

MAPPING CYLINDER NEIGHBORHOODS OF ONE-COMPLEXES IN FOUR-SPACE

BY

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Abstract. We prove the following theorem: *If K is a 1-complex topologically embedded in S^4 , and if K has mapping cylinder neighborhoods in S^4 at almost all of its points, then K is tame.* The proof uses engulfing and the theory of proper, one-acyclic mappings of 3-manifolds onto the real line.

Suppose M^m is a (topological) manifold (without boundary) in the manifold N^n , $x \in M^m$. We say that M^m has a mapping cylinder neighborhood in N^n at x if there exist

- (i) an open neighborhood V of x in M^m ;
- (ii) an open $(n-1)$ -manifold U ;
- (iii) a proper mapping ϕ of U onto V ; and
- (iv) a homeomorphism ψ of Z_ϕ onto a neighborhood of V in N such that $\psi(v)=v$ for each $v \in V$.

The requirement that the image of ψ be a neighborhood of V forces U to have two components when $m=n-1$ (and V is small).

In the above definition, Z_ϕ denotes the mapping cylinder of ϕ , with U and V identified as subsets of Z_ϕ as is customary. A proper map is one under which inverse images of compact sets are compact. We use R^n to denote euclidean n -space, B^n is the closed unit ball centered at 0 in R^n , and $S^n = \text{Bd } B^{n+1}$. The symbol “ \approx ” means “is homeomorphic to”.

It has been conjectured that an m -manifold M^m is locally flat in an n -manifold N^n if and only if M^m has mapping cylinder neighborhoods in N^n at each point. When $n \leq 3$, this conjecture has been proved by Nicholson in [17]. The case $n=4$, $m=3$ is proved in [14]. The main purpose of the present paper is to prove the conjecture in case $n=4$, $m=1$ (see Corollary 4.2 below). The cases $n-m=2$, $n \geq 4$ are false (because there are nonlocally flat piecewise linear embeddings). When $n \geq 5$, the cases $n-m \neq 2$ remain open.

The truth of the above mapping cylinder conjecture in case $n=k+l$, $m=l-1$ implies the truth of the following well-known conjecture: If the l -fold suspension

Received by the editors December 21, 1970.

AMS 1970 subject classifications. Primary 55A99, 57A30, 57A35, 57A40, 57A50.

Key words and phrases. Topological embeddings of one-complexes, locally flat embeddings, locally tame embeddings, mapping cylinder neighborhoods, UV properties, engulfing.

⁽¹⁾ Partially supported by grant NSF GP-19964.

⁽²⁾ Alfred P. Sloan Research Fellow, partially supported by grant NSF GP-19964.

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of a homology k -sphere H^k is a manifold, then H^k is simply connected. Thus, for $n \geq 5$, the cases $n-m=1$ might seem the more tractable of the mapping cylinder conjectures. See [7] for the latest on the homology sphere conjecture.

1. Acyclicity of certain maps.

DEFINITION. Let Z be a space, Y a closed subspace, G an abelian group. We say that Z is $\text{lc}^k(G)$ mod Y if, for any point $y \in Y$ and any neighborhood U of y in Z , there exists a neighborhood V of y in U such that the inclusion-induced map $\tilde{H}_i(V-Y; G) \rightarrow \tilde{H}_i(U-Y; G)$ is zero (on reduced singular homology) for $0 \leq i \leq k$.

The following result is well known. (\mathbb{Z} is the group of integers.)

THEOREM 1.1. *Suppose that M^m is an m -manifold topologically embedded in the interior of the n -manifold N^n . Then N^n is $\text{lc}^{n-m-2}(\mathbb{Z})$ mod M^m .*

Proof. Let $x \in M$, and let U be an open n -cell neighborhood of x in N . Consider the reduced homology sequence of the pair $(U, U-M)$:

$$\cdots \rightarrow H_{i+1}(U, U-M) \rightarrow \tilde{H}_i(U-M) \rightarrow \tilde{H}_i(U) \rightarrow \cdots.$$

We have $\tilde{H}_i(U)=0$ for all i and $H_{i+1}(U, U-M) \cong H_c^{n-i-1}(U \cap M)=0$ for $n-i-1 > m$. Therefore $\tilde{H}_i(U-M)=0$ for $i \leq n-m-2$.

DEFINITION. A compact set X in the ANR M is said to have *property $uv^k(G)$* (or be *strongly k -acyclic over G*) if for any neighborhood U of X in M there exists a neighborhood V of X in U such that $\tilde{H}_i(V; G) \rightarrow \tilde{H}_i(U; G)$ is zero for $0 \leq i \leq k$.

