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INVERSE LIMITS, ENTROPY AND WEAK ISOMORPHISM FOR DISCRETE DYNAMICAL SYSTEMS

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Abstract. A categorical approach is taken to the study of a single measure-preserving transformation of a finite measure space and to inverse systems and inverse limits of such transformations. The questions of existence and uniqueness of inverse limits are settled. Sinaĭ's theorem on generators is recast and slightly extended to say that entropy respects inverse limits, and various known results about entropy are obtained as immediate corollaries, e.g. systems with quasi-discrete or quasi-periodic spectrum have zero entropy. The inverse limit Φ of an inverse system $\{\Phi_\alpha : \alpha \in J\}$ of dynamical systems is (1) ergodic, (2) weakly mixing, (3) mixing (of any order) iff each Φ_α has the same property. Finally, inverse limits are used to lift a weak isomorphism of dynamical systems Φ_1 and Φ_2 to an isomorphism of systems $\hat{\Phi}_1$ and $\hat{\Phi}_2$ with the same entropy.

1. **Introduction.** By a *discrete*, *abstract dynamical* system (or simply dynamical system) we shall mean a quadruple $\Phi = (X, \mathscr{B}, \mu, \varphi)$, where X is a nonempty set, $\mathscr{B}$ is a σ-algebra of subsets of X, μ is a normalized (total measure one) measure defined on $\mathscr{B}$, and φ is a measurable and measure-preserving (but not necessarily invertible) mapping of X into itself.

For practical reasons we shall, in fact, be interested not in such quadruples but in equivalence classes of such quadruples. Clearly, there is no reason to distinguish between the mapping $x \to 2x \bmod 1$ of the unit interval and the mapping $z \to z^2$ of the unit circle in the complex plane. Moreover, we shall want to identify, for example, the measure space $X = \{0, 1\}$, where $\mathscr{B}$ is the class of all subsets of X and μ is the counting function, with the measure space $X = [0, 1]$,

$$\mathscr{B} = \{\varnothing, [0, \tfrac{1}{2}], [\tfrac{1}{2}, 1], X\},$$

where μ is the restriction to $\mathscr{B}$ of Lebesgue measure. For these reasons we shall (more or less consistently) identify any two dynamical systems $\Phi = (X, \mathscr{B}, \mu, \varphi)$ and $\Phi' = (X', \mathscr{B}', \mu', \varphi')$ for which there exists a mapping $\psi^*: \mathscr{B}' \to \mathscr{B}$ which is one-to-one and onto, and which satisfies

$$\mu(\psi^* B') = \mu'(B') \qquad (B' \in \mathscr{B}')$$

and

$$\mu(\varphi^{-1}(\psi^* B') \,\Delta\, \psi^*(\varphi'^{-1} B')) = 0 \qquad (B' \in \mathscr{B}').$$

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Of course, if $\psi: X \to X'$ is a measure-preserving, invertible map of X essentially onto X', such that ψ^{-1} is also measurable and $\psi\varphi = \varphi'\psi$ (mod zero), then $\psi^*(B) = \psi^{-1}(B)$ effects such an identification.

The principal advantage of this identification is that we may now assume, by an appropriate replacement of the underlying space or spaces, that set mappings $\rho^*: \mathscr{B}' \to \mathscr{B}$ connecting two dynamical systems Φ and Φ' are in fact given by point mappings $\rho: X \to X'$ (see Halmos and von Neumann [8] and D. Maharam [9]).

We shall say that the dynamical system $\Phi_1 = (X_1, \mathscr{B}_1, \mu_1, \varphi_1)$ is a *factor* of the system $\Phi = (X, \mathscr{B}, \mu, \varphi)$ if there exists a measure-preserving map $\psi: X \to X_1$ satisfying $\psi\varphi = \varphi_1\psi$ (mod zero)(1). In this case we write $\Phi_1 | \Phi$ and $\psi: \Phi \to \Phi_1$ or $\Phi \xrightarrow{\psi} \Phi_1$. Note that by our identification we may assume that $X = X_1$, $\mathscr{B}_1 \subseteq \mathscr{B}$ and μ_1 and φ_1 are merely the restrictions of μ and φ. Of course, $\mathscr{B}_1$ is then an *invariant* sub-σ-algebra of $\mathscr{B}$ in the sense that $\varphi^{-1}\mathscr{B}_1 \subseteq \mathscr{B}_1$ (mod zero). The study of factors of Φ is thus reduced to the study of invariant sub-σ-algebras of $\mathscr{B}$. In this regard, note that the factor Φ_1 is an *invertible* system in the sense that φ_1 has an equivalent representation as an invertible, bimeasurable measure-preserving transformation iff $\mathscr{B}_1$ is *totally invariant*, that is $\varphi^{-1}\mathscr{B}_1 = \mathscr{B}_1$ (mod zero).

It might be imagined that because of our identification of equivalent systems, if Φ_1 is a factor of Φ_2 and Φ_2 is a factor of Φ_1, then $\Phi_1 = \Phi_2$. However, it is not known whether this is true (see [14]). Under these conditions we follow Sinaĭ and say that Φ_1 and Φ_2 are *weakly isomorphic*.

