SLICES OF MAPS AND LEBESGUE AREA(1)

BY

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Abstract. For a large class of $k$ dimensional surfaces, $S$, it is shown that the Lebesgue area of $S$ can be essentially expressed in terms of an integral of the $k-1$ area of a family, $F$, of $k-1$ dimensional surfaces that cover $S$. The family $F$ is regarded as being composed of the slices of $F$. The definition of the $k-1$ area of a surface restricted to one of its slices is formulated in terms of the theory developed by H. Federer, [F3].

1. Introduction. There are many results in geometric measure theory that yield an inequality between the $k$ dimensional measure of a set $R \subseteq E^n$ and the integral of the $k-1$ dimensional measure of the intersection of $R$ with the level surfaces of certain real-valued functions defined on $E^n$. There are similar results in the theory of Lebesgue area.

In this paper we will show that the Lebesgue area of a large class of $k$ dimensional surfaces, $S$, can be essentially expressed in terms of an integral of the $k-1$ area of a family, $F$, of $k-1$ dimensional surfaces that cover $S$. An analogous result is obtained for the $k$ dimensional Hausdorff measure of a Hausdorff $k$-rectifiable set. We regard the family $F$ as being composed of the slices of the surface and one of the essential parts of our problem is to determine the appropriate definition of the $k-1$ area of a surface restricted to one of its slices. The definition that we employ is expressed in terms of the theory developed by H. Federer, [F3]. In case $k=2$, there are two other definitions considered and we show that they lead to the same results. One definition was created by L. Cesari [C] and it employs the concepts of the Carathéodory theory of prime ends. The other definition depends upon the notion of the length of a light mapping that was introduced in [F2].

For 2 dimensional surfaces, R. Rishel [RL] and R. Fullerton [FN] obtained results that are in the same spirit as ours by employing Cesari's definition of generalized length. However, Rishel's definition of the slice of a surface is different from ours and Fullerton's premature death precluded him from developing his results beyond polyhedral parametric surfaces and continuously differentiable non-parametric surfaces.

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(1) This work was supported in part by NSF grant GP 19694.

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2. Preliminaries and definitions. Euclidean $n$-space will be denoted by $E^n$. Lebesgue measure on $E^n$ will be denoted by $L_n$ and $H^k$ will stand for $k$ dimensional Hausdorff measure. If $A \subseteq E^n$, then the measure $H^k \lhd A$ is defined by $H^k \lhd A(E) = H^k(A \cap E)$. A set $R \subseteq E^n$ is called Hausdorff $k$-rectifiable if there is a Lipschitzian function $f$ on $E^k$ to $E^n$ such that

$$H^k(R - \text{range } f) = 0.$$ 

We refer the reader to [F4] for a thorough investigation of the properties possessed by Hausdorff $k$-rectifiable sets. Many of these properties will be employed below and the following will be especially useful. If $R \subseteq E^n$ is a Hausdorff $k$-rectifiable set, $g: R \to E^m$ is Lipschitzian, and $j = \min \{k, m\}$, then $g$ possesses an $(H^k \lhd R, k)$ approximate $j$ dimensional Jacobian $H^j \lhd (R - \text{range } g)$ almost everywhere, [F4, 3.2.19]. Denote this Jacobian by $ap \, J_j g$ and in case $k > m = 1$, we will use the notation $ap \, |\nabla g|$. The following theorem is of particular importance, [F4, 3.2.20, 3.2.22].

2.1. Theorem. If $R \subseteq E^n$ is Hausdorff $k$-rectifiable, $g: R \to E^m$ is Lipschitzian, and $j = \min \{k, m\}$, then

$$\int_R ap \, J_j g \, dH^k = \int_{E^m} H^a[g^{-1}(y) \cap R] \, dH^j(y)$$

where $a = \max \{0, k - m\}$.

Finally, we recall the definition of Hausdorff $k$ dimensional density

$$\Theta^k(H^k \lhd R, y) = \lim_{r \to 0} \alpha(k)^{-1} r^{-k} H^k[R \cap B(y, r)]$$

where $B(y, r)$ denotes the open $n$-ball of radius $r$ with center at $y$ and $\alpha(k)$ is the volume of the unit $k$-ball in $E^k$. If $R$ is Hausdorff $k$-rectifiable, then

(1)

$$\Theta^k(H^k \lhd R, y) = 1$$

for $H^k$ almost all $y \in R$.

We will now show that the Hausdorff $k$ measure of a rectifiable set can be expressed in terms of the integral of the $k - 1$ measure of its slices. The proof of this result will establish the method that is basic in the demonstration of the main theorem, 3.3, that appears below.

2.2. Theorem. Suppose $R \subseteq E^n$ is a Hausdorff $k$-rectifiable set. Then

$$H^k(R) = \sup \left\{ \int_{E^1} H^{k-1}[u^{-1}(r) \cap R] \, dL_1(r) \right\}$$

where the supremum is taken over all Lipschitz functions $u: E^n \to E^1$ that have Lipschitz constant 1.

