( 0 \leq k < n \). We let \( r_n(F, \delta) = r_n(F, \delta, f) \) denote the minimum cardinality of a set which \( (n, \delta) \)-spans \( F \). If \( K \) is compact, then the continuity of \( f \) guarantees \( r_n(K, \delta) < \infty \). For compact \( K \) we define

\[ \bar{r}_f(K, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(K, \delta) \]
and

\[ h(f, K) = \lim_{\delta \to 0} \bar{r}_{\delta}(K, \delta) \]

(notice that \( \bar{r}_{\delta}(K, \delta) \) increases as \( \delta \) decreases). Finally let \( h(f) = \sup_{K} h(f, K) \) where \( K \) varies over all compact subsets of \( X \). If \( X \) is compact, then \( h(f) = h(f, X) \) and we write \( h(f, \delta) = \bar{r}_{\delta}(X, \delta) \).

Let \( \Phi_{\varepsilon}(x) = \bigcap_{n \geq 0} f^{-n}B_{\varepsilon}(f^n(x)) \) and \( \bar{r}_{\varepsilon}(\delta) = \sup_{x \in X} h(f, \Phi_{\varepsilon}(x)) \). \( f \) is called \( h \)-expansive if \( \bar{r}_{\varepsilon}(\delta) = 0 \) for some \( \varepsilon > 0 \). In case \( f \) is a homeomorphism we set

\[ \Gamma_{\varepsilon}(x) = \bigcap_{n \in \mathbb{Z}} f^{-n}B_{\varepsilon}(f^n(x)) \]

and

\[ \bar{r}_{\varepsilon,\text{homeo}}(\delta) = \sup_{x \in X} h(f, \Gamma_{\varepsilon}(x)) \].

\textbf{Remark.} For \( f \) a homeomorphism, \( \Gamma_{\varepsilon}(x) \subseteq \Phi_{\varepsilon}(x) \) and so \( \bar{r}_{\varepsilon,\text{homeo}}(\delta) \leq \bar{r}_{\varepsilon}(\delta) \). The definition of \( h \)-expansiveness for homeomorphisms mentioned in the introduction, namely \( \bar{r}_{\varepsilon,\text{homeo}}(\delta) = 0 \), is actually equivalent to the above one in case \( X \) is compact.

For in 2.3 we prove \( \bar{r}_{\varepsilon}(\delta) = \bar{r}_{\varepsilon,\text{homeo}}(\delta) \) when \( X \) compact.

\textbf{Example 1.0.} Expansive maps.

\textbf{Example 1.1.} If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is linear and \( d \) comes from a norm, then \( h(f, e) = 0 \) for every \( e \).

\textbf{Proof.} \( f \) decomposes into a direct sum of linear maps \( f = f_1 \oplus f_2 : E_1 \oplus E_2 \to E_1 \oplus E_2 \) where \( f_1 \)'s eigenvalues have norm at most 1 and \( f_2 \)'s have norm greater than 1. If \( u \in E_2, u \neq 0 \), then \( d(f_2(u), 0) \to \infty \) as \( n \to \infty \). It follows that \( \Phi_{\varepsilon}(0) \subseteq E_1 \). But \( h(f|E_1) = h(f_1) = 0 \) by Theorem 15 of [4]. So \( h(f, \Phi_{\varepsilon}(0)) = 0 \). But \( \Phi_{\varepsilon}(x) = \Phi_{\varepsilon}(0) + x \) and \( h(f, K+x) = h(f, K) \) for any compact set \( K \).

\textbf{Example 1.2.} An endomorphism \( f \) of a Lie group \( G \) is \( h \)-expansive.

\textbf{Proof.} Here we use a right invariant metric \( d \). Then one checks \( \Phi_{\varepsilon}(x) = \Phi_{\varepsilon}(e)x \) and \( h(f, Kx) = h(f, K) \) for compact \( K \). So it is enough to see \( h(f, \Phi_{\varepsilon}(e)) = 0 \) for some \( e \).

