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THE SPACE OF HOMEOMORPHISMS ON A COMPACT TWO-MANIFOLD IS AN ABSOLUTE NEIGHBORHOOD RETRACT

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Abstract. The theorem mentioned in the title is proved.

1. **Introduction.** Throughout this paper M^n will denote a compact, metric, n-manifold, $n \geqq 1$. If M^n is without boundary $H(M^n)$ will denote the space (under the sup norm topology) of all homeomorphisms of M^n onto M^n. If M^n has nonempty boundary $H(M^n)$ will denote the space of all homeomorphisms of M^n onto M^n which leave the boundary pointwise fixed.

A result which has triggered a great deal of work recently is the following. The space of all orientation preserving homeomorphisms of the unit interval [0, 1] onto itself is homeomorphic to l_2, the separable hilbert space of square summable sequences [2]. A natural question is whether $H(M^n)$ is locally homeomorphic to l_2 for every M^n [13, p. 792], [24, Problem M1].

It is well known that $H(M^n)$ is a complete separable metric space [6, p. 265]. Mason [17] has shown that if K is a sigma-compact subset of $H(M^n)$, then $H(M^n)-K$ is homeomorphic to $H(M^n)$. Geoghegan [8] has shown that $H(M^n)\times l_2$ is homeomorphic to $H(M^n)$ (for a generalization see Keesling [14]). Černavskiĭ [4] and Edwards and Kirby [7] have shown that $H(M^n)$ is locally contractible, (earlier, Hamstrom and Dyer [10] showed that $H(M^2)$ was locally contractible).

The homotopy groups of $H(M^2)$ have been studied by Hamstrom [9] and McCarty [16] (see also Morton [19]). Mason [18] showed that if D is a 2-cell then $H(D)$ is an absolute retract (a problem originally raised by E. Michael, see [26, p. 229]).

In this paper we show that $H(M^2)$ is an absolute neighborhood retract (Theorem 18). During the course of the proof we show that various other function spaces are absolute neighborhood retracts. Some of these are: the space of embeddings of a 2-cell D into the plane E^2, the space of embeddings of Bd (D) into E^2, and the space of all embeddings of D into E^2 which are holomorphic on Int (D) (see §4).

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The basic idea of the proof is to use the result that $H(D)$ is an absolute retract, decompose M^2 into 2-cells, and use the techniques of Hamstrom and Dyer [10] on the decomposition.

All function spaces mentioned in this paper will be topologized by the "sup norm" distance function (see §2).

2. **Definitions and notation.** The statement that a metric space X is an *absolute neighborhood retract* (ANR) means that whenever X is embedded as a closed subset Z_0 of a metric space Z, there is a retraction of an open neighborhood of Z_0 onto Z_0.

A sequence $X_1, X_2, \ldots$ *converges* 0*-regularly* to a set X_0 if $X_1, X_2, \ldots$ converges to X_0, and for every $\varepsilon > 0$, a $\delta > 0$ and an integer $N > 0$ exist such that, if $n > N$, any two points x, y of X_n, with $\operatorname{dist}(x, y) < \delta$, lie together in a connected subset of X_n of diameter less than ε.

By *map* we mean continuous function. $f\colon A \twoheadrightarrow B$ means that the function f takes A onto B. If A and B are metric spaces, with A compact, and $f_1, f_2\colon A \to B$ are maps, then $\operatorname{dist}(f_1, f_2)$, *the distance between* f_1 *and* f_2, will be

$$\sup_{x \in A} \operatorname{dist}(f_1(x), f_2(x)).$$

If M is a manifold, $\operatorname{Bd}(M)$ denotes the boundary of M, and $\operatorname{Int}(M)$ denotes the interior of M.

E^2 denotes the Euclidean plane, and $\mathscr{C}^1$ the complex number plane. By 1 we shall often mean the point $1 + 0i \in \mathscr{C}^1$.

If $X \subset \mathscr{C}^1$ is a point set and $f\colon X \to \mathscr{C}^1$ is a map, then f is *holomorphic* at a point $x \in X$ if f has a (complex) derivative, $f'(z)$, for all z in some neighborhood (in $\mathscr{C}^1$) of x. If f is 1-1, f is *orientation preserving* on X, if given any simple closed curve J in X and a positive transversal α of J, then $f \circ \alpha$ is a positive transversal of $f(J)$ (see [23, Chapter 5, §2]).

An arc B is a *spanning arc* of a disk Z if $B \subset Z$ and $B \cap \operatorname{Bd}(Z)$ are the endpoints of B.

If A and B are point sets, $A + B$ denotes the union (sum) of A and B. $\{A_n\} \to A$ means that the sequence $A_0, A_1, \ldots$ converges to A.