Proof. If $u$ is such a function, then $ap \, |\nabla u|$ exists and is dominated by 1 at $H^k \lhd R$ almost all points. Therefore, by appealing to Theorem 2.1 it is clear that the above supremum is no more than $H^k(R)$. 

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In order to establish the opposite inequality, it will suffice to show that for every 
\( \epsilon > 0 \), there is a function \( u: E^n \to E^1 \) with Lipschitz constant 1 which satisfies

\[
\int_R \text{ap} |\nabla u| \, dH^k > (1 - \epsilon)H^k(R).
\]

To this end, first recall that \( R \) has an \( H^k \) approximate tangent \( k \) plane, \( P(y) \), at \( H^k \) almost all \( y \in R \), [F4, 3.3.18]. Moreover, there exists a \( H^k \subset R \) measurable function \( \alpha \) defined on \( R \) with values in the space of Grassmann simple \( k \) vectors of unit norm. In addition, for \( H^k \) almost all \( y \in R \), \( P(y) \) is the \( k \) dimensional vector-subspace of \( E^n \) associated with \( \alpha(y) \), [F4, 3.2.25]. Consequently, from Lusin's Theorem, the density theorem [F4, 2.9.11], and (1), it follows that at \( H^k \) almost all \( y \in R \) there is a set \( A \subset R \) such that

\[
\begin{align*}
(1) & \quad \Theta^k(H^k \subset A, y) = 1 \quad \text{and} \\
(2) & \quad \alpha \text{ is continuous at } y \text{ relative to } A.
\end{align*}
\]

Consider a point \( y \in R \) where \( P(y) \) exists and let \( N(y) \) be the \( n-k \) plane passing through \( y \) that is perpendicular to \( P(y) \). For every real number \( \beta > 0 \), let

\[
C(\beta) = \{ x : \text{dist} [x, N(y)] > \beta \text{ dist} [x, P(y)] \}.
\]

It follows from the definition of the \( H^k \) approximate tangent \( k \) plane, \( P(y) \), that

\[
\lim_{r \to 0} r^{-k}H^k[R \cap B(y, r) - C(\beta)] = 0
\]

for every \( \beta > 0 \).

Choose \( y \in R \) such that \( P(y) \) exists, and where (1) and (3) hold. This will be true at \( H^k \) almost all \( y \in R \). Let \( \epsilon > 0 \). Choose \( \zeta(\epsilon) = \zeta \) (which will be determined later) and select \( \beta \) large enough to ensure that

\[
|\pi_y([x - y]/|z - y|)| > 1 - \zeta
\]

whenever \( z \in C(\beta) \). Here \( \pi_y: E^n \to P(y) \) denotes the orthogonal projection. From (4), (3), and the equality

\[
\Theta^k(H^k \subset R, y) = \Theta^k(H^k \subset A, y) = 1
\]

follows the existence of \( r^*(y, \epsilon) = r^* \) such that, for \( 0 < r < r^* \),

\[
H^k[A \cap B(y, r) \cap C(\beta)] > (1 - \epsilon)^{1/2}H^k[R \cap B(y, r)] \quad \text{and} \\
|\alpha(x) - \alpha(y)| < \zeta \quad \text{whenever } x \in A \cap B(y, r).
\]

Select \( 0 < r < r^* \) and define \( u(x) = \text{dist} [x, E^n - B(y, r)] \). Since \( |\nabla u(x)| = 1 \) in \( B(y, r) \), (5) implies that

\[
|\pi_y[\nabla u(x)]| > 1 - \zeta \quad \text{for } x \in B(y, r) \cap C(\beta).
\]
If $\xi$ is chosen sufficiently small, then from (6) and (7) clearly follows \(|\pi_x[\nabla u(x)]| > (1-\varepsilon)^{1/2}\) for $x \in A \cap B(y, r) \cap C(\beta)$ where $P(x)$ exists. Therefore,

$$\int_{R \cap B(y, r)} ap |\nabla u| \, dH^k \geq (1-\varepsilon)^{1/2} H^k[A \cap B(y, r) \cap C(\beta)]$$

and thus (6) leads to

$$\int_{R \cap B(y, r)} ap |\nabla u| \, dH^k \geq (1-\varepsilon)H^k[R \cap B(y, r)]$$

whenever $0 < r < r^*$.

