THE SIZE FUNCTION ON ABELIAN VARIETIES

BY

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Abstract. The size function is defined for points in projective space over any field $K$, finitely generated field over $Q$, generalizing the height function for number fields. We prove that the size function on the $K$-rational points of an abelian variety is bounded by a quadratic function.

Introduction. In his book, *Introduction to transcendental numbers*, Lang showed how one can extend some of the theorems about the exponential function $e^x$ to theorems about the exponential map from complex $g$-space to the complex points of group varieties of dimension $g$, defined over the complex numbers. Looking at transcendental numbers in this general setting, he raised an arithmetic-geometric question about the addition formula of a group variety. In this paper, we shall answer this question in the case of an abelian variety.

In his report to *Seminaire Bourbaki* in May 1964, [6], Lang described the following result of Neron and Tate: If $A$ is an abelian variety defined over a number field $K$, there exist a quadratic function $Q$ and a linear function $L$ from $A(K)$, the $K$-rational points of $A$, to the real numbers, such that the logarithmic height function, $h: A(K) \rightarrow R$, defined with respect to any closed immersion in projective space, is additively equivalent to the function $Q+L$. Our main result, Theorem 3.5, is a generalization of this (albeit in a weaker form), to the size function, which is defined for an abelian variety defined over any field of characteristic 0. It states that there is a quadratic function $Q: A(K) \rightarrow R$ such that size($x$) is $Q(x)$ for all $x \in A(K)$.

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1. Let $K$ be a field which is finitely generated over $Q$. $K$ has a proper set of generators $\{t_1, \ldots, t_r, u\}$ over $Q$, denoted $\{t, u\}$, where proper means that $\{t_1, \ldots, t_r\}$ is a transcendence base of $K$ over $Q$ and $u$ is integral over $Z[t_1, \ldots, t_r]$. Let $q=[K: Q(t)]$. An element $\alpha \in K$ is said to be an integral coordinate with respect to $\{t, u\}$ if, when $\alpha$ is expressed as a linear combination of $\{1, u, \ldots, u^{q-1}\}$ with coefficients in $Q(t)$ in lowest terms, all coefficients lie in $Z[t]$. Note that if $\alpha, \beta \in K$ are integral coordinates with respect to $\{t, u\}$, then the sum $\alpha + \beta$ and the product $\alpha \beta$ are integral coordinates with respect to $\{t, u\}$.

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153
Let $\alpha \in K$ be an integral coordinate with respect to $\{t, u\}$, say $\alpha = f_0(t) + \cdots + f_{q-1}(t)u^{q-1}$. We define the measure of $\alpha$ with respect to $\{t, u\}$, denoted $|\alpha|_{\{t,u\}}$, to be the maximum of the absolute values of the coefficients of the polynomials $\{f_0, \ldots, f_{q-1}\}$. We define the degree of $\alpha$ with respect to $\{t, u\}$, denoted $\deg_{\{t,u\}}(\alpha)$, to be the maximum of the degrees of the polynomials $\{f_0, \ldots, f_{q-1}\}$. Finally, we define the size of $\alpha$ with respect to $\{t, u\}$, denoted $\size_{\{t,u\}}(\alpha)$, to be the maximum of the degree of $\alpha$ and the logarithm of the measure of $\alpha$.

**Proposition 1.1.** Let $\alpha_1, \ldots, \alpha_s$ be integral coordinates of $K$ with respect to $\{t, u\}$. Then

1. $\deg_{\{t,u\}}(\alpha_1 + \cdots + \alpha_s) \leq \max \{\deg_{\{t,u\}}(\alpha_i)\}$.
2. $|\alpha_1 + \cdots + \alpha_s|_{\{t,u\}} \leq |\alpha_1|_{\{t,u\}} + \cdots + |\alpha_s|_{\{t,u\}}$.
3. $\size_{\{t,u\}}(\alpha_1 + \cdots + \alpha_s) \leq \size_{\{t,u\}}(\alpha_1) + \cdots + \size_{\{t,u\}}(\alpha_s)$.
4. $\deg_{\{t,u\}}(\alpha_1 \cdots \alpha_s) \leq \deg_{\{t,u\}}(\alpha_1) + \cdots + \deg_{\{t,u\}}(\alpha_s) + \mathcal{A}(s-1)$

where $\mathcal{A}$ depends only on the set $\{t, u\}$.