Now

\[
\begin{array}{ccc}
T_eG & \xrightarrow{df} & T_eG \\
\downarrow & \exp & \downarrow \exp \\
G & \xrightarrow{f} & G
\end{array}
\]

commutes and \( \exp \) is a homeomorphism of a small neighborhood \( B_{\varepsilon}(0) \subseteq T_eG \) onto a neighborhood of some \( B_{\varepsilon}(e) \). Then \( \Phi_{\varepsilon}(e,f) \subseteq \exp \Phi_{\varepsilon}(0, df) \) and since \( f|\exp \Phi_{\varepsilon}(0, df) \) is a quotient of \( df|\Phi_{\varepsilon}(0, df) \) one has

\[ h(f, \Phi_{\varepsilon}(e,f)) \leq h(df, \Phi_{\varepsilon}(0, df)) = 0. \]

\textbf{Example 1.3.} Suppose \( f \) is \( h \)-expansive and \( T \) a uniformly continuous map so that \((T \cdot f)^n = T_n \cdot f^n \) for \( n \geq 0 \) where the \( T_n \) are isometries. Then \( T \cdot f \) is \( h \)-expansive.
**Proof.** One checks easily that \( \Phi_\varepsilon(x, T f) = \Phi_\varepsilon(x, f) \) and that a set which \((n, \delta)\)-spans some \( F \subset X \) with respect to \( f \) also \((n, \delta)\)-spans \( F \) with respect to \( T \cdot f \). It follows that

\[
h(T \cdot f, \Phi_\varepsilon(x, T \cdot f)) \leq h(f, \Phi_\varepsilon(x, f)) = 0.
\]

**Example 1.3.** Let \( G \) be a Lie group and for \( g, u \in G \) define \( L_g(u) = gu \) and \( R_g(u) = ug \). If \( f \) is an endomorphism of \( G \) and \( g \in G \), then the affine maps \( R_g \cdot f, L_g \cdot f \) and \( f \cdot L_g \) are all \( h \)-expansive.

**Proof.** If we set \( g_1 = g \) and \( g_{n+1} = f(g_n)g \), one sees that \( (R_g \cdot f)^n = R_{g_n} \cdot f^n \). As we use a right invariant metric, \( R_{g_n} \) is an isometry and 1.3 applies. Now \((L_g \cdot f)^n(u) = g(f(u)g^{-1})g = (R_g \cdot f^*)^n(u)\) where \( f^*(u) = g(f(u)g^{-1}) \) is an endomorphism. We leave \( f \cdot R_g \) and \( f \cdot L_g \) to the reader.

**Example 1.4.** Suppose \( H \) is a uniformly discrete subgroup of the Lie group \( G \), i.e. \( G/H \) is compact and \( \pi: G \to G/H \) given by \( \pi(x) = xH \) is a covering. For \( f \) an endomorphism of \( G \) with \( f(H) \subset H \) and \( g \in G \) define \( f^* \) on \( G/H \) by \( f^*(uH) = g(f(u)H) \). Then \( f^* \) is \( h \)-expansive.

**Proof.** For \( \delta \) small enough \( \pi \) maps \( B_\delta(x) \) isometrically onto \( B_\delta(xH) \) for every \( x \) and \( \pi \Phi_\varepsilon(x, L_g \cdot f) = \Phi_\varepsilon(xH, f^*) \). Then

\[
h(f^*, \Phi_\varepsilon(xH, f^*)) \leq h(L_g \cdot f, \Phi_\varepsilon(x, L_g \cdot f)) = 0.
\]

So \( f^* \) is \( h \)-expansive (see [4] for some more details).

**Example 1.5.** For the case of \( X \) compact define the nonwandering set

\[
\Omega(f) = \left\{ x \in X : \text{for every neighborhood } U \text{ of } x, U \cap \bigcup_{n>0} f^n(U) \neq \emptyset \right\}.
\]

Then \( f \Omega(f) \subset \Omega(f) \). If \( f|\Omega(f) \) is \( h \)-expansive, then so is \( f \). An example of this is one of Smale's Axiom A diffeomorphisms [15], where \( f|\Omega(f) \) is expansive.