A closely related concept to that of factor is direct product. If $\Phi_1 = (X_1, \mathscr{B}_1, \mu_1, \varphi_1)$ and $\Phi_2 = (X_2, \mathscr{B}_2, \mu_2, \varphi_2)$ are dynamical systems, we define their *direct product* $\Phi = \Phi_1 \otimes \Phi_2$ by $\Phi = (X_1 \times X_2, \mathscr{B}_1 \times \mathscr{B}_2, \mu_1 \times \mu_2, \varphi_1 \times \varphi_2)$, where $(\varphi_1 \times \varphi_2)(x_1, x_2) = (\varphi_1 x_1, \varphi_2 x_2)$. More generally, if $\Phi_\alpha = (X_\alpha, \mathscr{B}_\alpha, \mu_\alpha, \varphi_\alpha)$ is a dynamical system for each $\alpha \in J$, we define the *direct product* $\Phi = \bigotimes_{\alpha \in J} \Phi_\alpha$ by taking the product measure structure and defining

$$\varphi(x) = y \quad \text{where } y_\alpha = \varphi_\alpha x_\alpha.$$

We shall make use also of customary symbols such as $\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_n$ and $\bigotimes_{n=1}^{\infty} \Phi_n$.

If $\Phi = \Phi_1 \otimes \Phi_2$, then clearly Φ_1 is a factor of Φ. We shall say that it is a *direct factor*.

For a further discussion of the "arithmetic" of dynamical systems see [5]. We shall be more interested in some aspects of the "calculus" of such systems, as exemplified by the next section.

A word or two about originality of the results in this paper seems to be in order. We have attempted to make our discussion of inverse systems and their entropy more or less self-contained. Our approach is categorical and from that point of

(1) We use the phrase "mod zero" in the established sense that the relation holds after discarding appropriate sets of measure zero. The presence of this modifier will be assumed throughout when not specifically denied or inappropriate.

view the definitions and arguments are quite standard, though perhaps new to ergodic theory. Theorem 1 is new but not terribly exciting. It seems necessary for completeness. Theorem 2 is a new way of stating a theorem of J. R. Choksi [4] with some extra maps floating around. Theorem 3 is new. It elaborates on the representation theorem of Halmos and von Neumann [8] and anticipates the results of the present author on systems with quasi-discrete spectrum [3]. Theorems 4 through 7 are technical necessities.

Theorem 8 is an obvious reformulation of Sinaĭ's important theorem on generators. It is the most natural form of that theorem and contains as well the various extensions given by Billingsley [2]. This is, of course, a continuity theorem, and it is worth noting that countability of the inverse system is not required.

Theorem 9 is new.

2. **Inverse limits.** The direct product of infinitely many dynamical systems may be thought of as a limit of finite products in a way which will become clear in the following. On the other hand, the slightly more general notion of inverse limit is useful in calculation of entropy (see §3) and the analysis of complex dynamical systems (see the examples below and the paper [3]). We shall give a categorical definition (i.e. one involving only maps between dynamical systems) thus avoiding temporarily some of the sticky problems of existence. Note that the direct product in the previous section could also have been defined categorically.

By an *inverse system* of dynamical systems we shall mean a triple $(J, \Phi_\alpha, \psi_{\alpha\alpha'})$ such that J is a directed set, for each $\alpha \in J$, Φ_α is a dynamical system, and for each pair $\alpha, \alpha' \in J$ with $\alpha < \alpha'$, we have $\psi_{\alpha\alpha'}\colon \Phi_{\alpha'} \to \Phi_\alpha$. That is $\Phi_\alpha | \Phi_{\alpha'}$ whenever $\alpha < \alpha'$. An *upper bound* for such a system is a dynamical system Φ such that $\Phi_\alpha | \Phi$ for each $\alpha \in J$, say $\rho_\alpha\colon \Phi \to \Phi_\alpha$, and such that, moreover, for each $\alpha, \alpha' \in J$ with $\alpha < \alpha'$ the diagram

$$\begin{tikzcd} \Phi_\alpha & & \Phi_{\alpha'} \arrow[ll, "\psi_{\alpha\alpha'}"'] \\ & \Phi \arrow[ul, "\rho_\alpha"] \arrow[ur, "\rho_{\alpha'}"'] & \end{tikzcd}$$

commutes. Finally, an *inverse limit* $\hat{\Phi}$ of the inverse system $(J, \Phi_\alpha, \psi_{\alpha\alpha'})$ is an upper bound with maps $\hat{\rho}_\alpha\colon \hat{\Phi} \to \Phi_\alpha$ which is a factor of every other upper bound. That is whenever Φ is an upper bound with maps $\rho_\alpha\colon \Phi \to \Phi_\alpha$, there exists a map $\sigma\colon \Phi \to \hat{\Phi}$ such that the diagram

$$\begin{tikzcd} \Phi_\alpha & \hat{\Phi} \arrow[l, "\hat{\rho}_\alpha"'] \\ & \Phi \arrow[ul, "\rho_\alpha"] \arrow[u, "\sigma"'] \end{tikzcd}$$

commutes for each $\alpha \in J$. In this case we write

$$\hat{\Phi} = \operatorname*{inv\,lim}_{\alpha \in J} \Phi_\alpha \quad \text{or} \quad \hat{\Phi} = \operatorname*{inv\,lim}_{\alpha \in J} (\Phi_\alpha, \psi_{\alpha\alpha'}).$$

Clearly, if $\Phi = (X, \mathscr{B}, \mu, \varphi)$ is an upper bound, then we can represent the Φ_α as $(X, \mathscr{B}_\alpha, \mu, \varphi)$, where the $\mathscr{B}_\alpha$ for $\alpha \in J$ form an increasing net of invariant sub-σ-algebras of $\mathscr{B}$. The mappings $\psi_{\alpha\alpha'}$ and ρ_α become the identity mapping on X. Moreover, the inverse limit $\hat{\Phi}$, if it exists, can be identified with $(X, \hat{\mathscr{B}}, \mu, \varphi)$, where $\bigcup_{\alpha \in J} \mathscr{B}_\alpha \subseteq \hat{\mathscr{B}} \subseteq \mathscr{B}$. We shall show now that this identification leads to a proof of the existence and uniqueness of the inverse limit for any *bounded* inverse system.