5. $|\alpha_1 \cdots \alpha_s|_{\{t,u\}} \leq |\alpha_1|_{\{t,u\}} \cdots |\alpha_s|_{\{t,u\}} \prod (\deg_{\{t,u\}}(\alpha_i)^{y+1}\mathcal{B}^{s-1})$

where $\mathcal{B}$ depends only on the set $\{t, u\}$.

6. $\size_{\{t,u\}}(\alpha_1 \cdots \alpha_s) \leq \mathcal{C}(\size_{\{t,u\}}(\alpha_1) + \cdots + \size_{\{t,u\}}(\alpha_s))$

where $\mathcal{C}$ depends only on the set $\{t, u\}$.

**Proof.** Straightforward, cf. [5, p. 49].

Let $P = P^K_2$ be projective $n$-space over $K$, and $x \in P(K)$ a $K$-rational point of $P$. If $x = (\alpha_0, \ldots, \alpha_n)$ where all the $\alpha_i \in K$ are integral coordinates with respect to $\{t, u\}$, we say $(\alpha_0, \ldots, \alpha_n)$ are integral coordinates of $x$ with respect to $\{t, u\}$. The measure (resp. degree, size) of $x$ with respect to $\{t, u\}$ denoted $|x|_{\{t,u\}}$ (resp. $\deg_{\{t,u\}}(x)$, $\size_{\{t,u\}}(x)$), is defined to be the greatest lower bound, over all integral coordinates $(\alpha_0, \ldots, \alpha_n)$ of $x$, of the numbers $\max \{|\alpha_i|_{\{t,u\}}\}$ (resp. $\max \{\deg_{\{t,u\}}(\alpha_i)\}$, $\max \{\size_{\{t,u\}}(\alpha_i)\}$).

Let $S$ be a set and $f_1, f_2 : S \to R$ functions taking $S$ into the real numbers. The functions $f_1$ and $f_2$ are said to be *additively equivalent* if there exist numbers $C_1$ and $C_2$ such that $f_1(s) + C_1 \leq f_2(s) \leq f_1(s) + C_2$ for all $s \in S$. The functions $f_1$ and $f_2$ are said to be *multiplicatively equivalent*, denoted $f_1 \sim f_2$, if there exist numbers $C_1, C_2 > 0$ such that $C_1 f_1(s) \leq f_2(s) \leq C_2 f_1(s)$ for all $s \in S$. Clearly if $f_1$ and $f_2$ are $\geq \frac{1}{2}$ and additively equivalent, they are also equivalent. We shall be concerned mostly with equivalence classes of functions.

**Proposition 1.2.** Let $\{t, u\}$ and $\{t', u'\}$ be two proper sets of generators of $K$ over $Q$. Then $\size_{\{t,u\}} \sim \size_{\{t',u'\}}$ as functions from $P(K)$ to $R$. 

The size function on abelian varieties

Proof. Straightforward.

Proposition 1.2 shows that, up to equivalence, the size function is independent of the choice of a proper set of generators of \( K \) over \( Q \). From now on, we shall omit the subscripts referring to the proper set of generators and assume we have picked once and for all a proper set of generators \( \{ t, u \} \).

Lemma 1.3. Let \( f_1, \ldots, f_p \) be polynomials in \( n+1 \) variables with coefficients in \( K \). There exist numbers \( F', F'' > 0 \) such that if \( \{ a_0, \ldots, a_n \} \) is a set of integral coordinates of \( K \) and \( y = (f_0(a), \ldots, f_p(a)) \) is a point of \( P_{k^n}^{-1}(K) \), then \( \text{size}(y) \leq F' \max \{ \text{size}(a_i) \} + F'' \).

Proof. Straightforward.