**Proof.** Splice together the proof of Theorem 2.4 in [5] and that of 2.2 below for \( f|\Omega, a=0 \) and \( x \) staying in a neighborhood of \( \Omega \) up to time \( n \).

**Example 1.6.** Suppose \( \Phi = \{ \varphi_t : X \to X \}_{t \in \mathbb{R}} \) is a continuous flow on a compact metric space \( X \). Suppose also that there are \( \varepsilon > 0 \) and \( s > 0 \) so that

\[
\Gamma_\varepsilon(x, \Phi) = \{ y \in X : d(\varphi_t(y), \varphi_t(x)) \leq \varepsilon \text{ for all } t \in \mathbb{R} \} \subset \varphi_{t-s, s}(x) = \{ \varphi_s(x) : |r| \leq s \}.
\]

Then each \( \varphi_t \) is \( h \)-expansive.

**Proof.** For any \( t \in \mathbb{R} \) there is a \( \delta \) so that \( d(x, y) \leq \delta \) implies \( d(\varphi_t(x), \varphi_t(y)) \leq \varepsilon \) for all \( |r| \leq |t| \). Then

\[
\Gamma_\delta(x, \varphi_t) \subset \Gamma_\varepsilon(x, \Phi) \subset \varphi_{t-s, s}(x).
\]

For \( \beta > 0 \) choose \( \alpha > 0 \) so that for all \( x \in X \) and all \( |r| \leq \alpha \) we have \( d(x, \varphi_s(x)) \leq \delta \) (here we use \( X \) compact). Let \( K \) be a set of numbers so that every point in \([-s, s]\) is
within a of one of them. Then \(\{\varphi_u(x) : u \in K\}\) \((n, \delta)\)-spans \(\varphi_{\lfloor -s;3}(x)\) with respect to \(\varphi_t\). Hence

\[
r_n(\varphi_{\lfloor -s;3}(x), \delta, \varphi_t) \leq \text{card } K
\]

and

\[
h(\varphi_t, \Gamma(x, \varphi_t)) \leq h(\varphi_t, \varphi_{\lfloor -s;3}(x)) = 0.
\]

**Example 1.6*. Let \(\Phi = \{\varphi_t\}\) be one of Smale’s Axiom A flows [15]. Then \(\Phi|\Omega(\Phi)\) satisfies the condition of 1.6 [9]. By 1.5 and 1.6, each \(\varphi_t\) is \(h\)-expansive.

**Problem.** Find some differentiable maps which are not \(h\)-expansive.

2. Calculating topological entropy.

**Assumption.** For the remainder of the paper \(X\) is compact.

**Lemma 2.1.** Suppose \(0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = n\) and \(E_i(t_{i+1} - t_i, \alpha)\)-spans \(f^i(F)\) for \(0 \leq i < r\). Then

\[
r_n(F, 2\alpha) \leq \prod_{0 \leq i < r} \text{card } E_i.
\]

**Proof.** For \(x_i \in E_i\) write

\[
V(x_0, \ldots, x_{r-1}) = \{x \in F : d(f^{t_i}x, f^{t_i}x_i) \leq \alpha \text{ for } 0 \leq i < r\}.
\]

If \(x, y \in V(x_0, \ldots, x_{r-1})\), then by the triangle inequality \(d(f^s(x), f^s(y)) \leq 2\alpha\) for \(0 \leq s < n\). Since \(F = \bigcup V(x_0, \ldots, x_{r-1})\) we get an \((n, 2\alpha)\)-spanning set for \(F\) by taking one element from each nonempty \(V(x_0, \ldots, x_{r-1})\).