If $X \subset Z$ are manifolds in E^2 (or $\mathscr{C}^1$), then $E(X, Z)$ denotes the space of all embeddings of X into Z which are fixed on $X \cap \operatorname{Bd}(Z)$ (possibly $\operatorname{Bd}(Z) = \varnothing$), and which take $X \cap \operatorname{Int}(Z)$ into $\operatorname{Int}(Z)$. $E_0(X, Z) = \{f \in E(X, Z) : f \text{ is orientation preserving}\}$. $AE(X, Z) = \{f \in E(X, Z) : f \text{ is holomorphic on } \operatorname{Int}(X)\}$.

3. **Preliminaries on conformal mapping.** It is a well-known result of complex analysis that if D is the closed unit disk in $\mathscr{C}^1$, and J is a simple closed curve in $\mathscr{C}^1$ bounding a closed (topological) disk G, then there is a homeomorphism f of D onto G such that f is holomorphic on $\operatorname{Int}(D)$. Further, if $z_0 \in \operatorname{Int}(G)$ and $z_1 \in \operatorname{Bd}(G)$, there is a unique f such that $f(0) = z_0$, and $f(1) = z_1$ [20, Theorems 12.6, 14.8, 14.19].

The following continuity property of such f's is not so well known.

THEOREM 1. *Suppose that for each nonnegative integer i, J_i is a simple closed curve in $\mathscr{C}^1$ which bounds a closed (topological) disk G_i. Suppose*

(1) $\{J_i\} \to J_0$ *0-regularly,*

(2) $z_0 \in \operatorname{Int}(G_i)$ *for some fixed* z_0, $i=0, 1, 2, \ldots,$

(3) *for each i,* $i=0, 1, \ldots, f_i \colon D \twoheadrightarrow G_i$ *is a homeomorphism such that* $f_i(0)=z_0$ *and* f_i *is holomorphic on* $\operatorname{Int}(D)$,

(4) $\{f_i(1)\} \to f_0(1)$.

Then $\{f_i\} \to f_0$ *uniformly on D.*

(Proof omitted.)

A version of this theorem is given in [10]. A proof may be dug out of [5, p. 191]. Perhaps the most elementary method of proof is to use Lindelöff's lemma [21, Chapter 3, §10] and mimic the nice proof of Lemma 12.1 in [21, Chapter 3, §12].

In order to state the next lemma we need the following notation. This notation will be used in §4 also. Suppose $f \in E_0(\operatorname{Bd}(D), E^2)$, where $D=\{z \in \mathscr{C}^1 : |z| \leqq 1\} =\{(x, y) \in E^2 : x^2+y^2 \leqq 1\}$. Let $A(f)$ be an annulus in E^2 such that $f(\operatorname{Bd}(D)) \subset \operatorname{Int}(A(f))$, and $f(\operatorname{Bd}(D))$ separates $\operatorname{Bd}(A(f))$. Let $x(f)$ be a point in the bounded component of $E^2-A(f)$. Define

$$w(A(f)) = \{h \in E_0(\operatorname{Bd}(D), E^2) : h(\operatorname{Bd}(D)) \subset \operatorname{Int}(A(f)) \text{ and } h(\operatorname{Bd}(D)) \text{ separates } \operatorname{Bd}(A(f))\},$$

$$V(A(f)) = \{g \in AE(D, E^2) : g(0) = x(f) \text{ and } g|\operatorname{Bd}(D) \in w(A(f))\}.$$

Theorem 1 and the remarks preceding Theorem 1 give us

LEMMA 2. *There is a map* $\sigma' \colon w(A(f)) \to V(A(f))$ *such that* $\sigma'(h)(1)=h(1)$, *and* $\sigma'(h)(\operatorname{Bd}(D))=h(\operatorname{Bd}(D))$, *for all* $h \in w(A(f))$.

LEMMA 3 (SEE [10]). *There is a map* $\sigma \colon w(A(f)) \to E(D, E^2)$ *such that* $\sigma(h)|\operatorname{Bd}(D) =h$, *for all* $h \in w(A(f))$.

Proof. For each $h \in w(A(f))$, let $h_1 \colon \operatorname{Bd}(D) \twoheadrightarrow \operatorname{Bd}(D)$ be given by

$$h_1 = [\sigma'(h)]^{-1} \circ h$$

(σ' is the map of Lemma 2). Extend h_1 to a homeomorphism $H_1 \colon D \twoheadrightarrow D$ by $H_1(r, \theta)=(r, \arg(h_1(1, \theta)))$, (r, θ) polar coordinates. Finally, let $\sigma(h)=\sigma'(h) \circ H_1$. The continuity of σ' implies the continuity of σ.

Noting in the above proof that $h_1(1)=1$, we obtain

COROLLARY 4. $w(A(f))$ *is homeomorphic to* $V(A(f)) \times \overline{H}(\operatorname{Bd}(D))$, *where* $\overline{H}(\operatorname{Bd}(D))$ *is the space of all orientation preserving homeomorphisms of* $\operatorname{Bd}(D)$ *onto* $\operatorname{Bd}(D)$ *which are fixed at* 1.