Let \( k \) be a field, \(((\text{Sch}/k))\) the category of \( k \)-schemes, \( P = \mathbb{P}_k^n \), projective \( n \)-space over \( k \). For each \( k \)-scheme \( X \), consider the set of \((n+2)\)-tuples \((L; s_0, \ldots, s_n)\) where \( L \) is an invertible sheaf on \( X \) and \( (s_0, \ldots, s_n) \) is a set of global sections of \( L \) which generate \( L \). We say \((L; s_0, \ldots, s_n)\) is isomorphic to \((L'; s'_0, \ldots, s'_n)\) if there is an isomorphism \( v: L \cong L' \) such that \( v(s_i) = s'_i \), for \( 0 \leq i \leq n \). If we let \( M(X) \) be the set of isomorphism classes, it is clear that \( M \) is a contravariant functor of \( X \).

For any \( k \)-morphism \( f: X \to P \), the element \((0_P(1); H_0, \ldots, H_n) \in M(P)\), where the \( H_i \) are generating hyperplanes of \( P \), yields \((f^*0_P(1); f^*H_0, \ldots, f^*H_n) \in M(X)\). Thus we have, for each \( k \)-scheme \( X \), a natural map \( T_X: P(X) \to M(X) \). In fact, the collection of the \( T_X \) is an isomorphism of functors [7, p. 31], so in particular each \( T_X \) is a bijection.

Let \( f: X \to P \) be a \( k \)-morphism, \( x \in X(k) \). Then \( f(x) \) is a \( k \)-rational point of \( P \). Let \((a_0, \ldots, a_n)\) be any set of coordinates of \( f(x) \) with \( a_i \in k \). If \((L; s_0, \ldots, s_n) = T_X(f)\), then \( s_i(x) = a a_i \) for some \( a \in k^* \), \( 0 \leq i \leq n \), depending on the identification \( u: L_x \cong k \). Furthermore, for any two such identifications \( u \) and \( u' \), we have \( u(s_i(x)) = b u'(s_i(x)) \), \( 0 \leq i \leq n \), for some unit \( b \in k^* \). Note that if \((L; s_0, \ldots, s_n)\) and \((L'; s'_0, \ldots, s'_n)\) are isomorphic and \( x \in X(k) \), then \((s_0(x), \ldots, s_n(x)) = (s'_0(x), \ldots, s'_n(x)) \in P(k) \).

Now let \( k \) be a field which is finitely generated over \( Q \), and let \( P = \mathbb{P}_k^n \).

Definition. Let \( X \) be a \( K \)-scheme, \( f: X \to P \) a \( K \)-morphism. The size function with respect to \( f \), denoted \( \text{size}_f: X(K) \to R \), is the function \( \text{size}_f(x) = \text{size}(f(x)) \) for all \( x \in X(K) \). Note that this depends only on the isomorphism class of \( T_X(f) \).

Proposition 1.4. Let \( X, Y \) be \( K \)-schemes, \( g: Y \to X, f: X \to P \), \( K \)-morphisms. Then the functions \( \text{size}_{f \circ g} \) and \( \text{size}_f \circ g \) from \( Y(K) \) to \( R \) are equal.

Proof. Let \((L; s_0, \ldots, s_n) = T_X(f)\), and let \( y \in Y(K) \). Then \( \text{size}_f(g(y)) \) is the size of \((s_0(g(y)), \ldots, s_n(g(y))) \in P(K) \), while \( \text{size}_{f \circ g}(y) \) is the size of \((g^*s_0(y), \ldots, g^*s_n(y)) \in P(K) \). But \( g^*s_0 = g^*f^*(H_i) = (f \circ g)^*(H_i) \); so, \( g^*s_0(y) = s_i(g(y)) \).