**Proposition 2.2.** Let \(a = h^*_f(\epsilon)\) or \(h^*_{\text{homeo}}(\epsilon)\) (in case \(f\) is a homeomorphism). Then for every \(\delta > 0\) and \(\beta > 0\) there is a \(c\) such that

\[
r_n\left(\bigcap_{k=0}^{n-1} f^{-k}B_\delta(f^k(x)), \delta\right) \leq ce^{(a + \beta)n}
\]

for all \(x \in X\).

**Proof.** We do the case where \(f\) is a homeomorphism and \(a = h^*_{\text{homeo}}(\epsilon)\). The case where \(a = h^*_f(\epsilon)\) is slightly simpler and we leave the necessary modifications to the reader.

For each \(y \in X\) pick \(m(y)\) so that \(a + \beta \geq (1/m(y)) \log \text{card } E(y)\) where \(E(y)\) is a set which \((m(y), \frac{1}{4}\delta)\)-spans \(\Gamma_\epsilon(y)\). Then \(U(y) = \{w \in X : \exists z \in E(y) \text{ such that } d(f^k(w), f^k(z)) < \frac{1}{8}\delta \text{ for all } 0 \leq k < m(y)\}\) is an open neighborhood of the compact set \(\Gamma_\epsilon(y)\). Let \(S_M = \bigcap_{m \leq M} f^{-1}B_\delta(f^m(y))\). Then \(S_0 \supset S_1 \supset \cdots\) is a decreasing chain of compact sets with intersection \(\Gamma_\epsilon(y)\); hence there is an integer \(N(y)\) so that \(S_{N(y)} \subset U(y)\). Consider the compact sets \(W_y = \bigcap_{m \leq N(y)} f^{-1}B_\delta(f^m(y))\). Then \(\bigcap_{y \in \mathcal{E}} W_y = W_\epsilon = S_{N(y)} \subset U(y)\); hence, \(W_\epsilon \subset U(y)\) for some \(\gamma > \epsilon\). Let \(V(y)\) be a neighborhood of \(y\) such that \(d(f^j(u), f^j(y)) < \gamma - \epsilon\) for \(|j| \leq N(y)\) when \(u \in V(y)\). Then \(B_\epsilon(f^j(u)) \subset B_\delta(f^j(y))\) and

\[
\bigcap_{|j| \leq N(y)} f^{-j}B_\delta(f^j(u)) \subset U(y).
\]
Let \( V(y_1), \ldots, V(y_s) \) cover the compact space \( X \) and
\[
N = \max \{ N(y_1), \ldots, N(y_s), m(y_1), \ldots, m(y_s) \} + 1.
\]

Consider now any \( x \in X \) and \( F_n = \bigcap_{k=0}^{n-1} f^{-k}B_{\epsilon}(f^k(x)). \) For any \( t \in [N, n-N] \), \( f^t(x) \) is in some \( V(y_i) \) and
\[
f^t(F_n) = \bigcap_{k=0}^{n-t-1} f^{-k}B_{\epsilon}(f^k(f^t(x))) \subset \bigcap_{|k| \leq N(y_i)} f^{-k}B_{\epsilon}(f^k(f^t(x))) \subset U(y_i).
\]

Now \( E(y_1) (m(y_1), \frac{1}{2}\delta) \)-spans \( U(y_1) \), so it does \( f^t(F_n) \) also.

We shall define integers \( 0 = t_0 < t_1 < \cdots < t_r = n \). If \( n \leq N \), let \( r = 1 \) and \( t_1 = n \). If \( n > N \), take \( t_1 = N \) and pick \( V(y_{i_1}) \) containing \( f^{t_1}(x) \). Suppose we have chosen \( t_1, \ldots, t_k \) and \( y_{i_1}, \ldots, y_{i_k} \) (with \( t_k < n \)). If \( t_k > n - N \), then set \( r = k + 1 \) and \( t_r = N \). If \( t_k \leq n - N \), then set \( t_{k+1} = t_k + m(y_{i_k}) < n \) and choose \( V(y_{i_{k+1}}) \) containing \( f^{t_{k+1}}(x) \). Eventually this process stops.