For $H^k$ almost every $y \in R$, consider the family of all $n$-balls $B(y, r)$ where $0 < r < r^*(y, \varepsilon)$. Then by a covering theorem due to Besicovitch and A. P. Morse that has been generalized in [F4, 2.8.15], there exist a countable number of $n$-balls, $B_1, B_2, \ldots$, whose closures are disjoint such that $H^k \subset R(E^n - \bigcup_{i=1}^{\infty} B_i) = 0$. With each $n$-ball $B_i$ is associated the function $u_i(x) = \text{dist } [x, E^n - B_i]$ so that (8) is satisfied. Letting $u(x) = \sum_{i=1}^{\infty} u_i(x)$, $x \in E^n$, it is clear that $u$ has Lipschitz constant 1 and that

$$(1-\varepsilon)H^k(R) = (1-\varepsilon)H^k \subset R(E^n) = (1-\varepsilon) \sum_{i=1}^{\infty} H^k \subset R(B_i)$$

$$\leq \sum_{i=1}^{\infty} \int_{R \setminus B_i} ap |\nabla u_i| \, dH^k = \int_{R} ap |\nabla u| \, dH^k.$$ 

According to (2), this concludes the proof.

3. Slices and Lebesgue area. In this section we will establish a result concerning Lebesgue area that is analogous to Theorem 2.2. Throughout this section we will consider a continuous map $f: X \rightarrow E^n$ of finite Lebesgue area, where $X$ is a $k$ dimensional smooth manifold, $k \leq n$. We will also assume that $k = 2$ or $H^{k+1}[f(X)] = 0$.

Our treatment relies heavily on the work of Federer [F3] and the notation and results of that paper will be employed here without change. Thus, the monotone-light factorization $f = l_{\infty} \circ m_f$ will be considered where $l_{\infty}$ is defined on the middle space, $M_f$. Moreover, there is a unique current-valued measure $\mu$ over $M_f$ whose total variation, $||\mu||$, is equal to the Lebesgue area of $f$. If $T$ is a current, then $M(T)$ is the mass of $T$ and $F(T)$ denotes the flat norm of $T$. The boundary of $T$ is denoted by $\partial T$. Define a measure $\nu$ over $E^n$ by $\nu(A) = ||\mu||(l_{\infty}^{-1}(A))$.

One of the main results of [F3] states that there is an integer valued function, $K$, such that

$$\mathcal{L}(f) = \int_{E^n} K(y) \, dH^k(y)$$

where $\mathcal{L}(f)$ is the Lebesgue area of $f$. For $y \in E^n$ and $r > 0$, let $Z(r)$ be the family of components of $l_{\infty}^{-1}[B(y, r)]$. For $\nu$ almost all $y \in E^n$, there are a finite number of
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"essential" points, z, of $l^{-1}_f(y)$ with the property that if r is sufficiently small, then with each $V \in Z(r)$ that contains an essential $z$ is associated an integer $m(V)$ such that

$$K(y) = \sum_{V \in Z(r)} |m(V)|.$$ 

Moreover, there is an oriented $k$ dimensional plane, $P(y)$, passing through $y$ such that, for each such $V \in Z(r)$,

$$\lim_{r \to 0} r^{-k} F[\mu(V) - m(V) \cdot P(y) \cap B(y, r)] = 0.$$ 

3.1. Definition. Let $u: E^n \to E^1$ be Lipschitz and let $Z(r)$ be the set of components of $M_f \cap \{z : u \circ l_f(z) < r\}$. Define

$$\lambda(f; u, r) = \sum_{V \in Z(r)} M[\partial \mu(V)].$$

As a function of $r$, $\lambda(f; u, r)$ is $L_1$ measurable; indeed,

$$\liminf_{t \to r^-} \lambda(f; u, t) \geq \lambda(f; u, r).$$

To see this, choose $r > 0$ and let $V_1, V_2, \ldots$ be the components that constitute $Z(r)$. For $t > r$ let $W_t(t)$ be the union of those components of $Z(t)$ that are contained in $V_i$, $i=1, 2, \ldots$. The sets $W_t(t)$ are nested and increase with $t$. Therefore, since $\bigcup_{t < r} W_t(t) = V_i$, it follows that, as currents, $\mu[W_t(t)] \to \mu(V_i)$ weakly as $t \to r^-$. From the facts that $\partial$ is continuous and mass is lower semicontinuous with respect to weak convergence follows

$$\liminf_{t \to r^-} M[\partial \mu(W_t(t))] \geq M[\partial \mu(V_i)], \quad i = 1, 2, \ldots.$$ 

Hence,

$$\liminf_{t \to r^-} \lambda(f; u, t) \geq \liminf_{t \to r^-} \sum_{i=1}^{\infty} M[\partial \mu(W_t(t))] \geq \sum_{i=1}^{\infty} M[\partial \mu(V_i)] = \lambda(f; u, r).$$