Proposition 1.5. Let \( X \) be a quasi-projective \( K \)-scheme (with a fixed embedding in \( P \)), \( f: X \to \mathbb{P}_k^n \) a \( K \)-morphism. There exists a number \( F_1 \) depending only on \( f \) such that \( \text{size}_f(x) \leq F_1 \text{size}(x) \) for all \( x \in X(K) \).
Proof. Given \( \varepsilon > 0 \), let \((a_0, \ldots, a_n)\) be integral coordinates of \( x \in X(K) \) such that \( \text{size}(a_i) \leq \text{size}(x) + \varepsilon \). The coordinates \( \beta_i \) of \( f(x) \) are polynomials in the \( a_i \) with coefficients in \( K \), so, by Lemma 1.3, there exists a number \( F_i \) independent of \( \varepsilon \) and \( x \) such that \( \text{size}_e(x) \leq F_i(\text{size}(x) + \varepsilon) \). Since \( \varepsilon \) was arbitrary, the conclusion follows.

Suppose \( X \) is a proper \( K \)-scheme; let \( L \) be an invertible sheaf on \( X \). By the finiteness theorem [2, III 3.2.1], \( H^0(X, L) \) is a finitely generated \( K \)-module. If the global sections of \( L \) generate \( L \), then a basis of \( H^0(X, L) \) defines \( K \)-morphisms \( \phi: X \to \mathbb{P}^r \) for some positive integer \( n \). Therefore if \( L \) is an invertible sheaf on \( X \) whose global sections generate it, we may define \( \text{size}_L \) to be the function size, where \( T_X(f) = (L; s_0, \ldots, s_n) \) and \( \{s_0, \ldots, s_n\} \) is a basis of \( H^0(X, L) \). If \( \{s'_0, \ldots, s'_m\} \) is any set of generators of \( H^0(X, L) \) and \( T_X(g) = (L; s'_0, \ldots, s'_m) \), then \( \text{size}_L \sim \text{size}_g \) by Proposition 1.5.

2. Let \( G \) be a commutative group scheme defined over \( K \) where \( K \) is a field, finitely generated over \( \mathbb{Q} \). Let \( P = \mathbb{P}^r_k \) be projective \( n \)-space \( K \), and let \( \{t_1, \ldots, t_r, u\} \) be a proper set of generators of \( K \) over \( \mathbb{Q} \). For each integer \( N \), we have an endomorphism \((N) = N_g \) of \( G \) which takes each \( x \in G(K) \) into \( Nx \in G(K) \).

**Proposition 2.1.** Let \( f: G \to P \) be a \( K \)-morphism, and suppose \( f \) has the property that, for some given positive integer \( N \), there exist \( n+1 \) homogeneous polynomials \( \sum c_{ik}x^i \in K[Z], 0 \leq i \leq n \), of degree \( N^2 \) such that, for \( x \in G(K) \), the \( i \)th coordinate of \( f(Nx) \) can be written \( \sum c_{ik}x^i(x) \), where \( T_G(f) = (L; s_0, \ldots, s_n) \). (Here \( x(k) \) denotes \( Z_1 \cdots Z_k ). \) Then there exists a number \( C' \) (depending on \( N \)) independent of \( x \) and \( m \) such that

\[
\text{size}_f(N^m x) \leq N^{2m} R \text{size}_e(x) + C'N^{2m}
\]

where \( R = 1 + N^2(r+1)/(N^2 - 1) \) for all integers \( m \geq 0 \).

**Proof.** Throughout the proof, \( \deg(x) \) (resp. \( |x| \)) will refer to \( \deg(f(x)) \) (resp. \( |f(x)| \)). We may assume that each coefficient \( c_{ik} \) is an integral coordinate and that \( s_i(Nx) = \sum c_{ik}s_{ik}(x) \). Furthermore, for any \( \varepsilon > 0 \), we may assume that the \( s_i(x) \) are integral coordinates and that \( \text{size}(s_i(x)) \leq \text{size}_e(x) + \varepsilon \); hence we may assume \( \text{size}(s_i(x)) \leq \text{size}_e(x), 0 \leq i \leq n \), for all \( x \in G(K) \).