Let \( K \) be a set which \( (N,\frac{1}{2}\delta) \)-spans \( X \). Then \( K (t_1-t_0, \delta) \)-spans \( F_n \) and also \( (t_r-t_{r-1}, \frac{1}{2}\delta) \)-spans \( f^{t_r-t_{r-1}}(F_n) \). From the way the \( t_k \)'s and \( y_{i_k} \)'s were chosen we see that, for \( 0 < k < r - 1 \), \( E(y_{i_k}) (t_{k+1} - t_k, \delta) \)-spans \( f^{t_{k+1}}(F_n) \). Lemma 2.1 applies to give
\[
r_n(F_n, \delta) \leq (\text{card } K)^2 \prod_{0 \leq k < r-1} \text{card } E(y_{i_k})
\]
\[
\leq (\text{card } K)^2 \prod_{0 \leq k < r-1} \exp ((a + \beta)(n(y_{i_k}))) \leq (\text{card } K)^2 e^{(a + \beta)n}.
\]

**Corollary 2.3.** If \( f \) is a homeomorphism, then \( h^*_{\text{homeo}}(e) = h^*_f(e) \).

**Proof.** Let \( a = h^*_{\text{homeo}}(e) \). Fixing \( \beta, \delta \) the proposition gives us \( r_n(\Phi_x(x), \delta) \leq ce^{(a + \beta)n} \). Hence \( \tilde{r}_f(\Phi_x(x), \delta) \leq a + \beta \) and \( h(f, \Phi_x(x)) \leq a + \beta \). As \( \beta > 0 \) was arbitrary, \( h(f, \Phi_x(x)) \leq a \) and \( h^*_f(e) \leq a = h^*_{\text{homeo}}(e) \). The reverse inequality we noted before.

**Theorem 2.4.** \( h(f) \leq h(f, e) + h^*_f(e) \). In particular, \( h(f) = h(f, e) \) if \( e \) is an \( h \)-expansive constant for \( f \).

**Proof.** Let \( \delta > 0 \) and \( \beta > 0 \). Let \( E_n(n, e) \)-span \( X \), i.e.
\[
X = \bigcup_{x \in E_n} \bigcap_{k=0}^{n-1} f^{-k}B_{\epsilon}(f^k(x)).
\]

By Proposition 2.2 there is a constant \( c \) so that each of the sets in the above union can be \( (n, \delta) \)-spanned by using at most \( ce^{(a + \beta)n} \) elements (where \( a = h^*_f(e) \)). Hence \( r_n(X, \delta) \leq \text{card } E_n ce^{(a + \beta)n} \leq r_n(X, e) ce^{(a + \beta)n} \). It follows that \( h(f, \delta) \leq h(f, e) + a + \beta \). Letting \( \beta \to 0 \), \( h(f, \delta) \leq h(f, e) + a \). Now letting \( \delta \to 0 \) we get our result.

If \( h^*_f(e) = 0 \), then \( h(f) \leq h(f, e) \). But \( h(f) \geq h(f, e) \) from the definition of \( h(f) \); hence \( h(f) = h(f, e) \).

**Corollary 2.5.** If \( h(f) = h(f, e) + h^*_f(e) \), then \((1/n) \log r_n(X, e) \to h(f, e) \). In particular, if \( h^*_f(e) = 0 \), then \((1/n) \log r_n(f, e) \to h(f) \).

**Proof.** Otherwise there is an increasing sequence of integers \( \{n_k\} \) so that \((1/n_k) \log r_{n_k}(X, e) \to b < h(f, e) \). Let \( a = h^*_f(e) \). Then \( h(f) > a + b \) and, for \( \gamma > 0 \)
small enough, \( h(f, \gamma) > a + b \). Choose \( \beta > 0 \) so that \( h(f, \gamma) > a + b + \beta \). For some \( c \), as in the proof of the theorem, we have \( r_n(X, \frac{1}{2} \gamma) \leq r_n(X, \epsilon)c \exp((a + \beta)n) \). So
\[
\limsup_{k \to \infty} \frac{1}{n_k} \log r_{n_k}(X, \frac{1}{2} \gamma) \leq b + a + \beta < h(f, \gamma).
\]