3.2. Lemma. If $f: X \to E^n$ and $u: E^n \to E^1$ has Lipschitz constant $N$, then

$$\int_{-\infty}^{\infty} \lambda(f; u, r) \, dL_1(r) \leq N \mathcal{L}(f).$$

Proof. Let

$$\gamma(r) = \|\mu\|((z : u \circ l_f(z) < r)).$$

Observe, for $L_1$ almost all $r$, that $\gamma'(r) < \infty$ and for each $V \in Z(r)$ that $\mu(V)$ is an integral current, [F3, 3.4]. Now by applying [FF, 3.9] to $T=\mu(V)$, the proof proceeds as in [F3, 3.2] and we obtain

$$M[\partial \mu(V)] \leq N\liminf_{h \to 0^+} h^{-1}\|\mu(V)\|((y : r-h \leq u(y) < r)).$$
From this follows

$$
\lambda(f; u, r) \leq N \liminf_{h \to 0^+} h^{-1} \sum_{V \in Z(r)} \|\mu(V)\| \{y : r-h \leq u(y) < r\}
$$

$$
\leq N \liminf_{h \to 0^+} h^{-1} \|\mu(V)\| \{z : r-h \leq u \circ l_I(z) < r\} \leq Ny'(r).
$$

Our purpose now is to show that the supremum of the left side of 3.2 over all functions \( u \) with \( N=1 \) equals the Lebesgue area of \( f \).

To this end let \( R \) be the set of those \( y \in E^n \) for which (10) and (11) hold and for which \( 0 < K(y) < \infty \). Then \( R \) is a Hausdorff \( k \)-rectifiable set and, without loss of generality, we may assume that

$$
(12) \quad \Theta^k(H^k \mathcal{L} R, y) = 1, \quad y \in R.
$$

Choose \( y \in R \) and to simplify notation, take \( y=0 \). Assume \( r \) to be taken small enough so that (10) holds. Select an essential \( z \in I^{-1}(0) \) and consider those \( V \in Z(r) \) that contain \( z \). Let \( T_r = \mu(V) \) and \( P = \partial B(0, r) \). Let \( \pi : E^k \to P \) be the orthogonal projection and define \( h_r : P \to P \) by \( h_r(x) = r^{-1} \cdot x \). The flat norm of a current is not increased under a projection and, for \( k \)-dimensional currents in \( E^k \), the flat norm and the mass norm agree. Therefore, it follows from (11) that

$$
\lim_{r \to 0} r^{-k} M[(h_r \circ \pi)_#(T_r) - m(V) \cdot B(0, r) \cap P] = 0.
$$

This implies

$$
\lim_{r \to 0} M[(h_r \circ \pi)_#(T_r) - m(V) \cdot B(0, 1) \cap P] = 0.
$$

Again, from the continuity of \( \partial \) and the lower semicontinuity of mass, for \( \eta > 0 \) and all sufficiently small \( r \),

$$
M[(h_r \circ \partial)_#(\partial T_r)] \geq |m(V)| M[\partial(B(0, 1) \cap P)] - \eta.
$$

Thus, for all small \( r \),

$$
M(\partial T_r) \geq M[(h_r \circ \partial)_#(\partial T_r)] \geq [m(V) - \eta \alpha(k-1)^{-1}] M[\partial(B(0, r) \cap P)].
$$

Choose \( \delta > 0 \). It follows from (10) and (13) that, for \( y \in R \), there is \( r^* = r^*(y, \delta) \) such that if \( B = B(y, r) \), \( 0 < r < r^* \), and \( u(x) = -\text{dist}(x, E^n - B) \), then

$$
\lambda(f; u, -t) = \sum_{V \in Z(-t)} M[\partial \mu(V)] \geq [K(y) - \delta] M[\partial(B(y, t) \cap P)]
$$

where \( Z(-t) \) is the set of components of \( \{z : u \circ l_I(z) < -t\} \) and \( 0 < t < r \). Therefore,

$$
\int_{-\infty}^{\infty} \lambda(f; u, t) \, dL_1(t) \geq [K(y) - \delta] \alpha(k)r^k.
$$

Define a measure \( \zeta \) over \( E^n \) by

$$
\zeta(E) = \int_{E^n \cap R} K(y) \, dH^k(y).
$$
for every Borel set $E$. Then

$$\zeta(E^n) = \mathcal{L}(f);$$

indeed, $\zeta = \nu$, but we will not use this fact. Appealing to [F4, 2.9.8] we have for $H^k$ almost all $y \in R$, that

$$\lim_{r \to 0} \frac{\zeta(B(y, r))}{H^k \cap R(B(y, r))} = K(y).$$

For $\varepsilon > 0$ and $y \in R$, it follows from (12), (14), and (16) that there exists $r^*(y, \varepsilon) = r^*$ such that

$$\int_{-\infty}^{\infty} \lambda(f; u, r) \, dL_1(t) \geq (1 - \varepsilon)\zeta(B(y, r))$$

whenever $0 < r < r^*$. Consider the family of $n$-balls, $B(y, r), y \in R$ and $0 < r < r^*(y, \varepsilon)$. Appealing to the covering theorem [F4, 2.8.15], there exist balls $B_1, B_2, \ldots$ whose closures are disjoint and that have the property