Proof. If $h \in w(A(f))$, the mapping sending h to $(\sigma'(h), [\sigma'(h)]^{-1} \circ h)$ is the required homeomorphism.

Recall that if Y is a spanning arc of the disk D then $E(Y, D)$ is the space of all embeddings of Y into D which are fixed on the endpoints of Y and which take $Y \cap \operatorname{Int}(D)$ into $\operatorname{Int}(D)$. For the next lemma we think of D as the set

$$\{(x, y) \in E^2 : 0 \leqq x \leqq 1, -1 \leqq y \leqq 1\},$$

Y as the arc

$$\{(x, y) \in E^2 : 0 \leqq x \leqq 1, y = 0\},$$

and define disks

$$\begin{aligned} Z_0 &= \{(x, y) \in E^2 : 0 \leqq x \leqq 1, -2 \leqq y \leqq 2\}, \\ Z_1 &= \{(x, y) \in E^2 : 0 \leqq x \leqq 1, -2 \leqq y \leqq 0\}, \\ Z_2 &= \{(x, y) \in E^2 : 0 \leqq x \leqq 1, 0 \leqq y \leqq 2\}. \end{aligned}$$

LEMMA 5. *There are maps* $\gamma: E(Y, D) \to E(Z_1, Z_0)$, *and* $\alpha: E(Y, D) \to E(Z_2, Z_0)$ *such that* $\alpha(f)|Y = \gamma(f)|Y = f$, $\gamma(f)|\operatorname{Bd}(Z_0) \cap \operatorname{Bd}(Z_1) = \operatorname{Id}$, *and* $\alpha(f)|\operatorname{Bd}(Z_0) \cap \operatorname{Bd}(Z_2) = \operatorname{Id}$, *all* $f \in E(Y, D)$.

Proof. Follows from Lemma 3.

COROLLARY 6. $E(Y, D) \times H(Z_1) \times H(Z_2)$ *is homeomorphic to* $\overline{H}(Z_0)$, *where* $\overline{H}(Z_0) = \{h \in H(Z_0) : h| Y \in E(Y, D)\}$.

Proof. If $h \in \overline{H}(Z_0)$, the map sending h to $(h|Y, h^{-1} \circ \gamma(h|Y), h^{-1} \circ \alpha(h|Y))$ is the required homeomorphism.

4. Some function spaces which are ANR's.

DEFINITION. Let $A(D, E^2)$ denote the space of all maps from $D = \{z \in \mathscr{C}^1 : |z| \leqq 1\}$ into E^2 which are holomorphic on $\operatorname{Int}(D)$.

LEMMA 7. $A(D, E^2)$ *is an ANR.*

Proof. $A(D, E^2)$ is a Banach space [20, Example 18.11], and hence an ANR [12, Chapter 3, Corollary 6.4].

DEFINITIONS. For r a real number, $0 < r < 1$, let $D(r) = \{z \in \mathscr{C}^1 : |z| \leqq r\}$. Let $A(D, E^2, r) = \{f \in A(D, E^2) : f \text{ is 1-1 on some neighborhood of } D(r)\}$.

LEMMA 8. $A(D, E^2, r)$ *is an ANR*, $0 < r < 1$.

Proof. We show that $A(D, E^2, r)$ is an open subset of $A(D, E^2)$. Suppose $f_1, f_2, \ldots$ is a sequence of elements of $A(D, E^2)$, f_0 is an element of $A(D, E^2, r)$, and $\{f_i\} \to f_0$. For each i, $i = 0, 1, 2, \ldots$, define $h_i: \operatorname{Int}(D) \times \operatorname{Int}(D) \to E^2$ by

$$\begin{aligned} h_i(x, y) &= (f_i(x) - f_i(y))/(x - y) \quad \text{if } x \neq y, \\ &= f_i'(x) \quad \text{if } x = y. \end{aligned}$$

For each i, h_i is continuous on $\operatorname{Int}(D) \times \operatorname{Int}(D)$ [23, p. 75]. Choose $\varepsilon > 0$ so that f_0 is 1-1 on $D(r + 2\varepsilon)$. Then $h_0(x, y) \neq 0$ for all $(x, y) \in D(r+\varepsilon) \times D(r+\varepsilon)$, since $f_0'(x) \neq 0$ for all $x \in D(r+\varepsilon)$ [23, p. 84]. But $\{h_i\} \to h_0$ uniformly on $D(r+\varepsilon) \times D(r+\varepsilon)$ (see proof of Theorem 7.3.1 in [23, p. 86]). Hence, for sufficiently large i,

$h_i(x, y) \neq 0$, all $(x, y) \in D(r+\varepsilon) \times D(r+\varepsilon)$, and so $f_i(x) \neq f_i(y)$. Hence, for large i, $f_i \in A(D, E^2, r)$. It follows that $A(D, E^2, r)$ is open in $A(D, E^2)$. Therefore, by [12, Chapter 3, Proposition 7.9] and Lemma 7, $A(D, E^2, r)$ is an ANR.