Choose \( R \) so that \( (1/R) \log r_R(X, \frac{1}{2} \gamma) = a < h(f, \gamma) \). This means there is an \((R, \frac{1}{2} \gamma)\)-spanning set for \( X \) with \( e^{R \alpha} \) elements. By Lemma 2.1 (using \( t_k = kR \)) one gets \( r_{R \alpha}(X, \gamma) \leq (e^{R \alpha})^p \). For \( 0 \leq q \leq R \),
\[
r_{R \alpha + q}(X, \gamma) \leq r_{R(p + 1)}(X, \gamma) \leq e^{R \alpha(p + 1)}.
\]
Hence
\[
h(f, \gamma) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(X, \gamma) \leq \limsup_{p \to \infty} \frac{(p + 1)R \alpha}{Rp} = a.
\]

But we chose \( a < h(f, \gamma) \), a contradiction.

If \( h_\alpha(\epsilon) = 0 \), then \( h(f) = h(f, \epsilon) \) by the theorem, and so the first statement applies.

**Remarks.** For expansive homeomorphisms the second part of 2.4 was proved in [11] and the second part of 2.5 in [6]. In the original definition of topological entropy using open covers [1] certain limits existed whose analogues might not exist when one uses spanning sets. 2.5 is a technical result giving us conditions which insure that these limits exist. It has an application in counting periodic orbits of the Axiom A diffeomorphisms and flows of Smale (see [6]).

3. **Measures.** We continue to assume \( f: X \to X \) is continuous and \( X \) a compact metric space. \( \mu \) denotes a Borel measure on \( X \) with \( \mu(X) = 1 \) which is \( f \)-invariant, i.e. \( \mu(f^{-1}(E)) = \mu(E) \) for Borel sets \( E \).

We call \( A = \{A_1, \ldots, A_r\} \) a (finite) Borel partition provided the \( A_i \) are pairwise disjoint Borel sets whose union is \( X \). (Note that any finite \( \mu \)-measurable partition is \( \mu \)-equivalent to a Borel partition.) We write
\[
H_\mu(A) = \sum_{i=1}^{r} -\mu(A_i) \log \mu(A_i).
\]

If \( A, B \) are two Borel partitions, so is \( A \vee B = \{A \cap B : A \in A, B \in B\} \). Setting \( A^n = A^n = A \vee f^{-1} A \vee \cdots \vee f^{-(n-1)} A \), one defines the entropies (of Kolmogorov and Sinai, see [3])
\[
h_\mu(f, A) = \lim_{n \to \infty} \frac{1}{n} H_\mu(A^n) \quad \text{and} \quad h_\mu(f) = \sup_A h_\mu(f, A).
\]

An important device for calculating \( h_\mu(f) \) in some examples is Goodwyn's theorem [8]: \( h_\mu(f) \leq h(f) \). We shall use his ideas to prove a stronger statement for the case of \( X \) finite dimensional: \( h_\mu(f) \leq h_\alpha(f, A) + h_\alpha(\epsilon) \) where \( A \) is a finite Borel partition with \( \text{diam } A = \max \{ \text{diam } A : A \in A \} \leq \epsilon \). This reduces to Goodwyn's theorem when we take \( A = \{X\} \) and \( \epsilon = \text{diam } X \). If \( \epsilon \) is an \( h \)-expansive constant for \( f \), it gives \( h_\mu(f) = h_\mu(f, A) \).
Lemma 3.1. If \( a_1, \ldots, a_n \geq 0 \) and \( s = \sum_{i=1}^{n} a_i \leq 1 \), then
\[
\sum -\mu(a_i) \log \mu(a_i) \leq s(\log n - \log s).
\]

Proof. This is a well-known case of Jensen's inequality [13, pp. 11-12].