If $\mathscr{B}_\alpha$ $(\alpha \in J)$ is any collection of sub-σ-algebras of a σ-algebra $\mathscr{B}$, we shall denote by $\bigvee_{\alpha \in J} \mathscr{B}_\alpha$ the smallest σ-algebra containing $\bigcup_{\alpha \in J} \mathscr{B}_\alpha$ and call it the *join* of the $\mathscr{B}_\alpha$. If Φ_α $(\alpha \in J)$ is a collection of factors of the dynamical system Φ, say $\rho_\alpha \colon \Phi \to \Phi_\alpha$, this allows us to define the *join* $\hat{\Phi} = \bigvee_{\alpha \in J} \Phi_\alpha$ to be $\hat{\Phi} = (X, \bigvee_{\alpha \in J} \rho_\alpha^{-1}(\mathscr{B}_\alpha), \mu, \varphi)$. Note that the join $\bigvee_{\alpha \in J} \Phi_\alpha$ coincides with the product $\bigotimes_{\alpha \in J} \Phi_\alpha$ iff the collection $\{\rho_\alpha^{-1}(\mathscr{B}_\alpha) \colon \alpha \in J\}$ is independent. We shall show that if $(J, \Phi_\alpha, \psi_{\alpha\alpha'})$ is an inverse system, then $\bigvee_{\alpha \in J} \Phi_\alpha = \operatorname{inv\,lim}_{\alpha \in J} \Phi_\alpha$.

THEOREM 1. *If $(J, \Phi_\alpha, \psi_{\alpha\alpha'})$ is an inverse system with upper bound Φ, then $\operatorname{inv\,lim}_{\alpha \in J} \Phi_\alpha = \bigvee_{\alpha \in J} \Phi_\alpha$. In particular, the inverse limit, when it exists, is uniquely determined (up to isomorphism).*

Proof. We may use the "internal characterization" of $(J, \Phi_\alpha, \psi_{\alpha\alpha'})$ whereby $\Phi = (X, \mathscr{B}, \mu, \varphi)$, $\Phi_\alpha = (X, \mathscr{B}_\alpha, \mu, \varphi)$, $\psi_{\alpha\alpha'} = \rho_\alpha = \text{identity}$. It is clear then that $\hat{\Phi} = (X, \bigvee_{\alpha \in J} \mathscr{B}_\alpha, \mu, \varphi)$ is an upper bound. We need only show that any other upper bound can be factored through $\hat{\Phi}$. Let $\Phi' = (X', \mathscr{B}', \mu', \varphi')$ be any other upper bound, relative say to the maps $\rho'_\alpha \colon \Phi' \to \Phi_\alpha$. Since each $\psi_{\alpha\alpha'}$ is the identity on X, it follows that the maps $\rho' = \rho'_\alpha \colon X' \to X$ are the same for each $\alpha \in J$. Moreover, $\rho'^{-1}(\bigcup_{\alpha \in J} \mathscr{B}_\alpha) = \bigcup_{\alpha \in J} \rho'^{-1}(\mathscr{B}_\alpha) \subseteq \mathscr{B}'$. Thus $\rho'^{-1}(\bigvee_{\alpha \in J} \mathscr{B}_\alpha) \subseteq \mathscr{B}'$ and Φ is a factor of Φ', $\rho' \colon \Phi' \to \Phi$. Moreover, since each ρ_α is the identity on X, the proper diagrams commute, and we have shown that $\hat{\Phi}$ is an inverse limit.

So far we have only shown that two inverse limits are weakly isomorphic. However, the above construction can be used now to yield an isomorphism. For suppose that Φ itself is an inverse limit. Then there must exist a map $\sigma \colon \hat{\Phi} \to \Phi$ such that the diagram

$$\begin{tikzcd} \Phi \arrow[dr, "\rho_\alpha"'] & & \hat{\Phi} \arrow[ll, "\sigma"'] \arrow[dl, "\hat{\rho}_\alpha"] \\ & \Phi_\alpha & \end{tikzcd}$$

commutes, where $\rho_\alpha(x) = \hat{\rho}_\alpha(x) = x$ $(x \in X)$. It follows that $\sigma(x) = x$. Measurability of σ thus implies that $\mathscr{B} \subseteq \bigvee_{\alpha \in J} \mathscr{B}_\alpha$, and hence that they coincide. Returning to the external description of the inverse limit, we have shown that, if the upper bound Φ is an inverse limit, then $\bigcup_{\alpha \in J} \rho_\alpha^{-1}(\mathscr{B}_\alpha)$ is dense in $\mathscr{B}$. This, of course, implies that any two inverse limits are equivalent (isomorphic). $\square$

Note that if it were not for our desire to pass back and forth from factors to invariant sub-σ-algebras, we could merely have insisted that all of our measure spaces be sufficiently decent to support "enough" measure-preserving point transformations. This is done, for instance, in [14] and elsewhere by restricting attention to Lebesgue spaces and defining factor transformations on the so-called measurable partitions. However, some important dynamical systems that we shall want to consider are *not* defined on Lebesgue spaces. (See, for example, Theorem 3 and the paper [3].) If such a policy were pursued, then the uniqueness theorem above would yield a point isomorphism. In this sense it is stronger than the uniqueness theorem of Choksi [4].