LEMMA 9. *$\bigcap_{n=2}^{\infty} A(D, E^2, 1-1/n)$ is an ANR.*

Proof. We shall show that $\bigcap_{n=2}^{\infty} A(D, E^2, 1-1/n)$ is a retract of $A(D, E^2, \frac{1}{2})$. It will then follow that $\bigcap_{n=2}^{\infty} A(D, E^2, 1-1/n)$ is an ANR [12, Chapter 3, Proposition 7.7]. Define $\lambda: A(D, E^2, \frac{1}{2}) \to (\frac{1}{2}, 1]$ by $\lambda(f) = \operatorname{Sup}\{r : f \text{ is 1-1 on } D(r)\}$. It will be shown below that λ is continuous. Define $R: A(D, E^2, \frac{1}{2}) \to \bigcap A(D, E^2, 1-1/n)$ by $R(f)(z) = f(\lambda(f)\cdot z)$, all $z \in D \subset \mathscr{C}^1$, all $f \in A(D, E^2, \frac{1}{2})$. Since $\lambda(f) = 1$ for all $f \in \bigcap A(D, E^2, 1-1/n)$, it follows that R is a retraction, provided λ is continuous.

Suppose λ is not continuous. Suppose $\{f_n\} \to f_0$, but $\{\lambda(f_n)\} \not\to \lambda(f_0)$ for some sequence $f_0, f_1, \ldots$ of elements of $A(D, E^2, \frac{1}{2})$.

Case 1. For infinitely many n, $\lambda(f_0) - \lambda(f_n) > \varepsilon$ for some $\varepsilon > 0$. Choose t so that $\lambda(f_0) - \varepsilon < t < \lambda(f_0)$. Then $f_0 | D(t)$ is 1-1. But then, as in the proof of Lemma 8, $f_n | D(t)$ is 1-1 for large n. This contradicts the assumption that $\lambda(f_n) < t$ for infinitely many n.

Case 2. For infinitely many n, $\lambda(f_n) - \lambda(f_0) > \varepsilon$ for some $\varepsilon > 0$. Choose t so that $\lambda(f_0) < t < \lambda(f_0) + \varepsilon$. Then for infinitely many $n, f_n | D(t)$ is 1-1. But $\{f_n\} \to f_0$, so by a simple argument using Rouche's theorem (or see [21, p. 91]), $f_0 | D(t)$ is 1-1. This contradicts the assumption that $\lambda(f_0) < t$. The proof of Lemma 9 is complete.

For our next proof we need the following theorem of Hanner.

THEOREM 10 (HANNER [11, THEOREM 7.2]). *A separable metric space X is an ANR provided there exist* (1) *a sequence of ANR's $Y_1, Y_2, \ldots$,* (2) *a sequence of maps $\phi_i: X \to Y_i$, $i = 1, 2, \ldots$,* (3) *a sequence of maps $\psi_i: Y_i \to X$, $i = 1, 2, \ldots$, and* (4) *a sequence of homotopies $H^i: X \times I \to X$ such that $H^i(x, 0) = x$, $H^i(x, 1) = \psi_i\phi_i(x)$, all $x \in X$, and $H^1, H^2, \ldots$ converges to the identity mapping $X \to X$.*

DEFINITION. *$H^1, H^2, \ldots$ converges to the identity mapping $X \to X$* if for any point $x_0 \in X$ and any neighborhood V of x_0 there is another neighborhood W of x_0 and an integer N such that $x \in W$ and $n \geqq N$ imply $H^n(x, t) \in V$ for all t.

Recall that $AE(D, E^2)$ is the set $\{f \in A(D, E^2) : f \text{ is an embedding}\}$.

LEMMA 11. *$AE(D, E^2)$ is an ANR.*

Proof. We shall use Theorem 10. Let $X = AE(D, E^2)$. For i a positive integer, let $Y_i = \bigcap_{n=2}^{\infty} A(D, E^2, 1-1/n)$, let $\phi_i: X \to Y_i$ be the inclusion map, let $\psi_i: Y_i \to X$ be defined by $\psi_i(f)(z) = f((1-1/i)\cdot z)$, all $z \in D \subset \mathscr{C}^1$, and let $H^i: X \times I \to X$ be defined by $H^i(f, t)(z) = f((1-t)\cdot z + t\cdot(1-1/i)\cdot z)$, all $z \in D$, $t \in I$. With these definitions it is easily checked that the hypotheses of Theorem 10 are satisfied, and so $X = AE(D, E^2)$ is an ANR.