Lemma 3.2. Let \( A_1, A_2, \ldots \) be finite Borel partitions of \( X \) with \( \text{diam } A_m \to 0 \). Then \( h_\mu(f, A_m) \to h_\mu(f) \).

Proof. This is a slight variation of a well-known result of Rohlin. Looking at 6.3, 8.6 and 9.5 of [16], one sees that our lemma is implied by the following statement:

Given a Borel partition \( \beta = \{B_1, \ldots, B_n\} \) and \( \epsilon > 0 \), then for large \( m \) we can find a partition \( \alpha = \{C_1, \ldots, C_n\} \) coarser than \( A_m \) (i.e. each \( C_i \) is the union of members of \( A_m \)) so that \( \mu(B_i \triangle C_i) < \epsilon \) for \( 1 \leq i \leq n \).

We now prove this statement. Since \( \mu \) is a Borel measure, one can choose compact sets \( K_i \subset B_i \) with \( \mu(B_i \setminus K_i) < \epsilon/n \). Choose \( \delta > 0 \) so that \( d(K_i, K_j) > \delta \) for \( i \neq j \) and suppose \( \text{diam } A_m < \delta \). Form \( \alpha = \{C_1, \ldots, C_n\} \) coarser than \( A_m \) by putting \( A \in A \) into
(a) \( C_i \) if \( A \cap K_i \neq \emptyset \) or
(b) \( C_k \) if \( A \cap K_i = \emptyset \) for all \( i \).

This makes sense, for if \( x \in A \cap K_i \) and \( y \in A \cap K_j \), then
\[
d(K_i, K_j) \leq d(x, y) \leq \text{diam } A < \delta
\]
and so \( i = j \).

Clearly \( C_i \supseteq K_i \). Hence \( \mu(B_i \setminus C_i) \leq \mu(B_i \setminus K_i) < \epsilon/n \). Since \( C_i \setminus B_i \subset \bigcup_{j \neq i} (B_j \setminus C_j) \), \( \mu(C_i \setminus B_i) < (n - 1)\epsilon/n \). Thus \( \mu(B_i \setminus C_i) < \epsilon \).

Remark. We used 3.2 in [4] but stated there (in the introduction) instead a stronger form—which we cannot prove.

Suppose now that \( \mathcal{B} \) is any finite cover of \( X \). For \( E \subseteq X \) let
\[
F(E, \mathcal{B}) = \{B \in \mathcal{B} : B \cap E \neq \emptyset \}.
\]

We give a very slight modification of Proposition 2 of [8].

Lemma 3.3. Let \( \mathcal{B} \) be a finite cover of \( X \) by closed sets such that each point \( x \in X \) lies in at most \( m \) elements of \( \mathcal{B} \). There is a \( \delta > 0 \) so that \( \text{card } F(E, \mathcal{B}^n) \leq r_n(\delta, E)n^m \) for all \( E \subseteq X, n \geq 0 \).

Proof. For each \( x \in X \) choose a neighborhood \( U_x \) intersecting at most \( m \) elements of \( \mathcal{B} \). Let \( U_{x_1}, \ldots, U_{x_n} \) cover \( X \) and \( \delta > 0 \) be a Lebesgue number for this open cover. For each \( n \) let \( K_n \) be a set which \((n, \delta)\)-spans \( E \) and has \( r_n(\delta, E)n^m \) elements. For each \( \beta \in F(E, \mathcal{B}^n) \) pick \( p(\beta) \in E \cap \beta \) and \( q(\beta) \in K_n \) so that \( d(f^t(q(\beta)), f^t(p(\beta))) \leq \delta \) for \( 0 \leq t \leq n \). If \( \beta = \bigcap_{t=0}^{n-1} B_{t+1} \), \( B_n \in \mathcal{B} \), then \( f^t(p(\beta)) \in B_s(f^t(q(\beta))) \cap B_n \neq \emptyset \).