COROLLARY 1.2. *Suppose that $f: U^{n-1} \rightarrow V^m$ is a proper, surjective map between manifolds. If Z_f embeds (locally) in euclidean n -space then $f^{-1}(y)$ has property $uv^{n-m-2}(\mathbb{Z})$ for each $y \in V^m$.*

Proof. It follows immediately from (1.1) that Z_f is $\text{lc}^{n-m-2}(\mathbb{Z})$ mod V . The result follows from Theorem 2.2 of [13].

2. One-acyclic maps of 3-manifolds onto R . Let \mathbb{Z}_* be either the integers or the integers modulo two, R the real line.

THEOREM 2.1. *Let W^3 be an open 3-manifold, and suppose that $f: W^3 \rightarrow R$ is a proper, surjective map. Suppose further that $f^{-1}(t)$ has property $uv^1(\mathbb{Z}_*)$ for each $t \in R$. Then there exists a locally finite subset F of R such that if J is an open interval of R which contains no point of F then $f^{-1}(J) \approx S^2 \times R$.*

Before proving (2.1) we need a definition and a preliminary result.

DEFINITION. A compact set X in the ANR M is said to have *property 1-UV* if each neighborhood U of X in M contains a neighborhood V of X such that every loop in V is null-homotopic in U .

THEOREM 2.2. *Let f be a proper, monotone map of the open 3-manifold W^3 onto R . Suppose that, for each $t \in R$, $f^{-1}(t)$ has a neighborhood which contains no fake cubes and $f^{-1}(t)$ has property 1-UV. Then $W^3 \approx S^2 \times R$.*

Proof. Since $f|f^{-1}(U)$ induces an isomorphism $\pi_1(f^{-1}(U)) \rightarrow \pi_1(U)$ for any connected open set $U \subset \mathbf{R}$ (see [1] or [12]) we have that $f^{-1}(J)$ is simply connected for any open interval $J \subset \mathbf{R}$, and, moreover, $f^{-1}(J)$ has two simply connected ends.

Choose J such that $f^{-1}(J)$ contains no fake cubes. Applying [9], we see that either end of $f^{-1}(J)$ has a collar neighborhood; thus, there exists an embedding

$$h: S^2 \times ((-\infty, -1) \cup (1, +\infty)) \rightarrow f^{-1}(J)$$

such that the complement of the image of h is compact. Let $W_- = h(S^2 \times (-\infty, -2])$, $W_+ = h(S^2 \times [2, +\infty))$, and $W_0 = f^{-1}(J) - W_-^\circ - W_+^\circ$. W_0 is a compact simply connected 3-manifold whose boundary is the union of two 2-spheres. Moreover, W_0 contains no fake cubes. Such a 3-manifold must be homeomorphic to $S^2 \times I$; for, by boring a hole from one boundary component to the other, we can write W_0 as the union of a 3-cell and a homotopy 3-cell meeting along an annulus common to their boundaries. Thus, $f^{-1}(J) \approx S^2 \times \mathbf{R}$.

Now, let $\dots < t_{i-1} < t_i < t_{i+1} < \dots$ be a locally finite set in \mathbf{R} such that $f^{-1}(t_i, t_{i+2}) \approx S^2 \times \mathbf{R}$ for each i . We can engulf any compact set in W with $f^{-1}(t_{i-1}, t_{i+1})$ using the structure of $f^{-1}(t_i, t_{i+2})$ to pull $f^{-1}(t_{i-1}, t_{i+1})$ along. Thus, W contains no fake cubes, and the result follows from the argument in the preceding paragraph.

Proof of (2.1). Let $f: W^3 \rightarrow \mathbf{R}$ be given as in the hypothesis of (2.1). Define

$$\begin{aligned} F' &= \{t \in \mathbf{R} \mid f^{-1}(t) \text{ does not have property 1-UV}\}, \\ F'' &= \{t \in \mathbf{R} \mid \text{every neighborhood of } f^{-1}(t) \text{ contains a fake cube}\}, \\ F &= F' \cup F''. \end{aligned}$$

By (2.2), we need only show that F is locally finite. A theorem of Kneser [11] shows that F'' is locally finite. The local finiteness of F' follows from work of McMillan [15] (see also [16]) and Wright [18]. Let $F'_0 = \{t \in \mathbf{R} \mid f^{-1}(t) \text{ does not have arbitrarily small neighborhoods with free fundamental group}\}$. It follows from Theorem 2 of [15] that F'_0 is locally finite (for otherwise a compact submanifold of W has nonfinitely generated fundamental group by Grushko's theorem). But the arguments [18, Theorem 1] show that $F'_0 \supset F'$. (If $t \notin F'_0$, then $f^{-1}(t)$ has small neighborhoods $V \subset U$, where $H_1(V; \mathbf{Z}_2) = 0$ and $\pi_1(U)$ is free. Let $S_0G = G$, $S_{n+1}G = \text{subgroup of } S_nG$ generated by all squares of elements of S_nG , and $S_\omega G = \bigcap_{n=1}^\infty S_nG$. It follows that $S_\omega \pi_1(V) = \pi_1(V)$ while $S_\omega \pi_1(U) = \{1\}$, so that every homomorphism $\pi_1(V) \rightarrow \pi_1(U)$ is trivial.)