Let \( C_1 = \max \{ |c_{ik}| \} + N^{2m} \). Then one shows by an easy induction on \( m \) that \( \deg(N^m x) \leq N^{2m} \deg(x) + D_mC_1 \) where \( D_m = (N^m - 1)/(N^2 - 1) \). Thus we have

\[
\deg(N^m x) \leq N^{2m}(\deg(x) + C_1).
\]

In particular, there is a number \( C_2 \), independent of \( x \) and \( m \) such that \( \deg(N^m x) \leq C_2N^{2m} \deg(x) \). Now let \( C_3 = \max \{|c_{ik}| \} C_1^{r+1}(n+1)BC_2^{r+1}N^2 \). Then by a similar induction on \( m \), one shows that

\[
|N^m x| \leq (C_3(\deg(x))N^{2(r+1)}D_mN_{E_m}|x|N^{2m}
\]

where \( E_m = N^{2(r+1)}(N^{2m} - mN^2 + m - 1)/(N^2 - 1)^3 \). Combining inequalities (7) and (8), one obtains Proposition 2.1.
By Lemma 1.3, the addition map $s_{12}: G \times G \to G$ induces an inequality

$$
\text{size}_f(x + y) \leq T(\text{size}_f(x) + \text{size}_f(y))
$$

for all $x, y \in G(K)$ where $T$ is independent of $x$ and $y$. Let $p$ be an integer such that $T \leq N^p$.

**Proposition 2.2.** Under the hypotheses of Proposition 2.1, there exists a number $D$, independent of $m \in Z$ and $x \in G(K)$, such that

$$
\text{size}_f(mx) \leq Dm^2 \text{size}_f(x).
$$

**Proof.** It follows from (9) that $\text{size}_f(m'x) \leq \{T(T^{s'-p})/(T-1)\} \text{size}_f(x)$. Let $\sigma(x) = \max\{\text{size}_f(m'x)\}$, $0 \leq m' \leq N^p$. Then

$$
\sigma(x) \leq \{T(T^{N^{p+1}} - 1)/(T-1)\} \text{size}_f(x).
$$

By Proposition 1.5, there exists a number $C_4$ such that $\text{size}_f(-x) \leq C_4 \text{size}_f(x)$. Therefore to prove Proposition 2.2, it suffices to show that

$$
\text{size}_f(mx) \leq (RN^2 + 1)m^2 \sigma(x) + m^2 C'N^2.
$$

**Lemma 2.3.** If $m < N^{s+1}$, then

$$
\text{size}_f(mx) \leq TN^{2(s-p)}(R\sigma(x) + C') + T \text{size}_f(m_x),
$$

where $m_1 < N^{s-p}$.

**Proof.** For some $s' \leq s$, we have $N^{s'} \leq m < N^{s'+1}$. Divide the interval, $[N^{s'}, N^{s'+1}]$, into $N^{s'+1} - N^p$ equal intervals, $[N^{s'} + MN^{s'-p}, N^{s'} + (M + 1)N^{s'-p}]$ for $0 \leq M < N^{s'+1} - N^p$. Then for some $M$, we have $N^{s'} + MN^{s'-p} \leq m < N^{s'} + (M + 1)N^{s'-p}$. Therefore we have

$$
\text{size}_f(mx) \leq TN^{2(s'-p)}(R \text{size}_f((N^{p} + M)x) + C') + T \text{size}_f(((m-(N^{p}+M))N^{s'-p})x).
$$

This proves Lemma 2.3.

**Lemma 2.4.** If $i$ is a nonnegative integer, then

$$
\text{size}_f(mx) \leq (R\sigma(x) + C')N^{2(s+1)} + T^i \text{size}_f(m_i x),
$$

where either $m_i < N^{s-(p+1)+1}$ or $m_i < N^{p+1}$.

**Proof.** One first proves by induction on $i$, using Lemma 2.3, that

$$
\text{size}_f(mx) \leq (R\sigma(x) + C')(TN^{2(s-p)} + \cdots + T^i N^{2(s-i(p+1)+1)}) + T^i \text{size}_f(m_i x)
$$

where $m_i < N^{s-(p+1)+1}$ or $m_i < N^{p+1}$. Then since $\sum_{i=1}^t T^i N^{2(s-i(p+1)+1)} \leq N^{2(s+1)}/(N^2 - 1)$ the proof of Lemma 2.4 is complete.