LEMMA 12. *$E(\mathrm{Bd}\,(D), E^2)$ is an ANR.*

Proof. $E(\mathrm{Bd}\,(D), E^2)$ is the union of the space $E_0(\mathrm{Bd}\,(D), E^2)$ of orientation preserving embeddings of $\mathrm{Bd}\,(D)$ into E^2 and the space of orientation reversing embeddings of $\mathrm{Bd}\,(D)$ into E^2. Since these two spaces are homeomorphic and open in $E(\mathrm{Bd}\,(D), E^2)$, it suffices to show that $E_0(\mathrm{Bd}\,(D), E^2)$ is an ANR [12, Chapter 3, Theorem 8.1].

Given $f \in E_0(\mathrm{Bd}\,(D), E^2)$, choose an annulus $A(f)$ such that $f(\mathrm{Bd}\,(D)) \subset \mathrm{Int}\,(A(f))$, and $f(\mathrm{Bd}\,(D))$ separates $\mathrm{Bd}\,(A(f))$. Define $x(f)$, $w(A(f))$, $V(A(f))$ as in the paragraph preceding Lemma 2. By Corollary 4, $w(A(f))$ is homeomorphic to $V(A(f)) \times \overline{H}(\mathrm{Bd}\,(D))$. It is clear that $\overline{H}(\mathrm{Bd}\,(D))$ is homeomorphic to the space $H(I)$ of orientation preserving homeomorphisms of the interval [0, 1] onto itself. But $H(I)$ is homeomorphic to l_2 [2], and thus is an ANR [12, Chapter 3, Corollary 6.4]. The space $Q = \{h \in AE(D, E^2) : h(0) = x(f)\}$ is clearly a retract of $AE(D, E^2)$ and thus an ANR by Lemma 11. $V(A(f))$ is an open subset of Q, hence $V(A(f))$ is an ANR [12, Chapter 3, Proposition 7.9]. Therefore $w(A(f))$ being the product of two ANR's, is an ANR [12, Chapter 3, Proposition 7.6].

Finally, $w(A(f))$ is open in $E_0(\mathrm{Bd}\,(D), E^2)$, and

$$E_0(\mathrm{Bd}\,(D), E^2) = \sum_{f \in E_0(\mathrm{Bd}(D, E^2))} w(A(f)),$$

hence $E_0(\mathrm{Bd}\,(D), E^2)$ is an ANR [12, Chapter 3, Theorem 8.1]. The proof of Lemma 12 is complete.

Lemma 13. *$E(D, E^2)$ is an ANR.*

Proof. As in Lemma 12 it suffices to show that $E_0(D, E^2)$ is an ANR.

Given $g \in E_0(D, E^2)$, define $g|\mathrm{Bd}\,(D) = g_1 \in E_0(\mathrm{Bd}\,(D), E^2)$. For each $g \in E_0(D, E^2)$, let $A(g_1)$ be an annulus such that $g_1(\mathrm{Bd}\,(D)) \subset \mathrm{Int}\,(A(g_1))$, and $g_1(\mathrm{Bd}\,(D))$ separates $\mathrm{Bd}\,(A(g_1))$. Define $w(A(g_1)) \subset E_0(\mathrm{Bd}\,(D), E^2)$ as in the paragraph preceding Lemma 2. Define $T(A(g_1)) = \{h \in E_0(D, E^2) : h_1 \in w(A(g_1))\}$. Then $T(A(g_1))$ is open in $E_0(D, E^2)$, and $E_0(D, E^2) = \sum_{g \in E_0(D, E^2)} T(A(g_1))$. It suffices, therefore, to show that $T(A(g_1))$ is an ANR. Let $\sigma\colon w(A(g_1)) \to E_0(D, E^2)$ be the map of Lemma 3 such that $\sigma(f)|\mathrm{Bd}\,(D) = f$, all $f \in w(A(g_1))$. Define $\gamma\colon T(A(g_1)) \twoheadrightarrow w(A(g_1)) \times H(D)$ by $\gamma(h) = (h_1, h^{-1} \circ \sigma(h_1))$. It is easily checked that γ is a homeomorphism. $w(A(g_1))$ is an ANR by the proof of Lemma 12; $H(D)$ is an ANR by [18]. Hence $T(A(g_1))$ is an ANR and the proof of Lemma 13 is complete.

Suppose Y is a spanning arc of the disk D. Then

Lemma 14. *$E(Y, D)$ is an ANR.*

Proof. By Corollary 6, $E(Y, D) \times H(Z_1) \times H(Z_2)$ is homeomorphic to $\overline{H}(Z_0)$, where Z_0, Z_1, Z_2 are disks, $D \subset Z_0$, and $\overline{H}(Z_0) = \{h \in H(Z_0) : h|Y \in E(Y, D)\}$. $H(Z_0)$ is an ANR by [18], so $\overline{H}(Z_0)$, being an open subset of $H(Z_0)$, is an ANR. But then $E(Y, D)$ is a retract of $\overline{H}(Z_0)$, hence $E(Y, D)$ is an ANR.

5. **More on conformal mapping.** In this section we describe a procedure for extending, in a canonical way, an embedding of the boundary of an annulus to an embedding of the entire annulus. This procedure is used in [10], [15], [19], and [22].