$$\zeta \left( E^n - \bigcup_{i=1}^{\infty} B_i \right) = 0.$$ 

As in the proof of 2.2 we define

$$u(x) = \sum_{i=1}^{\infty} u_i(x), \quad x \in E^n,$$

where $u_i(x) = - \text{dist} (x, E^n - B_i), i = 1, 2, \ldots$. Then $u$ has Lipschitz constant 1 and $\lambda(f; u, r) = \sum_{i=1}^{\infty} \lambda(f; u_i, r), r \in E^1$. Therefore, (17) and (18) yield

$$\int_{-\infty}^{\infty} \lambda(f; u, r) \, dL_1(t) = \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} \lambda(f; u_i, r) \, dL_1(t)$$

$$= \sum_{i=1}^{\infty} (1 - \varepsilon)\zeta(B_i) = (1 - \varepsilon)\zeta(E^n) = (1 - \varepsilon)\mathcal{L}(f).$$

Thus, the following theorem has been established.

3.3. Theorem. Suppose $f: X \to E^n$ has finite Lebesgue area and suppose $k = 2$ or $H^{k+1}[f(X)] = 0$. Then $\sup \{ \int_{-\infty}^{\infty} \lambda(f; u, r) \, dL_1(r) \} = \mathcal{L}(f)$ where the supremum is taken over all functions $u: E^n \to E^1$ that have Lipschitz constant 1.

4. Mappings from a 2-cell. The results of [F3] are also valid in case $X$ is a manifold with boundary, [G], [M], and therefore Theorem 3.3 also holds in this case. In this section we consider the special situation when $X$ is the unit square, $Q$, in $E^2$ for then there are two other reasonable definitions for the 1-area or “length” of $f$ restricted to a slice determined by $u$. In order for 3.3 to remain valid, Definition 3.1 of $\lambda(f; u, r)$ is modified so that only those components $V \in Z(r)$ are considered for which closure $V \cap m_t(bdry Q) = 0$. Let $C(r)$ be the set of such components.

4.1. Definition. If $U$ is a subset of a separable metric space, let

$$\delta(U) = \text{bdry} U \cap \{ z : \text{dimension}(\text{bdry} U, z) > 0 \}.$$
4.2. Definition. For $V \in C(r)$, let

$$|V| = \int_{\mathbb{R}^n} N[I, \delta(V), y] \, dH^1(y)$$

where $N[I, \delta(V), y]$ denotes the number of points (possibly $\infty$) in $l_{I^{-1}}(y) \cap \delta(V)$. Define

$$\rho(f; u, r) = \sum_{V \in C(r)} |V|.$$ 

Similarly, for $V \in C(r)$, let $\|V\|$ denote the length of $f$ restricted to the boundary of $m_{I^{-1}}(V)$ as defined by Cesari in [C, 20.2]. Define

$$\sigma(f; u, r) = \sum_{V \in C(r)} \|V\|.$$ 

Finally, define

$$\mathcal{C}(f) = \sup \left\{ \int_{-\infty}^{\infty} \sigma(f; u, r) \, dL_0(r) \right\}$$

where the supremum is taken over all functions $u: \mathbb{E}^n \to \mathbb{E}^1$ that have Lipschitz constant 1. By replacing $\sigma(f; u, r)$ by $\rho(f; u, r)$, we define $\mathcal{D}(f)$ in a similar manner.

The following result is an immediate consequence of [C, 20.5] and [RL, 1.6].

4.2. Theorem. If $f: Q \to \mathbb{E}^n$ is continuous, then $\mathcal{L}(f) \leq \mathcal{C}(f) \leq \mathcal{D}(f)$.

In order to prove that $\mathcal{D}(f) \leq \mathcal{L}(f)$, we first observe that if $f$ and $g$ are Fréchet equivalent maps of $Q$ into $\mathbb{E}^n$, then $\mathcal{D}(f) = \mathcal{D}(g)$. The same is true for the functionals $\mathcal{L}$ and $\mathcal{C}$, [C, 31.7]. These facts will be very useful since we later employ Morrey's representation theorem. Undoubtedly, it is possible to prove $\mathcal{D}(f) \leq \mathcal{L}(f)$ without resorting to Morrey's theorem, but we prefer this method since we anticipate an application of (24) below in future work.

In this regard recall that a continuous map $f: Q \to \mathbb{E}^n$ is almost conformal provided that

(i) the coordinate functions of $f$ are absolutely continuous in the sense of Tonelli (ACT) on $Q$, and their partial derivatives are square integrable on $Q$, and

(ii) the partial derivatives $D_1f, D_2f$ satisfy $D_1f(x) \cdot D_2f(x) = 0$ for $L_2$ almost all $x \in Q$.