REMARK. Examples show that F in (2.1) may well be nonvoid. Using connected sum, stick homology spheres onto $S^2 \times \mathbf{R}$ to form W^3 ; the map f is to shrink spines of the homology spheres to points and then project to \mathbf{R} .

3. A special case of the main theorem.

THEOREM 3.1. Suppose that $\phi: S^2 \times \mathbf{R} \rightarrow \mathbf{R}$ is a proper, surjective map such that $\phi^{-1}(J) \approx S^2 \times \mathbf{R}$ for any open interval $J \subset \mathbf{R}$. Suppose further that $\psi: Z_\phi \rightarrow \mathbf{R}^4$ is an embedding of Z_ϕ onto a neighborhood of $\psi(\mathbf{R})$ in \mathbf{R}^4 . Then $\psi(\mathbf{R})$ is locally flat in \mathbf{R}^4 .

Throughout the remainder of this section we shall adhere to the following notation:

$$(a) \phi: S^2 \times \mathbf{R} \rightarrow \mathbf{R}$$

$$(b) \psi: Z_\phi \rightarrow \mathbf{R}^4$$

are as in the hypothesis of Theorem 3.1.

$$(c) S_0 = \psi(Z_\phi).$$

(d) For $0 \leq t < 1$, S_t denotes the portion of $\psi(Z_\phi)$ between the levels t and 1, the range of ϕ being the 1-level of the mapping cylinder Z_ϕ .

We may assume that $\psi(S^2 \times \mathbf{R}) = \text{Bd } S_0$ is locally flat in \mathbf{R}^4 , since S_0 may be replaced by S_t ($t > 0$).

LEMMA 3.2. *Given a finite subset $\{s_1, s_2, \dots, s_n\}$ of \mathbf{R} with $s_1 < s_2 < \dots < s_n$, there exists an embedding $g: B^3 \times [1, n]$ into S_0 such that $g^{-1}(\text{Bd } S_0) = S^2 \times [1, n]$ and $g(0, k) = \psi(s_k)$ for $k = 1, \dots, n$.*

Proof. We first show how to find a 3-cell B in S_0 . Let $\alpha: S^2 \times (-1, 1) \rightarrow \phi^{-1}(-1, 1)$ be a homeomorphism. Let Σ_t be $\alpha(S^2 \times \{t\}) \times \{t\}$, copied in Z_ϕ ($0 \leq t < 1$). Clearly $B = \psi(\bigcup_{0 \leq t < 1} \Sigma_t \cup \{1\})$ is a 3-cell in S_0 that is locally flat in S_0 except possibly at $\psi(1)$. A result of Kirby [10] then implies that B is locally flat. From the construction of B it is also clear that B meets $\text{Bd } S_0$ "nicely".

Now, for $k = 1, \dots, n$, let B_k be a 3-cell in $\psi(Z_{\phi|\phi^{-1}(s_{k-1}, s_{k+1})})$ as constructed above with $B_i \cap B_j = \emptyset$ if $i \neq j$ and $B_k \cap \psi(\mathbf{R}) = \psi(s_k)$. Let A_k be the annulus in $\text{Bd } S_0$ bounded by $\text{Bd } B_k$ and $\text{Bd } B_{k+1}$ ($k = 1, \dots, n-1$). Then $B_k \cup A_k \cup B_{k+1}$ is a locally flat 3-sphere in S_0 that bounds a 4-cell C_k in S_0 by the generalized Schoenflies Theorem [2]. Let $g_k: B^3 \times [k, k+1] \rightarrow C_k$ be a homeomorphism that takes $B^3 \times \{k\}$ onto B_k and $B^3 \times \{k+1\}$ onto B_{k+1} . Construct the g_k 's inductively so that $g_k|B^3 \times \{k+1\} = g_{k+1}|B^3 \times \{k+1\}$ ($k = 1, \dots, n-1$). Defining $g: B^3 \times [1, n] \rightarrow S_0$ by $g|B^3 \times [k, k+1] = g_k$, we complete the proof of the lemma.