The question of existence of the inverse limit is somewhat more difficult. The usual approach is to define the inverse limit set

$$X_\infty = \left\{x \in \bigtimes_{\alpha\in J} X_\alpha : \psi_{\alpha\alpha'}x_{\alpha'} = x_\alpha \text{ for all } \alpha, \alpha' \in J, \alpha < \alpha'\right\},$$

define the projections $\rho_\alpha\colon X_\infty \to X_\alpha$ in the obvious way and attempt to extend the measures $\mu_\alpha \circ \rho_\alpha$ from $\bigcup_{\alpha\in J} \rho_\alpha^{-1}(\mathscr{B}_\alpha)$ to $\bigvee_{\alpha\in J} \rho_\alpha^{-1}(\mathscr{B}_\alpha)$. However, it is known (see e.g. [7, p. 214]) that this is not always possible even when each φ_α is the identity and the measurable spaces $(X_\alpha, \mathscr{B}_\alpha)$ are partial products. However, our "free substitution rule" makes it possible to assert that the inverse limit in our sense always exists. Note that Theorem 1 implies that the inverse limit of a *sequence* of Lebesgue systems is a Lebesgue system. In general the inverse limit of Lebesgue systems need not be a Lebesgue system.

THEOREM 2. *The inverse limit of any inverse system of dynamical systems exists.*

Proof. In [4] J. R. Choksi has shown (in the proof of Theorem 5.2) that the given system can be replaced by a system $(J, \Phi_\alpha^1, \psi_{\alpha\alpha'}^1)$ with the measure algebras $\mathscr{B}_\alpha(\mu)$ and $\mathscr{B}_\alpha^1(\mu^1)$ isomorphic under a system of isomorphisms σ_α that commute with the maps $\psi_{\alpha\alpha'}$ and $\psi_{\alpha\alpha'}^1$, i.e. $\psi_{\alpha\alpha'}^1\sigma_{\alpha'} = \sigma_\alpha\psi_{\alpha\alpha'}$, and such that (1) the spaces X_α^1 are compact Hausdorff and (2) the maps $\psi_{\alpha\alpha'}^1$ are continuous. His Theorem 2.2 then gives the existence of an inverse limit measure space $(X, \mathscr{B}, \mu)$, with X being the inverse limit set of the system $(J, \Phi_\alpha^1, \psi_{\alpha\alpha'}^1)$. Next we observe that $\varphi_\alpha^{-1}(A) = \sigma_\alpha\varphi_\alpha^{-1}\sigma_\alpha^{-1}(A)$ defines a measure-preserving set mapping in X_α^1. Since X_α^1 is a "decent" measure space, φ_α^{-1} must be essentially the inverse of a point mapping $\varphi_\alpha^1\colon X_\alpha^1 \to X_\alpha^1$. Moreover, the appropriate diagrams commute, and so $\psi_{\alpha\alpha'}^1\colon \Phi_{\alpha'}^1 \to \Phi_\alpha^1$ whenever $\alpha < \alpha'$. Finally, we define $\varphi\colon X \to X$ by $\varphi(x) = y$ where $y_\alpha = \varphi_\alpha^1 x_\alpha$ for each $x \in X$. It follows that $\Phi = (X, \mathscr{B}, \mu, \varphi)$ is the inverse limit of the system $(J, \Phi_\alpha^1, \psi_{\alpha\alpha'}^1)$ and hence of the system $(J, \Phi_\alpha, \psi_{\alpha\alpha'})$. $\square$

3. **Examples.** The most obvious example of an inverse limit is the direct product defined in the first section. In this case we can take I to be the set of finite subsets of J, directed by inclusion. Then

$$\bigotimes_{\alpha\in J} \Phi_\alpha = \operatorname*{inv\,lim}_{(\alpha_1,\ldots,\alpha_n)\in I} \Phi_{\alpha_1} \otimes \Phi_{\alpha_2} \otimes \cdots \otimes \Phi_{\alpha_n}.$$

While this is a relatively uninteresting observation, it is useful in extending the addition formula for entropy (see §5). In addition, this example is, of course, the prototype for the notion of inverse limit.

A somewhat more interesting example is the following. For each $n=1, 2, 3, \ldots$ let $Y_n=\{0, 1, \ldots, k_n-1\}$ be a finite cyclic group of order k_n. Let $\mathscr{A}_n$ denote all the subsets of Y_n, and let $\nu_n=\{p_{n0}, p_{n1}, \ldots, p_{n,k_n-1}\}$ be a probability measure. We define $X_n=Y_1\times\cdots\times Y_n$, $\mathscr{B}_n=\mathscr{A}_1\times\cdots\times\mathscr{A}_n$, $\mu_n=\nu_1\times\cdots\times\nu_n$ and

$$\begin{aligned}\varphi_n(x_1, \ldots, x_n) &= (x_1+1, x_2, \ldots, x_n) \quad \text{if } x_1 < k_1-1,\\ &= (0, \ldots, 0, x_j+1, x_{j+1}, \ldots, x_n) \quad \text{if } j \text{ is the smallest integer for which } x_j < k_j-1,\\ &= (0, 0, \ldots, 0) \quad \text{if } x_j = k_j-1 \text{ for all } j.\end{aligned}$$

Then φ_n is an ergodic measure-preserving (invertible) transformation iff $p_{nj}=1/k_n$ is independent of j. The sequence $\Phi_n=(X_n, \mathscr{B}_n, \mu_n, \varphi_n)$ is an inverse system with respect to the projection maps

$$\psi_{n,n+m}(x_1, \ldots, x_{n+m}) = (x_1, \ldots, x_n).$$

The inverse limit of the sequence is $\Phi=(X, \mathscr{B}, \mu, \varphi)$, where $X=\times_{n=1}^{\infty} X_n$, $\mathscr{B}=\times_{n=1}^{\infty}\mathscr{B}_n$, $\mu=\times_{n=1}^{\infty}\mu_n$, and φ is defined like φ_n with the obvious modifications.