The formal differential, $df_i$, which is defined as the linear transformation associated with the matrix of partial derivatives of $f_i$, exists for $L_2$ almost all points in $Q$. Thus, for all such $x$, $df_i(x)$ is a linear transformation from $E^2$ into $E^n$ and it induces a linear transformation on the space of Grassmann 2-vectors. Thus, if $\alpha$ is a 2-vector in $E^2$, then $df_i(x)(\alpha) = df_i(x, \alpha)$ is a 2-vector in $E^n$. Later, we will also use the fact that if $\{f_i\}$ is a sequence of mollifiers of $f$, then

(i) $f_i \to f$ uniformly on compact subsets of $Q$ and

$$\int_Q |df_i - df|^2 \, dL_2 \to 0$$

where $|df(x)|$ denotes the norm of $df(x)$. 
If \( u: E^n \to E^1 \) is Lipschitz and \( f: Q \to E^n \) is almost conformal, then it is not difficult to show that \( F = u \circ f \) is ACT on \( Q \) and \( |\nabla F| \) is square integrable on \( Q \). Moreover, the following facts were established in [Z]:

(i) If \( g \) is square integrable, then

\[
\int_Q |\nabla F| g \, dL_2 = \int_{E^n} \int_{F^{-1}(y)} g(y) \, dH^1(y) \, dL_1(r).
\]

(ii) If \( W \) is a component of \( \{ x : F(x) < r \} \), then for \( L_1 \) almost all \( r \), \( W \) has finite perimeter. That is, if \( W \) is considered as a current, then in fact, it is an integral current.

(iii) Let \( \beta(W) \) be the reduced boundary of \( W \), that is, \( \beta(W) \) consists of those points \( x \) where \( n(x) \), the exterior normal to \( W \), exists at \( x \). Observe that \( \beta(W) \subseteq \text{bdry } W \supseteq F^{-1}(r). \) Following the proof of [Z, 3.3], we have \( H^1[\beta(W) - \delta(W)] = 0. \)

4.3. Lemma. If \( f: Q \to E^n \) is almost conformal and \( u: E^n \to E^1 \) is Lipschitz, then for \( L_1 \) almost all \( r \), \( f^{-1}(y) \) is totally disconnected for \( H^1 \) almost all \( y \in u^{-1}(r) \).

Proof. In view of the fact that the coordinate functions of \( f \) are ACT on \( Q \), it follows that there is a countable set of vertical line segments in \( Q \), \( \Lambda_1 \), such that \( \Lambda_1 \) is dense in \( Q \) and if \( \lambda \in \Lambda_1 \) then \( H^1[f(\lambda)] < \infty \). Likewise, there is a set \( \Lambda_2 \) corresponding to the horizontal direction. Setting \( \Lambda = \Lambda_1 \cup \Lambda_2 \), it follows that if \( N = f(\Lambda) \) then \( H^2[N] = 0 \). However, [F1, 3.2] or 2.1 implies that \( H^1[N \cap u^{-1}(r)] = 0 \) for \( L_1 \) almost all \( r \). Hence, if \( y \notin N \cap u^{-1}(r) \), then \( f^{-1}(y) \) has only point components.

4.4. Lemma. With the same hypotheses as 4.3, \( \lambda(f; u, r) \leq p(f; u, r) \) for \( L_1 \) almost all \( r \in E^1 \).

Proof. Let \( \alpha \) be a continuous function on \( Q \) whose values are unit Grassmann 2-vectors in \( E^2 \). Select \( r \in E^1 \) so that the results of (20) and 4.3 hold, and let \( W \) be a component of \( \{ x : F(x) < r \} \) such that \( \text{(closure } W \rangle \cap \text{bdry } Q = 0 \). Since the mollifiers, \( f_\gamma \), of \( f \) are \( C^\infty \), (19) implies for every \( C^\infty \) differential 2-form \( \varphi \) that

\[
\lim_{\gamma \to \infty} f_\gamma(W)(\varphi) = \lim_{\gamma \to \infty} \int_W \varphi[f(x)] \cdot df(x, \alpha(x)) \, dL_2(x)
\]

\[
= \int_W \varphi[f(x)] \cdot df(x, \alpha(x)) \, dL_2(x).
\]

On the other hand, [F3, 3.4] implies that, for \( L_1 \) almost all \( r \), \( \lim_{\gamma \to \infty} f_\gamma(W)(\varphi) = \mu(V)(\varphi) \) where \( V = m_\gamma(W) \). Thus,

\[
\mu(V)(\varphi) = \int_W \varphi[f(x)] \cdot df(x, \alpha(x)) \, dL_2(x).
\]
Since \( \int_Q |df_i - df|^2 \, dL_2 \to 0 \) it follows from [(20)(i)] that there exists a subsequence (which will still be denoted as \( \{df_i\} \)) such that

\[
\int_{\beta^{-1}(r)} |df_i - df| \, dH^1 \to 0.
\]