REMARK 1. It is clear that in the statement of Lemma 3.2 we could have replaced S_0 by S_t for any $t \in [0, 1]$.

REMARK 2. The proof of Theorem 3.1 could now be completed by appealing to the technique of [3] as explained in [4]. The reason this is so is that for t sufficiently close to 1 and for suitably chosen $s_1 < 0 < s_2 < \dots < s_{n-1} < 1 < s_n$, we can get a small push of $\psi([0, 1])$ off of any 2-complex K by first placing $g(\{0\} \times [1, n])$ and K in general position. We have a complete, elementary proof, however, that uses a construction similar to one found in [6].

LEMMA 3.3. *Suppose that $\{s'_1, \dots, s'_m\}$ and $\{s''_1, \dots, s''_n\}$ are subsets of \mathbf{R} with $s'_1 < \dots < s'_m$, $s''_1 < \dots < s''_n$, and $s'_i \neq s''_j$ (for any i, j), that $t', t'' \in [0, 1]$, and that $g': B^3 \times [1, m] \rightarrow S_{t'}$ and $g'': B^3 \times [1, n] \rightarrow S_{t''}$ are the associated embeddings obtained from Lemma 3.2. Let*

$$\{s_1, \dots, s_{m+n}\} = \{s'_1, \dots, s'_m\} \cup \{s''_1, \dots, s''_n\}$$

arranged so that $s_1 < \dots < s_{m+n}$. Then there exist $t \in [0, 1)$ and an embedding $g: B^3 \times [1, m+n] \rightarrow S_t$ such that

$$\begin{aligned} g(B^3 \times \{k\}) &= g'(B^3 \times \{i\}) \cap S_t \quad \text{if } s_k = s'_i, \\ &= g''(B^3 \times \{j\}) \cap S_t \quad \text{if } s_k = s''_j. \end{aligned}$$

Proof. Observe that the 3-cell B constructed at the beginning of the proof of Lemma 3.2 separates each S_t in the same manner in which it separates S_0 . Furthermore, given $g': B^3 \times [1, m] \rightarrow S_{t'}$ and $g'': B^3 \times [1, n] \rightarrow S_{t''}$, there exists $t \geq \max\{t', t''\}$, $t < 1$, so that $g'(B^3 \times \{i\}) \cap g''(B^3 \times \{j\}) \cap S_t = \emptyset$ for any i, j . Thus the proof of Lemma 3.2 can be carried out using the 3-cells $g'(B^3 \times \{i\}) \cap S_t$ ($i = 1, \dots, m$) and $g''(B^3 \times \{j\}) \cap S_t$ ($j = 1, \dots, n$) in the construction of g .

Proof of (3.1). We are going to construct a homeomorphism H of \mathbf{R}^4 that carries $\psi[0, 1]$ onto a locally flat arc. Since, in effect, $\psi[0, 1]$ is an arbitrary subinterval in $\psi(\mathbf{R})$, this will prove that $\psi(\mathbf{R})$ is locally flat.

Let $\varepsilon_1, \varepsilon_2, \dots$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} \varepsilon_i < \infty$.

Inductively we can use (3.2) and (3.3) to find numbers $0 \leq t'_1 < t'_2 < \dots < 1$ and embeddings $g'_k: B^3 \times [-1, 2^k+1] \rightarrow S_{t'_k}$ such that $g'_k(0, i) = \psi(i/2^k)$ and

$$g'_{k+1}(B^3 \times \{2i\}) = g'_k(B^3 \times \{i\}) \cap S_{t'_{k+1}}.$$

Let g_1, g_2, \dots be a subsequence of g'_1, g'_2, \dots (with corresponding subsequences t_1, t_2, \dots of t'_1, t'_2, \dots and n_1, n_2, \dots of $2^1+1, 2^2+1, \dots$) chosen so that

$$\text{diam } g_i(B^3 \times [j, j+1]) < \varepsilon_i \quad (-1 \leq j < n_i).$$

Set $J_i = g_i(\{0\} \times [0, n_i-1])$, $B_{ij} = g_i(B^3 \times [j, j+1])$, and $B_i = \bigcup_{j=-1}^{n_i-1} B_{ij}$. Observe that whenever $k > i$ and $0 \leq j \leq n_i-2$, $J_i \cap B_{ij}$, $J_k \cap B_{ij}$, and $\psi([0, 1]) \cap B_{ij}$ are properly embedded arcs in B_{ij} sharing common endpoints. Let $h''_i: J_i \rightarrow \psi([0, 1])$ be a homeomorphism taking $J_i \cap B_{ij}$ onto $\psi([0, 1]) \cap B_{ij}$ that fixes the endpoints of $J_i \cap B_{ij}$, and let h'_i be a homeomorphism of \mathbf{R}^4 such that $h'_i|Bd B_{ij}$ and $h'_i|R^4 - B_i$ are the identity and $h'_i|J_i = (h''_{i+1})^{-1}h''_i$. Thus h'_i is an ε_i -homeomorphism and $h'_i(J_i) = J_{i+1}$.