Let $s$ be the integer such that $N^s \leq m < N^{s+1}$. If $s \leq p$, then (11) is trivial. Hence we may assume $s > p$. Let $i$ be the integer such that $s/p \geq i > (s/p) - 1$. Then $m_i < N^{p+1}$ and hence $\text{size}_f(m_i x) \leq \sigma(x)$. Furthermore $T^i \leq N^4$. Therefore, by Lemma 2.4, we
have size,$\{mx\} \leq (R\sigma(x) + C')N^{2(s+1)} + N^s\sigma(x) \leq (RN^2 + 1)m^2\sigma(x) + m^2C'N^2$. The proof of Proposition 2.2 is now complete.

3. Let $k$ be a field, $P = P^n_r$.

**Proposition 3.1.** Let $X$ be a projective scheme over $k$, $f: X \hookrightarrow P$ a $k$-immersion and let $(L; s_0, \ldots, s_s) = T_X(f)$.

Then there exists a positive integer $N_0$ such that for all integers $N' \geq N_0$, the $k$-module of global sections of $L^\otimes N'$ is generated by monomials of degree $N'$ of the global sections $\{s_i\}$ of $L$.

**Proof.** For each integer $N'$, we have an exact sequence of coherent $\mathcal{O}_P$-modules

$$0 \to I(N') \to \mathcal{O}_P(N') \to L^\otimes N' \to 0$$

where $I$ is the sheaf of ideals defining the closed immersion $f: X \hookrightarrow P$. By Serre's theorem [2, III 2.2.2(iii)], there exists an integer $N_0$ such that $H^1(P, I(N')) = 0$ for all $N' \geq N_0$. Furthermore, $H^0(P, \mathcal{O}_P(N'))$ is equal to the $k$-module generated by the monomials of degree $N'$ of global sections of $\mathcal{O}_P(1)$ [2, III 2.1.12(ii)]. Hence for $N' \geq N_0$, the map

$$H^0(P, \mathcal{O}_P(1))^\otimes N' \to H^0(X, L^\otimes N')$$

is surjective; Proposition 3.1 now follows easily.

Let $W_1, W_2$ be algebraic $k$-schemes. Let $p_i: W_1 \times W_2 \to W_i$, $i = 1, 2$, be the projection on the $i$th factor. If $w$ is a $k$-rational point of $W_1$, then let $p_w$ be the composition

$$W_1 \longrightarrow \text{Spec}(k) \longrightarrow \text{Spec}(k(w)) \longrightarrow W_1$$

where $\{w\}: \text{Spec}(k(w)) \to W_1$ is the canonical closed immersion.

**Proposition 3.2.** Let $X, Y, Z$ be proper varieties defined over $k$; let $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$ be $k$-rational points and let $M$ be an invertible sheaf on $X \times Y \times Z$. Suppose $M|_{X \times Y \times \{z_0\}}$ (i.e., $(\text{id}_X \times \text{id}_Y \times \{z_0\})^*M$), $M|_{X \times \{y_0\} \times Z}$, and $M|_{\{x_0\} \times Y \times Z}$ are trivial. Then $M$ itself is trivial.

**Proof.** Let $d = (\text{id}_X, p_{x_0}) \times (\text{id}_Y, p_{y_0}) \times (\text{id}_Z, p_{z_0})$, $d: X \times Y \times Z \to X \times X \times Y \times Y \times Z \times Z$, and let $p_{ijk} = p_i \times p_j \times p_k$, $p_{ijk}: X \times X \times Y \times Y \times Z \times Z \to X \times Y \times Z$ for $i, j, k = 1, 2$. Then $M' = \sum p_{ijk}^*M^\otimes (i+j+k-3) \otimes 0$ by the theorem of the cube [3, p. 68], so $d^*M' \simeq \mathcal{O}_{X \times Y \times Z}$. On the other hand, $d^*p_{111}^*M \simeq M$ and $d^*p_{ijk}^*M \simeq \mathcal{O}_{X \times Y \times Z}$ if $ijk \neq 1$ by assumption. Therefore $M \simeq \mathcal{O}_{X \times Y \times Z}$.