From Hölder’s inequality and (i) of (20), it is clear that (23) holds for \( L_1 \) almost all \( r \). Since \( \beta(W) \subset F^{-1}(r) \) for all \( r \) under consideration, (19), (iii) of (20), (21), (22), (23), and the Gauss-Green Theorem [F4, 4.5.6] imply that, for \( L_1 \) almost all \( r \),

\[
\partial \mu(V)(\varphi) = \lim_{i \to \infty} \partial f_{i_a}(W)(\varphi) = \lim_{i \to \infty} f_{i_a}(\partial W)(\varphi)
\]

\[
= \lim_{i \to \infty} \int_{\beta(W)} \varphi[f_i(x)] \cdot df_i(x, v(x)) \, dH^1
\]

\[
= \int_{\beta(W)} \varphi[f(x)] \cdot df(x, v(x)) \, dH^1(x)
\]

\[
= \int_{\beta(W) \cap \delta(W)} \varphi[f(x)] \cdot df(x, v(x)) \, dH^1(x)
\]

where \( v(x) \) is the unit vector perpendicular to \( n(x) \) chosen so that \( v(x) \wedge n(x) = \alpha(x) \). Since the partial derivatives of the coordinate functions of \( f \) exist \( L_2 \) almost everywhere in \( Q \), we apply [F4, 3.1.8] to find disjoint sets \( Q = \bigcup_{i=0}^n A_i \) such that \( L_2(A_0) = 0 \) and \( f \) restricted to \( A_i, i > 0 \), is Lipschitzian. From [FU, Theorem 3(d)] and (i) of (20), we may assume that \( H^1[\delta(W) \cap A_i] = 0 \). Thus, [F4, 3.2.20] and (24) yield

\[
M[\partial \mu(V)] \leq \int_{\mathbb{R}^n} N[f, \delta(W), y] \, dH^1(y).
\]

However, 4.3 allows us to assume that

\[
\int_{\mathbb{R}^n} N[f, \delta(W), y] \, dH^1(y) = \int_{\mathbb{R}^n} N[f, \delta(V), y] \, dH^1(y)
\]

and the lemma follows directly from this.

4.5. Theorem. Suppose \( f: Q \to E^n \) has finite Lebesgue area and, in addition, suppose \( f \) has the property that for each \( y \in E^n \), no component of \( f^{-1}(y) \) disconnects \( Q \). Then \( \mathcal{C}(f) = \mathcal{D}(f) = \mathcal{L}(f) \).

Proof. With the conditions imposed on \( f \) it follows that the middle space, \( M_f \), is either a 2-cell or a 2-sphere, [R, II.2.91].

In the event that \( M_f \) is a 2-cell, Morrey’s representation theorem [MO1], [MO2] asserts the existence of an almost conformal map \( g: Q \to E^n \) that is Fréchet equivalent to \( f \). Now \( \mathcal{D}(f) = \mathcal{D}(g) \) and \( \mathcal{L}(f) = \mathcal{L}(g) \) while 4.2, 4.4, and 3.3 imply that \( \mathcal{D}(g) = \mathcal{L}(g) \).

If \( M_f \) is a 2-sphere, then \( f \) has a Fréchet equivalent \( g: Q \to E^n \) with the property that \( g \) is constant on the boundary of \( Q \) while \( f \) is not constant on any non-degenerate continuum in the interior of \( Q \), [R, II.3.28]. However, this case is
treated essentially the same way as the preceding one and, thus, the proof is complete.

In order to establish Theorem 4.5 for any map \( f: Q \rightarrow E^n \) of finite Lebesgue area, it will suffice to prove that \( \mathcal{D} \) is cyclically additive because it is known that Lebesgue area possesses this property, [R, V.2.55].

Let \( C_1, C_2, \ldots \) be the cyclic elements of \( M_f \) and let \( r_i: M_f \rightarrow C_i \) be the monotone retraction, [R, II.2.40]. The mappings \( f_i = l_f \circ r_i \circ m_f \) are the cyclic components of \( f \) and they satisfy the hypotheses of Theorem 4.5. Let \( C_1, C_2, \ldots, C_k \) be any finite number of cyclic elements of \( M_f \). It will be sufficient to show that

\[
\mathcal{D}(f) \geq \sum_{i=1}^{k} \mathcal{D}(f_i)
\]

in view of the fact that \( \mathcal{D}(f) \leq \sum_{i=1}^{\infty} \mathcal{D}(f_i) \), which follows from the cyclic additivity of \( \mathcal{L} \), 4.2, and 4.5.

Choose \( \varepsilon > 0 \). From the construction that appears in the proof of Theorem 3.3, it follows that there is a function \( u: E^n \rightarrow E^1 \) with Lipschitz constant 1 such that, for \( i=1, 2, \ldots, k \),

\[
\int_{-\infty}^{\infty} \lambda(f_i; u, r) \, dL_x(r) > \omega(f_i) - \varepsilon/k.
\]

In order to see this we will only consider the case of \( k=2 \), since the general case is handled in the same way and is no more difficult except for complications in notation. Let \( \xi \) be the measure associated with \( f_i \) as in (15), \( i=1, 2 \), and let \( R \) be the Hausdorff 2-rectifiable set that is determined by \( f_i \) as in the proof of Theorem 3.3. Choose compact sets \( K_1 \subset R_1 - R_2, K_2 \subset R_2 - R_1, \) and \( K_3 \subset R_1 \cap R_2 \) such that

\[
\begin{align*}
\omega_1(K_1) &> \omega_1(R_1 - R_2) - \varepsilon/3, \\
\omega_2(K_2) &> \omega_2(R_2 - R_1) - \varepsilon/3, \\
\omega_3(K_3) &> \omega_3(R_1 \cap R_2) - \varepsilon/3,
\end{align*}
\]

for \( i=1, 2 \).