Since \( B_s(f^t(q(\beta))) \) lies inside some \( U_{x_p} \), for a given \( q(\beta) \) there are at most \( m \) possibilities for \( B_k \). It follows that, for \( z \in K_n \), \( z \neq q^{-1}(z) \leq m^n \). Hence card \( F(E, \mathcal{B}^n) \leq (\text{card } K_n)n^m \).
Definition. For $A, B$ two Borel partitions let
\[ b(A, B) = \max_{A \in \mathcal{A}} \text{card } F(A, B). \]

Lemma 3.4. \[ h_u(f, A \vee B) \leq h_u(f, A) + \liminf_{n \to \infty} \frac{1}{n} \log b(A^n, B^n). \]

Proof. Since $(A \vee B)^n = A^n \vee B^n$,
\[ H_u((A \vee B)^n) = \sum_{\alpha \in A^n} \sum_{\beta \in F(\alpha, B^n)} -\mu(\alpha \cap \beta) \log \mu(\alpha \cap \beta). \]
By Lemma 3.1
\[ \sum_{\beta \in F(\alpha, B^n)} -\mu(\alpha \cap \beta) \log \mu(\alpha \cap \beta) \leq \mu(\alpha)(\log b(\alpha, B^n) - \log \mu(\alpha)) \]
and so
\[ H_u((A \vee B)^n) \leq \log b(A^n, B^n) + h_u(A^n). \]
Divide by $n$ and let $n \to \infty$.

Theorem 3.5. Assume $X$ is finite dimensional. Let $A$ be a Borel partition of $X$ with diameter $A \leq \varepsilon$. Then $h_u(f) \leq h_u(f, A) + h^*_u(\varepsilon)$ for any normalized $f$-invariant Borel measure $\mu$. If $\varepsilon$ is an $h$-expansive constant for $f$, then $h_u(f) = h_u(f, A)$.

Proof. Say $\dim X = m - 1$. Then for each $\gamma > 0$ we can find a finite closed cover $\mathcal{B} = \mathcal{B}(\gamma)$ with diameter $\mathcal{B} \leq \gamma$ and no point of $X$ in more than $m$ elements of $\mathcal{B}$ (see [10]). Let $M$ be a fixed positive integer.
Let $B = \{B_1^*, \ldots, B_p^*\}$ be a Borel partition of $X$ where $B_1^* \subset B_1$ and $\mathcal{B} = \{B_1, \ldots, B_p\}$. We consider $f^M$ with respect to the partition $B \vee A_f^M$. If $\alpha \in (A_f^M)^nM$ and $x \in \alpha$, then $x \in \bigcap_{n=0}^{M-1} f^{-n}B_x(f^n(x))$. Let $\delta > 0$ be as in Lemma 3.3 and $\beta > 0$ arbitrary. By 2.2 we have
\[ r_n(\alpha, \delta, f^M) \leq r_M(\alpha, \delta, f) \leq ce^{(a + \beta)nM} \]
where $a = h^*_u(\varepsilon)$. Using Lemma 3.3 we get (the first inequality is obvious)
\[ \text{card } F(\alpha, B_f^M) \leq \text{card } F(\alpha, \mathcal{B}_f^M) \leq ce^{(a + \beta)nM}M^n \]
Applying Lemma 3.4,
\[ h(f^M, B \vee A_f^M) \leq h_u(f^M, A_f^M) + M(a + \beta) + \log m. \]
Letting $\gamma \to 0$, diam $B \vee A_f^M \leq$ diam $B \leq$ diam $\mathcal{B}(\gamma) \to 0$ and so by Lemma 3.2
\[ h_u(f^M) \leq h_u(f^M, A_f^M) + M(a + \beta) + \log m. \]
Now $h_u(f^M) = Mh_u(f)$ and $h_u(f^M, A_f^M) = Mh_u(f, A)$. So
\[ h_u(f) \leq h_u(f, A) + a + \beta + \frac{1}{M} \log m. \]
Letting $\beta \to 0$ and then $M \to \infty$, we get our result.
REFERENCES