**Corollary 3.3.** Let $f, g, h: S \to A$ be $k$-morphisms from a $k$-scheme $S$ to an abelian variety $A$, and let $L$ be an invertible sheaf on $A$. Then

$$(fg)\star L \simeq (fg)*L \otimes (fh)*L \otimes (gh)*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1},$$

where $fg$ means the product of $f$ and $g$. 

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Proof. Let $e$ be the unit element of $A$ and $s_{123} : A \times A \times A \to A$ be the group addition. Let $s_{12} = s_{123} \circ (id_A \times id_A \times p_3)$, $s_{13} = s_{123} \circ (id_A \times p_2 \times id_A)$ and $s_{23} = s_{123} \circ (p_2 \times id_A \times id_A)$. Then applying Proposition 3.2 to $X = Y = Z = A$, $x_0 = y_0 = z_0 = e$ and $M = s_{123} \ast L \otimes s_{12} \ast L^{-1} \otimes s_{13} \ast L^{-1} \otimes s_{23} \ast L^{-1} \otimes p_1 \ast L \otimes p_2 \ast L \otimes p_3 \ast L$ we obtain $s_{123} \ast L \cong s_{12} \ast L \otimes s_{13} \ast L \otimes s_{23} \ast L \otimes p_1 \ast L^{-1} \otimes p_2 \ast L^{-1} \otimes p_3 \ast L^{-1}$. Pulling this isomorphism back to $S$ via $(f, g, h) : S \to A \times A \times A$, we obtain (13).

Corollary 3.4. Let $A$ be an abelian variety over $k$, $L$ an invertible sheaf on $A$, and let $L' = (-1) \ast L$ then

$$(14) \quad (N) \ast L \cong L^{\otimes (N^2 + N)/2} \otimes L' \otimes (N^2 - N)/2$$

for all integers $N > 0$. In particular, if $L$ is symmetric (i.e., $L = L'$), then $(N) \ast L \cong L^{\otimes N^2}$.

Proof. The proof proceeds by induction on $N$. The conclusion is trivial for $N = 1$. Assume (14) holds for all positive integers $< N + 1$. By (13) with $f = (N), g = (1), \text{and } h = (-1)$, $(N + 1) \ast L \cong (N) \ast L \otimes (N - 1) \ast L^{-1} \otimes L \otimes L'$. Hence by induction we have the result.

Let $K$ be a field, finitely generated over $Q$. Let $A$ be an abelian variety defined over $K$. Since $A$ is projective [3, p. 87], there exists a very ample symmetric invertible sheaf $F$ on $A$. Indeed, if $F_1$ is an invertible sheaf on $A$, then $F_1 \otimes F_1$ is symmetric; furthermore, if $F_1$ is very ample, so is $F_0 \otimes F_0$ by [2, II 4.4.9].

Theorem 3.5. Let $A$ be an abelian variety defined over $K$. If $L$ is a very ample invertible sheaf on $A$, then the size function $size_L : A(K) \to R$ is bounded by a quadratic function, i.e. there exists a quadratic function $Q : A(K) \to R$ such that $size_L(x) \leq Q(x)$ for all $x \in A(K)$.

Proof. We may assume $L$ is symmetric. Indeed, any two closed immersions of $A$ differ from one another by an isomorphism, so their corresponding size functions are equivalent by Proposition 1.5. Let $f$ be a corresponding immersion.

By Corollary 3.4, for each $N > 0$ we have $(N) \ast L \cong L^{\otimes N^2}$. If $N^2 \geq N_0$, then by Proposition 3.1 each of the global sections $s_i \circ (N) \in L^{\otimes N^2}$ (where $s_i$ is a basis of $H^0(A, L)$) can be expressed as a polynomial of degree $N^2$ of the sections $s_i$ with coefficients in $K$, i.e. $s_i(Nx) = \sum c_{(k)}(x) e^k$ where $c_{(k)}(x) \in K$ and $(k) \in Z^{N^2}$. Therefore, $f : A \to P^r_K$ satisfies the conditions of Proposition 2.1.