Let \( U_i \) be open sets that are mutually disjoint that contain \( K_i \), \( i=1, 2, 3 \). For each \( y \in K_i \) and all sufficiently small balls centered at \( y \), (17) holds with \( f \) and \( \xi \) replaced by \( f_i \) and \( \xi_i \), \( i=1, 2 \). Similarly, at \( H^2 \) almost all \( y \in K_3 \), (17) is satisfied simultaneously for \( i=1, 2 \) for all sufficiently small balls centered at \( y \). We may assume that all balls considered are contained in some \( U_i \), \( i=1, 2, 3 \). Thus, we have a Vitali cover of \( K = \bigcup_{i=1}^{3} K_i \) and, therefore, there exist balls \( B_1, B_2, \ldots \) such that

\[
H^2 \subset K \left( E^n - \bigcup_{i=1}^{\infty} B_i \right) = 0.
\]

As in the proof of Theorem 3.3, we define a Lipschitz function \( u: E^n \rightarrow E^1 \) in terms of the \( B_i \). Since \( \xi_i \) is absolutely continuous with respect to \( H^2 \), it follows for \( i=1, 2 \) that

\[
\int_{-\infty}^{\infty} \lambda(f_i; u, r) \, dL_x(r) \geq \sum_{j=1}^{\infty} (1-\varepsilon)\xi_i(B_j) = (1-\varepsilon)\xi_i(K_i \cup K_3) \geq (1-\varepsilon)[\xi_0(R_i) - 2\varepsilon/3] = (1-\varepsilon)[\mathcal{L}(f_i) - 2\varepsilon/3].
\]
By redefining \( \varepsilon \) appropriately, (26) is now established. Therefore, (26), 4.4, and 4.5 imply that

\[
\int_{-\infty}^{\infty} \rho(f_i; u, r) \, dL_4(r) > \mathcal{D}(f_i) - \varepsilon/k, \quad i = 1, 2, \ldots, k.
\]

Let \( W \) be a component of \( \{ x : u \circ f(x) < r \} \) and let \( V = m_f(W) \). Now \( r(V) = V \cap C_i \) and the boundary of \( V \cap C_i \) relative to \( C_i \) is contained in \( (\text{bdry } V) \cap C_i \). Consequently, if \( V_1 = V \cap C_i, \delta(V_1) \subseteq \delta(V), i = 1, 2, \ldots, k \). Any two cyclic elements intersect at most in one point, [R, II.2.24], and therefore

\[
N[I, \delta(V), y] \geq \sum_{i=1}^{k} N[I, \delta(V_i), y]
\]

for all but finitely many \( y \in E^n \). Hence,

\[
\rho(f; u, r) \geq \sum_{i=1}^{k} \rho(f_i; u, r)
\]

and (27) now leads to

\[
\mathcal{D}(f) = \int_{-\infty}^{\infty} \rho(f; u, r) \, dL_4(r) \geq \sum_{i=1}^{k} \int_{-\infty}^{\infty} \rho(f_i; u, r) \, dL_4(r) > \sum_{i=1}^{k} \mathcal{D}(f_i) - \varepsilon.
\]

Since \( \varepsilon \) was arbitrarily chosen, (25) is now established and we have established that if \( f : Q \to E^n \) has finite Lebesgue area, then

\[
(28) \quad \mathcal{L}(f) = \mathcal{C}(f) = \mathcal{D}(f).
\]

Moreover, by considering \( u : E^n \to E^1 \) as the distance function from a fixed hyperplane in \( E^n \), it follows from Theorems 4.6 and 7.16 of [F2] that there is a constant \( k \) such that \( \mathcal{L}(f) \leq k \mathcal{D}(f) \). Thus, if \( \mathcal{D}(f) < \infty \) then so is \( \mathcal{L}(f) < \infty \). Therefore 4.2 and (28) yield the following result.

4.6. Theorem. If \( f : Q \to E^n \) is continuous, then \( \mathcal{L}(f) = \mathcal{C}(f) = \mathcal{D}(f) \).

The results and techniques employed by Slepian [S1], [S2] will allow the extension of this theorem to maps defined on any 2-dimensional manifold.

BIBLIOGRAPHY


(\( \odot \)) The author is indebted to the referee for this observation.
1972] SLICES OF MAPS AND LEBESGUE AREA 151


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