It is not hard to see that the X of this example, considered as the inverse limit of a sequence of topological groups, is a monothetic, compact abelian group with topological generator $a=(1, 0, 0, \ldots)$, and that $\varphi(x)=x+a$. Thus this example is a special case of the following theorem.

THEOREM 3. *Any ergodic dynamical system with discrete spectrum* (*see* [6]) *is the inverse limit of direct products of ergodic translations of the unit interval* (*addition* mod 1) *and cyclic permutations of finite sets.*

Proof. The pertinent facts about duality of topological groups may be found, for example, in [12]. It is proved in [6] that such a system is isomorphic to translation by a topological generator a on a monothetic, compact abelian group G. Let $\Gamma=\hat{G}$ be the dual of the group G. Then Γ is discrete, and the mapping ψ^*, defined by $\psi^*\gamma=\gamma(a)$, is a one-to-one homomorphism of Γ onto a subgroup of K_d, the circle group with the discrete topology. Thus we may assume that $\Gamma\subseteq K_d$ (in which case ψ^* is the identity map). Suppose that $\Gamma_1\subseteq\Gamma$ is finitely generated. Then $\Gamma_1=(\alpha_1)\oplus(\alpha_2)\oplus\cdots\oplus(\alpha_n)$ is a direct sum (equals direct product) of cyclic groups with generators $\alpha_j\in K_d$. Thinking of K_d now as the unit interval with addition mod 1, each α_j is either rational, hence of finite order, or irrational, and so a topological generator of K (usual topology). (As a matter of fact, not more than one α_j can be rational.) In the first case, the dual of (α_j) is isomorphic to (α_j), that is to the finite cyclic group Z_t of integers mod t for some t, and in the second case, since (α_j) is infinite cyclic, to K. Thus the dual G_1 of Γ_1 is a direct product of such factors. Since $\Gamma_1 \subseteq \Gamma$, it follows that G_1 is a homomorphic image (in the sense of topological groups) of G. Let $\rho_1: G \to G_1$ be the homomorphism. Then ρ_1 is a measure-preserving (Haar measure in each case) mapping of G onto G_1. It induces the factor dynamical system Φ_1, where $\varphi_1(\rho_1 x) = \rho_1(\varphi x)$. Since $\varphi(x) = x + a$, this gives

$$\varphi_1(\rho_1 x) = \rho_1(x+a) = \rho_1(x) + \rho_1(a).$$

Thus φ_1 is translation on G_1 by $\rho_1(a)$.

Let us calculate $\rho_1(a)$. Since ρ_1 is the adjoint of the restriction of the embedding map $\rho_1^*: \Gamma_1 \to \Gamma \subseteq K_a$, we have for each $\gamma \in \Gamma_1$ that

$$\gamma(\rho_1(a)) = \rho_1^*\gamma(a) = \gamma(a) = \psi^*\gamma = \gamma.$$

Thus $\rho_1(a)$, considered as an element of $(\alpha_1)^\wedge \otimes (\alpha_2)^\wedge \otimes \cdots \otimes (\alpha_n)^\wedge$ is $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n$. It follows that φ_1 is a direct product of translations of the type described in the theorem.

Finally, since Γ is the union (equals inductive limit) of its finitely generated subgroups, it follows that G is the inverse limit of factors of the type G_1. From this it follows immediately that Φ is the inverse limit of the corresponding factors Φ_1. $\square$

One more example of particular importance is due to V. A. Rohlin [10]. Let $\Phi = (X, \mathscr{B}, \mu, \varphi)$ be a dynamical system, where φ is, in general, not invertible. In particular, if $\bigcap_{n=0}^{\infty} \varphi^{-n}\mathscr{B}$ is trivial, Φ is called *exact.* Rohlin defines the *natural extension* $\hat{\Phi}$ of Φ in a way that is equivalent to the following. For each $n = 1, 2, \ldots$ let $\Phi_n = \Phi$, and for each $k = 1, 2, \ldots$ let $\psi_{n,n+k}: \Phi_{n+k} \to \Phi_n$ be defined by $\psi_{n,n+k} = \varphi^k$. The inverse limit $\hat{\Phi}$ of this sequence is an *invertible* dynamical system, called the natural extension of Φ. Rohlin actually considered only Lebesgue systems Φ, in which case $\hat{\Phi}$ is also a Lebesgue system. He showed that an invertible dynamical system $\hat{\Phi}$ has completely positive entropy (that is, every factor of $\hat{\Phi}$ has a non-invertible factor) iff $\hat{\Phi}$ is the natural extension of an exact system (see [11]).

To see that $\hat{\Phi}$ is invertible, note that by the proof of Theorem 2 we can assume that $\hat{X}$ is the inverse limit set

$$\hat{X} = \left\{ x \in \bigtimes_{n=1}^{\infty} X_n : x_n = \varphi x_{n+1} \text{ for each } n \right\}.$$

But then

$$\hat{\varphi}(x_1, x_2, x_3, \ldots) = (\varphi x_1, \varphi x_2, \varphi x_3, \ldots) = (\varphi x_1, x_1, x_2, \ldots).$$

Thus $\hat{\varphi}$ is one-to-one, and its inverse

$$\hat{\varphi}^{-1}(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

is also measurable. That is, $\hat{\Phi}$ is invertible. Of course, if Φ is invertible, then $\hat{\Phi}$ is isomorphic to Φ.