By the Mordell-Weil Theorem [4, p. 71], $A(K)$ is a finitely generated group. Choose a direct sum decomposition of $A(K)$ into a free subgroup and a torsion subgroup. Since both direct summands are finitely generated, we may choose a basis $\{x_1, \ldots, x_i\}$ of the free summand and a constant $C$ such that $size_L(x_i) \leq C$ for all $x_i$ in the torsion summand.

Each element $x \in A(K)$ can be uniquely expressed in the form $x = n_1 x_1 + \cdots + n_i x_i + x_t$ where $n_i \in Z$ and $x_t$ is a torsion element of $A(K)$. Therefore by Lemma 1.3 there exists a number $T$ such that

$\text{size}_L(x) \leq T(\text{size}_L(n_1 x_1) + \cdots + \text{size}_L(n_i x_i) + C)$. 

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By Proposition 2.2, we have $\text{size}_L(x) \leq Q(x)$ where

$$Q(x) = TD(n_1^2 \text{size}_L(x_1) + \cdots + n_k^2 \text{size}_L(x_k) + C)$$

and $Q$ is clearly quadratic.

Let $G$ be an affine group variety. Then one can use the results of §2 to prove a result similar to Theorem 3.5 for affine group varieties. However, a direct proof is easier.

**Proposition 3.6.** Let $G$ be an affine group variety defined over $K$. Then there exists a number $C'$ such that

$$\text{size}(x^N) \leq C'|N| \text{ size}(x)$$

for all $N \in \mathbb{Z}$ and all $x \in G(K)$.

**Proof.** By Proposition 1.5, we may assume $N > 0$. The group variety $G$ is contained in $GL(M, K)$ for some integer $M > 0$ [1, p. 4-03]. Let $x = \|a_{ij}\|$, $1 \leq i, j \leq M$. Under the usual embedding $\iota: A^n \to P^n$, we have $\iota(x) = (1, a_{ij})$. A direct computation now gives the result.

**Corollary 3.7.** Suppose $H = G \times A$ is the product of an affine group variety $G$ and an abelian variety $A$, both defined over $K$. Then there exists a constant $C''$ such that, for all $x \in H(K)$, $\text{size}(x^N) \leq C''N^2 \text{ size}(x)$.

As a corollary to Theorem 3.5, we obtain the theorem mentioned by Lang [5, p. 54].

**Theorem 3.8.** Let $Q(t)$ be a purely transcendental extension of $Q$ of transcendence type $\leq \tau$ for some integer $\tau \geq 2$. Let $K$ be the algebraic closure of $Q(t)$, $A$ an abelian variety defined over $K$, $\varphi: C \to A(C)$ a 1-parameter subgroup of $A$ of algebraic dimension $d$, and $\Gamma$ a subgroup of $C$. Suppose $\Gamma$ contains at least $2m + 2$ elements $z_1, \ldots, z_{2m+2}$ which are linearly independent over $Q$, and such that $\varphi(\Gamma) \subset A(K)$. If $m \geq d\tau$, then $\tau \geq d$.

**Remark.** The methods used here do not extend to arbitrary commutative group varieties because Proposition 3.1 fails. For example if $G$ is affine, the embedding of $G \hookrightarrow A^{m^2} \hookrightarrow P^{m^2}$ has $L = f^*O_f(1)$, isomorphic to the structure sheaf of $G$. Since $O_G^{\otimes N} \cong O_G$, Proposition 3.1 does not hold. On the other hand, if $G$ is the commutative group variety of dimension 2 parametrized by $(1, \varphi(t), \varphi'(t), u - \zeta(t))$ where $\zeta$ is the Weierstrass zeta function [5, p. 43], then using the addition formulas for $\varphi$ and $\zeta$, one can verify directly that Proposition 2.1 holds in this case. Thus Theorem 3.8 holds not only for abelian varieties, but also for products $G \times A$ by Corollary 3.7 and for the group just mentioned.
BIBLIOGRAPHY


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