Observe also that if U is any neighborhood of J_i ($i \geq 2$), then there exists an ε_i -homeomorphism θ_i of \mathbf{R}^4 such that $\theta_i|(\mathbf{R}^4 - B_{i-1}) \cup J_i = \text{identity}$ and $\theta_i(U) \supset B_i$.

For $0 < r \leq 1$ let $B_r^3 = \{rx \mid x \in B^3\}$ and let $C_i = g_i(B_{1/i}^3 \times [-1/i, n_i-1+1/i])$.

Let $h_1 = h'_1$. Assuming $h_k: \mathbf{R}^4 \rightarrow \mathbf{R}^4$ is defined such that $h_k(J_1) = J_{k+1}$, let θ_{k+1} be an ε_{k+1} -homeomorphism of \mathbf{R}^4 such that $\theta_{k+1}|(\mathbf{R}^4 - B_k) \cup J_{k+1} = \text{identity}$ and $\theta_{k+1}h_k(C_{k+1}) \supset B_{k+1}$, and let $h_{k+1} = h'_{k+1}\theta_{k+1}h_k$. Then h_{k+1} moves no point of \mathbf{R}^4 farther than $2 \sum_{i=1}^{k+1} \varepsilon_i$. Thus the sequence h_1, h_2, \dots converges to a map h of \mathbf{R}^4 onto itself. By construction $h|J_1 = \lim h_i|J_1 = h'_1: J_1 \rightarrow \psi([0, 1])$ is a homeomorphism and $h|R^4 - B_1 = \text{identity}$. Let p be a point of $B_1 - J_1$. There exists i such that $p \in B_1 - C_i$. Thus $h_{i+1}(p) \notin B_{i+1}$ and so $h(p) = h_{i+2}(p)$. This implies that $h|B_1 - J_1$ is a homeomorphism of $B_1 - J_1$ onto $B_1 - \psi([0, 1])$, and hence h is a homeomorphism. The homeomorphism $H = h^{-1}$ takes $\psi([0, 1])$ onto an arc (J_1) that is known to be locally flat. Hence $\psi([0, 1])$ is locally flat.

4. Proof of the main theorem. In the following, K^0 denotes the set of vertices of the complex K .

THEOREM 4.1. *Let K be a finite 1-complex topologically embedded in the interior of the PL 4-manifold M^4 . Suppose that $K - K^0$ has mapping cylinder neighborhoods in M^4 at each point. Then K is tame in M^4 .*

Proof. For each $x \in K - K^0$, let V_x be an open arc in $K - K^0$ containing x such that there exist an open 3-manifold U_x , a proper map ϕ_x of U_x onto V_x , and an embedding ψ_x of Z_{ϕ_x} into M such that $\psi_x(v) = v$ for all $v \in V_x$. Applying (1.2), we see that $\phi_x^{-1}(t)$ has property $uv^1(Z)$ for each $t \in V_x$. Applying (2.1), we see that there is a locally finite subset F_x of V_x such that if J is an open interval in $V_x - F_x$ then $\phi_x^{-1}(J) \approx S^2 \times \mathbf{R}$. By shrinking the interval V_x somewhat, we may assume that F_x is actually finite.

Now, let $\{V_{x_1}, V_{x_2}, \dots\}$ be a locally finite subcollection of $\{V_x\}$ which covers $K - K^0$, and let $F = K^0 \cup \bigcup_{i=1}^{\infty} F_{x_i}$. F is a compact countable subset of K . The crucial property that F enjoys is the following: For each point $x \in K - F$, V_x , U_x , ϕ_x , ψ_x may be chosen so that $\phi_x^{-1}(J) \approx S^2 \times \mathbf{R}$ for any open interval $J \subset V_x$. By the special case (3.1), $\psi_x(V_x)$ is locally flat in M^4 . Hence, K is locally tame in M^4 at each point of $K - F$. It follows from [5] and [8] that K is tame in M^4 .

COROLLARY 4.2. *If S is a 1-sphere in S^4 which has mapping cylinder neighborhoods in S^4 at every point, then S is flat.*

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