4. **Properties of the inverse limit.** In this section, we make some elementary but useful observations about inverse limits.

THEOREM 4. *Let* $(J, \Phi_\alpha, \psi_{\alpha\alpha'})$ *be an inverse system of dynamical systems, and let* $J_0 \subseteq J$ *have the property that for each* $\alpha \in J$ *there is a* $\beta \in J_0$ *such that* $\alpha < \beta$. *Then* $(J_0, \Phi_\alpha, \psi_{\alpha\alpha'})$ *is an inverse system, and* $\operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha = \operatorname{inv}\lim_{\alpha\in J_0} \Phi_\alpha$.

Proof. Since $\bigvee_{\alpha\in J} \Phi_\alpha$ is clearly the same as $\bigvee_{\alpha\in J_0} \Phi_\alpha$, the result is immediate from the internal characterization of the Φ_α as factors of $\operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha$. Externally, the theorem is simply the observation that we can fill in "gaps" in the system of projections by setting $\rho_\alpha = \psi_{\alpha\beta}\rho_\beta$. □

THEOREM 5. *If* $\Sigma_\alpha | \Phi_\alpha$ *for each* $\alpha \in J$, *then* $\operatorname{inv}\lim_{\alpha\in J} \Sigma_\alpha | \operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha$. *In particular, if* φ_α *and* ψ_α *are commuting transformations on* X_α *for each* $\alpha \in J$, *then*

$$\varphi = \operatorname*{inv\,lim}_{\alpha\in J} \varphi_\alpha \quad \textit{and} \quad \psi = \operatorname*{inv\,lim}_{\alpha\in J} \psi_\alpha$$

also commute.

THEOREM 6. *If* Φ_α^1 *and* Φ_α^2 *are factors of* Φ_α *for each* $\alpha \in J$, *then*

$$\operatorname*{inv\,lim}_{\alpha\in J} (\Phi_\alpha^1 \vee \Phi_\alpha^2) = \left(\operatorname*{inv\,lim}_{\alpha\in J} \Phi_\alpha^1\right) \vee \left(\operatorname*{inv\,lim}_{\alpha\in J} \Phi_\alpha^2\right),$$

where the latter join is as factors of $\operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha$. *In particular,* $\operatorname{inv}\lim_{\alpha\in J} (\Phi_\alpha^1 \otimes \Phi_\alpha^2) = (\operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha^1) \otimes (\operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha^2)$. *Moreover, if* Φ_α^1 *and* Φ_α^2 *are disjoint, in the sense of Furstenberg* [5], *for each* $\alpha \in J$, *then* $\operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha^1$ *and* $\operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha^2$ *are disjoint.*

The proofs of Theorems 5 and 6 are routine verifications and will be omitted.

THEOREM 7. *The inverse limit* $\operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha$ *is* (1) *ergodic,* (2) *weakly mixing,* (3) *mixing* (*of any order*) *iff each* Φ_α *has the same property.*

Proof. As in the proof of Theorem 1, we may assume that $\Phi = \operatorname{inv}\lim_{\alpha\in J} \Phi_\alpha = (X, \mathscr{B}, \mu, \varphi)$, $\Phi_\alpha = (X, \mathscr{B}_\alpha, \mu, \varphi)$, where the $\mathscr{B}_\alpha$ $(\alpha \in J)$ form an increasing net, and $\mathscr{B} = \bigvee_{\alpha\in J} \mathscr{B}_\alpha$. Now it is well known that Φ is ergodic iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap \varphi^{-k}B) = \mu(A)\mu(B) \tag{1}$$

for each pair $A, B \in \mathscr{B}$. As a matter of fact, it is sufficient to show that (1) holds for each pair $A, B \in \mathscr{B}_0$, where $\mathscr{B}_0$ is some algebra dense in $\mathscr{B}$. For if $\mu(A \,\Delta\, A_0) < \varepsilon$ and $\mu(B \,\Delta\, B_0) < \varepsilon$, then

$$\begin{aligned} |\mu(A \cap \varphi^{-k}B) - \mu(A_0 \cap \varphi^{-k}B_0)| &\leqq \mu[(A \cap \varphi^{-k}B) \,\Delta\, (A_0 \cap \varphi^{-k}B_0)] \\ &\leqq \mu[(A \,\Delta\, A_0) \cup (\varphi^{-k}B \,\Delta\, \varphi^{-k}B_0)] < 2\varepsilon \end{aligned}$$

for each k. Taking $\mathscr{B}_0 = \bigcup_{\alpha\in J} \mathscr{B}_\alpha$ shows that Φ is ergodic iff each Φ_α is ergodic. (An interesting alternate proof uses the definition of ergodicity and the martingale theorem. The above proof, of course, uses the ergodic theorem.) A similar argument shows that Φ is weakly mixing or mixing iff each Φ_α is. Alternatively, Φ is weakly mixing iff $\Phi \otimes \Phi$ is ergodic, which by Theorem 6 and the first part of this theorem is true iff each $\Phi_\alpha \otimes \Phi_\alpha$ is ergodic, that is iff each Φ_α is weakly mixing. □

5. **Entropy.** In this section we discuss the entropy of an abstract dynamical system Φ, particularly as it relates to inverse limits. In many cases, the calculation of the entropy of Φ is accomplished by representing $\Phi = \operatorname{inv} \lim_{\alpha \in J} \Phi_\alpha$ as an inverse limit, calculating the entropy of Φ_α for each $\alpha \in J$, and passing to the limit. This is the case, for example, in [10], where the entropy of the natural extension of Φ is shown to be equal to the entropy of Φ, in [1] and in [13], where the entropy of systems with quasi-discrete spectrum and quasi-periodic spectrum, respectively, is shown to be zero. In each of these cases, the inverse systems involved are sequences. However, Theorem 3 of §3 and the results in [3] show that the same approach can be fruitful for more general inverse systems.