It is well known that if G is a closed (topological) annulus in $\mathscr{C}^1$ then there is a unique real number $r>1$ and a homeomorphism f of the annulus $A(C_1, C_r) = \{z \in \mathscr{C}^1 : 1 \leqq |z| \leqq r\}$ onto G such that f is holomorphic on $\operatorname{Int}(A(C_1, C_r))$. Further, f is uniquely determined by the image of one boundary point [1, Chapter 5, §3.1].

As in §1, we have a continuity property for such f's. In the statement below we let $A(J, L)$ denote the closed annulus bounded by the simple closed curves J and L, with J in the bounded complementary domain of L, and we let

$$C_r = \{z \in \mathscr{C}^1 : |z| = r\},$$

r a real number.

THEOREM 15. *Given*: (1) *annulii* $A(J_n, L_n)$, $n=0, 1, 2, \ldots$, (2) *homeomorphisms* $f_n\colon A(C_1, C_{r_n}) \twoheadrightarrow A(J_n, L_n)$, *with* f_n *holomorphic on* $\operatorname{Int}(A(C_1, C_{r_n}))$, $n=0, 1, 2, \ldots$, (3) $\{J_n\} \to J_0$, $\{L_n\} \to L_0$, *0-regularly, and* (4) $\{f_n(1)\} \to f_0(1)$.

Then: (a) $\{r_n\} \to r_0$ (*radii of outer boundaries of* $A(C_1, C_{r_n})$ *converge*), *and* (b) *given a number* $\varepsilon>0$, *there is a number* $\delta>0$ *and an integer* N *such that if*, $\operatorname{dist}(x, y) < \delta$, *then* $\operatorname{dist}(f_n(x), f_0(y)) < \varepsilon$ *whenever* $n>N$, $x \in A(C_1, C_{r_n})$, $y \in A(C_1, C_{r_0})$.

This theorem may be proved by methods similar to those in the proof of Lemma 12.1 in [21, Chapter 3, §12].

(★) DEFINITION. Let AN denote the space of all orientation preserving embeddings g of C_1 into $\operatorname{Int}(A(C_{1/2}, C_2))$ such that: (a) $g(C_1)$ separates $C_{1/2}$ and C_2, and (b) $g(1)$ lies in the interior of a small disk O, centered at 1, so that the angle $\theta(g)$ at the origin from $g(1)$ to 2 ($=2+0i$) satisfies $-\pi/4 < \theta(g) < \pi/4$.

Now if $g \in AN$ there is an annulus $A(C_1, C_r)$ and an embedding $G\colon A(C_1, C_r) \to E^2$, holomorphic on $\operatorname{Int}(A(C_1, C_r))$, such that $G(1)=g(1)$, $G(C_1)=g(C_1)$, and $G(C_r)=C_2$. If we precede G with a radial homeomorphism R taking $A(C_1, C_2)$ onto $A(C_1, C_r)$ and let $\lambda'(g)=G \circ R$, then Theorem 15 gives us

LEMMA 16. *There is a map* $\lambda'\colon AN \to E(A(C_1, C_2), E^2)$ *such that* $\lambda'(g)(1)=g(1)$, $\lambda'(g)(C_1)=g(C_1)$, *and* $\lambda'(g)(C_2)=C_2$ *for all* $g \in AN$.

Suppose again that $g \in AN$. Let $g_1\colon C_1 \twoheadrightarrow C_1$ be the homeomorphism $g_1=g^{-1} \circ \lambda'(g)|C_1$. Let $g_2\colon C_2 \twoheadrightarrow C_2$ be $g_2=\lambda'(g)|C_2$. We may extend g_1 and g_2 to a homeomorphism $G(n, m)$ of $A(C_1, C_2)$ onto itself by sending (r, θ) (polar coordinates), to $(r, (2-r)(g_1(1, \theta)+2n\pi)+(r-1)(g_2(2, \theta)+2m\pi))$ where n and m are integers. If we let $\lambda(g, n, m)=\lambda'(g) \circ [G(n, m)]^{-1}$ we see that $\lambda(g, n, m)=g$ on C_1, and $\lambda(g, n, m)=\mathrm{Id}$ on C_2. Finally, let $\lambda(g)=\lambda(g, n(g), m(g))$, where the integers $n(g)$ and $m(g)$ are chosen as follows. If X is the segment of the real axis from 1 to 2, choose $n(g), m(g)$ so that the "angle change" along $\lambda(g, n(g), m(g))(X)$ is equal to the angle $\theta(g)$, $-\pi/4 < \theta(g) < \pi/4$, from $g(1)$ to 2 (see [10, p. 522] for a description of angle change); equivalently, choose $n(g)$, $m(g)$ so that the "circulation index" (see [23, Chapter 5, §1]) of $\lambda(g, n(g), m(g))|X$ about the origin has imaginary part $\theta(g)$. The continuity of $\lambda'(g)$ and $\theta(g)$ as functions of g imply the continuity of λ. Thus we have

LEMMA 17. *There is a map* $\lambda\colon AN \to E(A(C_1, C_2), E^2)$ *such that* $\lambda(g)|C_1 = g$, *and* $\lambda(g)|C_2 = \mathrm{Id}$ *for all* $g \in AN$.