The definition and basic properties of the entropy of a dynamical system may be found, for example, in [2], where the fundamental lemma below is also proved.

Let $\Phi = (X, \mathscr{B}, \mu, \varphi)$ be a dynamical system. For each finite algebra $\mathscr{A} \subseteq \mathscr{B}$, let $\dot{\mathscr{A}}$ denote the collection of its atoms. Thus $\dot{\mathscr{A}} = \{A_1, \ldots, A_n\}$ is a *partition* of X, in the sense that its elements are pairwise disjoint and $\mu(X \sim \bigcup_{j=1}^n A_j) = 0$. For such an algebra $\mathscr{A}$ let

$$H(\mathscr{A}) = -\sum_{A \in \dot{\mathscr{A}}} \mu(A) \log \mu(A),$$

and

$$h(\varphi, \mathscr{A}) = \lim_{n \to \infty} \frac{1}{n} H(\mathscr{A} \vee \varphi^{-1}\mathscr{A} \vee \cdots \vee \varphi^{-(n-1)}\mathscr{A}).$$

We define the *entropy* of the dynamical system Φ to be the number

$$h(\Phi) = \sup_{\mathscr{A} \subseteq \mathscr{B}} h(\varphi, \mathscr{A}). \tag{1}$$

It then follows (see [2]) that $0 \leqq h(\Phi) \leqq \infty$, $h(\Phi^n) = h(\Phi)|n|$ (for positive n in general, for positive or negative n in case Φ is invertible), $h(\Phi_1 \otimes \Phi_2) = h(\Phi_1) + h(\Phi_2)$, and that $h(\Phi_1) \leqq h(\Phi_2)$ whenever $\Phi_1 | \Phi_2$.

Aside from these "arithmetical" properties of h, the principal tool in calculating the entropy of familiar dynamical systems has been the following theorem of Sinaĭ and various extensions of it (see [2, Theorem 7.3 and its corollaries]).

THEOREM (SINAĬ). *If $\mathscr{A}$ is a finite subalgebra of $\mathscr{B}$ such that* (1) $\bigvee_{n=0}^\infty \varphi^{-n}\mathscr{A} = \mathscr{B}$ *or* (2) Φ *is invertible and* $\bigvee_{n=-\infty}^\infty \varphi^{-n}\mathscr{A} = \mathscr{B}$, *then* $h(\Phi) = h(\varphi, \mathscr{A})$.

Note that, for any finite algebra $\mathscr{A} \subseteq \mathscr{B}$, the σ-algebra $\mathscr{B}_1 = \bigvee_{n=0}^\infty \varphi^{-n}\mathscr{A}$ is invariant, and $h(\varphi, \mathscr{A}) = h(\Phi_1)$, where $\Phi_1 = (X, \mathscr{B}_1, \mu, \varphi)$ is a factor of Φ. Thus the various extensions of Sinaĭ's theorem become special cases of our Theorem 8 below. We shall make use of the following lemma [2, p. 89], whose proof can be made independent of the Sinaĭ theorem.

LEMMA. *Let $\mathscr{B}_0$ be a dense subalgebra of $\mathscr{B}$. Then*

$$h(\Phi) = \sup_{\mathscr{A} \subseteq \mathscr{B}_0} h(\varphi, \mathscr{A}). \tag{2}$$

THEOREM 8. *If* $\Phi = \operatorname{inv} \lim_{\alpha\in J} \Phi_\alpha$, *then*

$$h(\Phi) = \lim_{\alpha} h(\Phi_\alpha). \tag{3}$$

REMARK. Of course, by the monotone character of the function h with regard to factors, the numbers $h(\Phi_\alpha)$ $(\alpha \in J)$ form an increasing net, and so $\lim_\alpha h(\Phi_\alpha) = \sup_{\alpha\in J} h(\Phi_\alpha) \leqq +\infty$ exists.

Proof. We may assume without loss of generality that $\Phi = (X, \mathscr{B}, \mu, \varphi)$ and $\Phi_\alpha = (X, \mathscr{B}_\alpha, \mu, \varphi)$ for each $\alpha \in J$. Since $\Phi_\alpha | \Phi$, we have $h(\Phi_\alpha) \leqq h(\Phi)$ for each α, and so $h(\Phi) \geqq \sup_{\alpha\in J} h(\Phi_\alpha) = \lim_\alpha h(\Phi_\alpha)$. To prove the reverse inequality, we use the lemma with $\mathscr{B}_0 = \bigcup_{\alpha\in J} \mathscr{B}_\alpha$. Then if $\mathscr{A}_0$ is a *finite* subalgebra of $\mathscr{B}_0$, there must exist an $\alpha \in J$ with $\mathscr{A}_0 \subseteq \mathscr{B}_\alpha$. But then

$$h(\varphi, \mathscr{A}_0) \leqq \sup_{\mathscr{A} \subseteq \mathscr{B}_\alpha} h(\varphi, \mathscr{A}) = h(\Phi_\alpha).$$

It follows that

$$h(\Phi) = \sup_{\mathscr{A} \subseteq \mathscr{B}_0} h(\varphi, \mathscr{A}) \leqq \sup_{\alpha\in J} h(\Phi_\alpha),$$

and the proof is complete. □

COROLLARY 1. *If* $\Phi = \bigotimes_{\alpha\in J} \Phi_\alpha$, *then*

$$h(\Phi) = \sum_{\alpha\in J} h(\Phi_\alpha). \tag{4}$$

This sum is, of course, interpreted to be $+\infty$ if more than a countable number of the Φ_α have positive entropy (or if the sum diverges).

From Theorem 3 we obtain the following corollary, which of course is well known.

COROLLARY 2. *Any ergodic dynamical system with discrete spectrum has zero entropy.*

As indicated earlier, similar analyses of systems with quasi-discrete spectrum or quasi-periodic spectrum reveal that they also have zero entropy. We might note also the following theorem of Rohlin [10].

COROLLARY 3. *If* $\hat{\Phi}$ *is the natural extension of* Φ, *then* $h(\hat{\Phi}) = h(\Phi)$.

REMARK. An immediate consequence of Theorem 6 is that any inverse limit of dynamical systems with zero entropy has zero entropy. At the opposite extreme, it is proved in [11] that the inverse limit of a *sequence* of *Lebesgue* systems with completely positive entropy (every nontrivial factor has positive entropy) has completely positive entropy. It seems likely that this is true for arbitrary inverse limits of systems with completely positive entropy. However, the present author has been unable to demonstrate this.

6. **Weak isomorphism.** Following Sinaĭ [14] we have defined two dynamical systems Φ_1 and Φ_2 to be *weakly isomorphic* if each is a factor of the other. While it is not known whether this implies isomorphism of Φ_1 and Φ_2, we shall now show that whenever Φ_1 and Φ_2 are weakly isomorphic, there exist dynamical systems $\hat{\Phi}_1$ and $\hat{\Phi}_2$, such that Φ_i is a factor of $\hat{\Phi}_i$ $(i=1, 2)$ with the same entropy, and a "lifting" of the weak isomorphism to an isomorphism of $\hat{\Phi}_1$ and $\hat{\Phi}_2$.

THEOREM 9. *Suppose that* $\sigma_1: \Phi_1 \to \Phi_2$ *and* $\sigma_2: \Phi_2 \to \Phi_1$. *Define* $\psi_1 = \sigma_2\sigma_1: \Phi_1 \to \Phi_1$ *and* $\psi_2 = \sigma_1\sigma_2: \Phi_2 \to \Phi_2$. *Let J be the positive integers, and for each* $n \in J$, *let* $\Phi_n^1 = \Phi_1$, $\Phi_n^2 = \Phi_2$, $\psi_{n,n+k}^1 = \psi_1^k$, $\psi_{n,n+k}^2 = \psi_2^k$. *Then* $\hat{\Phi}_1 = \operatorname{inv\,lim}_{n\in J} (\Phi_n^1, \psi_{nn'}^1)$ *is isomorphic to* $\hat{\Phi}_2 = \operatorname{inv\,lim}_{n\in J} (\Phi_n^2, \psi_{nn'}^2)$. *In fact, there exist isomorphisms* $\hat{\sigma}_1: \hat{\Phi}_1 \to \hat{\Phi}_2$ *and* $\hat{\sigma}_2: \hat{\Phi}_2 \to \hat{\Phi}_1$ *such that* $\hat{\sigma}_2\hat{\sigma}_1$ *is the natural extension of* ψ_1.

Proof. The proof may best be summarized in the following diagram.

$$\begin{tikzcd}
\Phi_1 & \Phi_1 \arrow[l, "\psi_1"'] & \cdots \arrow[l, "\psi_1"'] & \hat{\Phi}_1 \arrow[l] \\
\Phi_2 \arrow[u, "\sigma_2"'] & \Phi_2 \arrow[l, "\psi_2"'] \arrow[u, "\sigma_2"'] & \cdots \arrow[l, "\psi_2"'] & \hat{\Phi}_2 \arrow[l] \arrow[u, "\hat{\sigma}_2"'] \\
\Phi_1 \arrow[u, "\sigma_1"'] & \Phi_1 \arrow[l, "\psi_1"'] \arrow[u, "\sigma_1"'] & \cdots \arrow[l, "\psi_1"'] & \hat{\Phi}_1 \arrow[l] \arrow[u, "\hat{\sigma}_1"']
\end{tikzcd}$$

The existence of the isomorphism follows from Theorems 1 and 4 and the observation that the first and second rows of the above diagram represent subsequences of the system

$$\Phi_1 \xleftarrow{\sigma_2} \Phi_2 \xleftarrow{\sigma_1} \Phi_1 \xleftarrow{\sigma_2} \Phi_2 \xleftarrow{\sigma_1} \cdots \longleftarrow \hat{\Phi},$$

so that $\hat{\Phi}$, $\hat{\Phi}_1$ and $\hat{\Phi}_2$ are all isomorphic. The mappings $\hat{\sigma}_1$ and $\hat{\sigma}_2$ in the first diagram above are defined in the obvious way (as in the second half of Theorem 5), and the fact that $\hat{\sigma}_2\hat{\sigma}_1$ is the natural extension $\hat{\psi}_1$ of ψ_1 is obtained by deleting the middle row of this diagram. Since $\hat{\psi}_1$ is invertible, so are $\hat{\sigma}_1$ and $\hat{\sigma}_2$. □

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