For further discussion see [19].

6. $H(M^2)$ **is an** ANR. In this section we drop the superscript 2 and let M denote a compact, metric, 2-manifold.

THEOREM 18. $H(M)$ *is an ANR.*

Proof. Since $H(M)$ is a topological group it is homogeneous. It is sufficient, therefore, to find an open subset of $H(M)$ which is an ANR and which contains the identity map [12, Chapter 3, Theorem 8.1].

As in [10] we proceed by induction on the number of cells in a cellular decomposition of M. Let $D_1, \ldots, D_n$ be a finite collection of disks such that (a) $M = \sum_{i=1}^n D_i$, (b) $D_i \cap D_j$ is either empty or an arc in $\mathrm{Bd}\,(D_i) \cap \mathrm{Bd}\,(D_j)$, $i \neq j$, (c) $\mathrm{Bd}\,(D_i) \cap \mathrm{Bd}\,(M)$ is either empty, a simple closed curve, or a finite collection of arcs, and (d) each D_i is the underlying point set of a subcomplex in a triangulation T of M. Such a decomposition may be obtained, for example, by taking each D_i to be the star, in the second barycentric subdivision of T, of the barycenter of a simplex of T.

If there is only one cell in the decomposition, then $H(M) = H(D_1)$ is an ANR by [18]. Assume that the theorem is true for manifolds which have a decomposition into fewer than n elements. Let $\{D_1, \ldots D_n\}$ be a cellular decomposition of M with n elements. Let $D = D_1$.

Case 1. $D \cap \mathrm{Bd}\,(M) = \varnothing$. Let N be a regular neighborhood of $\mathrm{Bd}\,(D)$ in M (see [13, p. 57]), such that $N \cap \mathrm{Bd}\,(M) = \varnothing$. Then $D + N$ is a disk, hence N is an annulus ($D + N = D + N'$, where N' is a regular neighborhood of D in M, and $D + N' \searrow D \searrow 0$, so $D + N'$ is a 2-cell, see [13, p. 57]). We may think of $D + N$ as being embedded in E^2, with $N = A(C_{1/2}, C_2)$ and $\mathrm{Bd}\,(D) = C_1$ (unit circle). Let AN be the subset of $E_0(C_1, \mathrm{Int}\,(A(C_{1/2}, C_2)))$, given in §5, Definition (★). Let

$$H_I(M) = \{F \in H(M) : F|\mathrm{Bd}\,(D) \in AN \text{ and } F(D) \subset \mathrm{Int}\,(D+N)\}.$$

Note that $H_I(M)$ is open in $H(M)$ and contains the identity map.

If we let $M' = \sum_{i=2}^n D_i$, and let

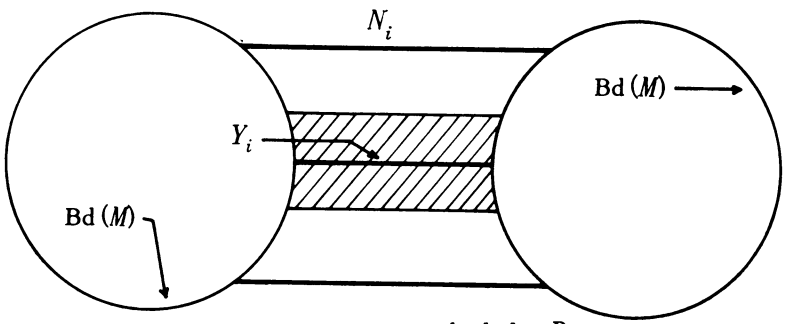
$$E_I(D, \mathrm{Int}\,(D+N)) = \{f \in E(D, \mathrm{Int}\,(D+N)) : f|\mathrm{Bd}\,(D) \in AN\},$$

then $H_I(M)$ is homeomorphic to $H(M') \times E_I(D, \mathrm{Int}\,(D+N))$. To see this let $\lambda\colon AN \to E(A(C_1, C_2), E^2)$ be the map of Lemma 17 such that $\lambda(f)|C_1 = f$, $\lambda(f)|C_2 = \mathrm{Id}$, all $f \in AN$. Define $\bar{\lambda}\colon AN \to E(M', M'+N)$ by $\bar{\lambda}(f) = \lambda(f)$ on $M' \cap N$, and $\bar{\lambda}(f) = \mathrm{Id}$ on $M' - N$. Then the map sending $F \in H_I(M)$ to $(F^{-1} \circ \bar{\lambda}(F|\mathrm{Bd}\,(D)), F|D)$ is the required homeomorphism.

But $E_I(D, \operatorname{Int}(D+N))$ is an open subset of $E(D, E^2)$, and thus an ANR by Lemma 13. $H(M')$ is an ANR by our inductive hypothesis. Therefore $H_I(M)$ is an ANR, being the product of two ANR's.

Case 2. $D \cap \operatorname{Bd}(M) \neq \varnothing$. If $D \cap \operatorname{Bd}(M)$ is a simple closed curve, then D is a component of M. Hence $H(M)=H(D)\times H(M')$, where $M'=\sum_{i=2}^{n} D_i$, and $H(M)$ is an ANR by [18] and our inductive hypothesis.

Suppose, then, $D \cap \operatorname{Bd}(M)$ is a finite collection of arcs. Let $Y_1, \dots, Y_m$ be the (disjoint) arcs making up the closure of $\operatorname{Bd}(D)-\operatorname{Bd}(M)$. For each i, $1 \leqq i \leqq m$, we may choose a regular neighborhood N_i of Y_i in M such that $N_i \cap N_j=\varnothing$, $i\neq j$, and $N_i \cap \operatorname{Bd}(M)$ is a regular neighborhood of $\operatorname{Bd}(Y_i)$ in $\operatorname{Bd}(M)$ and hence consists of two arcs in $\operatorname{Bd}(M)$ (see [13, p. 64]). Thus N_i is a 2-cell (since $N_i \searrow Y_i \searrow 0$) which meets $\operatorname{Bd}(M)$ in two disjoint arcs. Let R_i be a smaller regular neighborhood of Y_i such that $\operatorname{Bd}(R_i) \cap \operatorname{Bd}(N_i)=\operatorname{Bd}(R_i) \cap \operatorname{Bd}(M) \subset \operatorname{Int}(\operatorname{Bd}(N_i) \cap \operatorname{Bd}(M))$.



shaded = R_i

Let $E(Y_i, R_i)$, $1 \leqq i \leqq m$, be the space of embeddings of Y_i into R_i which are fixed on the endpoints of Y_i and which take $Y_i \cap \operatorname{Int}(R_i)$ into $\operatorname{Int}(R_i)$. By Lemma 5 there are maps $\gamma_i\colon E(Y_i, R_i) \to E(D \cap N_i, N_i)$ and $\alpha_i\colon E(Y_i, R_i) \to E(M' \cap N_i, N_i)$, $M'=\sum_{i=2}^{n} D_i$, such that $\gamma_i(f)=\alpha_i(f)=f$ on Y_i, $\gamma_i(f)=\mathrm{Id}$ on $D \cap \operatorname{Bd}(N_i)$, and $\alpha_i(f)=\mathrm{Id}$ on $M' \cap \operatorname{Bd}(N_i)$, all $f \in E(Y_i, R_i)$.

Let $H_I(M)=\{F \in H(M) : F\mid Y_i \in E(Y_i, R_i),\ 1 \leqq i \leqq m\}$. Note that $H_I(M)$ is open in $H(M)$ and contains the identity map.

Define $\gamma\colon H_I(M) \to E(D, D+\sum N_i)$ by $\gamma(F)=\gamma_i(F\mid Y_i)$ on $D \cap N_i$, $1 \leqq i \leqq m$, and $\gamma(F)=\mathrm{Id}$ on $D-\sum N_i$. Define $\alpha\colon H_I(M) \to E(M', M'+\sum N_i)$ by $\alpha(F)=\alpha_i(F\mid Y_i)$ on $M' \cap N_i$, $1 \leqq i \leqq m$, and $\alpha(F)=\mathrm{Id}$ on $M'-\sum N_i$. But now the map sending $F \in H_I(M)$ to $(F\mid Y_1, \dots, F\mid Y_m, F^{-1}\circ \gamma(F), F^{-1}\circ \alpha(F))$ is a homeomorphism of $H_I(M)$ onto $E(Y_1, R_1)\times \cdots \times E(Y_m, R_m)\times H(D)\times H(M')$. $E(Y_i, R_i)$, $1 \leqq i \leqq m$, is an ANR by Lemma 14; $H(D)$ is an ANR by [18], and $H(M')$ is an ANR by our induction hypothesis. Therefore $H_I(M)$ is an ANR, and the proof of Theorem 18 is complete.

COROLLARY 19. *If no component of M is a 2-sphere, torus, projective plane, or Klein bottle, then the identity component of $H(M)$ is an absolute retract (and thus is contractible).*

Proof. The identity component of $H(M)$ is homotopically trivial by [9]. But every homotopically trivial connected ANR is an absolute retract [12, Corollary 8.5, p. 219].

REMARK. In [25, p. 34] Earle and Eells remark that if $H_0(M)$, the identity component of $H(M)$, is an ANR, then the inclusion map of the identity component of the space of diffeomorphisms on (a suitably smooth) M into $H_0(M)$ is a homotopy equivalence.

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