

THE OSCILLATION OF AN OPERATOR

BY
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Abstract. Foiaş and Singer introduced the oscillation of a bounded linear operator mapping $C(S)$ into a Banach space. Using this concept we define a generalization of the Fredholm operators T with $\mathcal{K}(T) < \infty$ and a corresponding perturbation class which contains the weakly compact operators. We show that a bounded linear operator on c is a conservative summability matrix which sums every bounded sequence if and only if it has zero oscillation at infinity.

The oscillation of a bounded linear operator mapping $C[0, 1]$ into a Banach space X is defined in [6] and is used there to discuss two classes of operators, almost diffuse operators and countably almost diffuse operators, each class containing the weakly compact operators. These operators have recently been useful in approximation theory [2, §3].

In the following S will always denote a compact Hausdorff space, X a Banach space, and we will consider the bounded linear operators mapping $C(S)$ into X . Such operators have interesting properties which can be defined in terms of their oscillation: a generalized notion of a Fredholm operator T with index $\mathcal{K}(T) < \infty$, a corresponding perturbation class and a basic type of conservative summability method.

We restate the definitions of Foiaş and Singer [6]. Let $T: C(S) \rightarrow X$ be a bounded linear operator, the *oscillation of T at a point s in S* , $\omega_T(s) = \omega(T, s)$, is the supremum over all positive α satisfying the following: for every neighborhood U of s there is a function f of norm one in $C(S)$, which vanishes outside of U , with $\|Tf\| \geq \alpha$. The operator T is *almost diffuse* if the set of diffusion points of T , $D(T) = \omega_T^{-1}(0)$, is dense and T is *countably almost diffuse* if the set of concentration points, $\gamma(T) = \omega_T^{-1}(0, \infty)$, is countable.

It is easy to show that the set $\omega_T^{-1}([a, \infty))$ is a closed set, so $\omega_T(s)$ is a nonnegative upper semicontinuous function of s which is bounded by $\|T\|$. Consequently $D(T) = \omega_T^{-1}(0)$ is a G_δ and thus, if $D(T)$ is dense, i.e. if T is almost diffuse, its complement $\gamma(T)$ is of first category [6, Theorem 1, p. 437]. In general any nonnegative upper semicontinuous function may occur as the oscillation of a linear operator. For example let h be a nonnegative upper semicontinuous function on $[0, 1]$. Define A mapping $C[0, 1]$ to $B[0, 1]$, the bounded functions on $[0, 1]$ with

Received by the editors March 5, 1971.

AMS 1970 subject classifications. Primary 47A55, 47B30, 47B99, 40H05.

Key words and phrases. Bounded linear operator on $C(S)$, oscillation of an operator, Fredholm operator, ramming operator, weakly compact operator, strictly singular operator, φ_+ operator, conservative summability method, coercive summability method.

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the supremum norm, by $Af(s) = h(s)f(s)$. Since the range of A is separable there is a linear isometry B of it into $C[0, 1]$ [1, p. 185]. The operator BA on $C[0, 1]$ has $\omega(BA, \cdot) = h(\cdot)$. By [11, Corollary 3, p. 288], or [12, Theorem 1, p. 656], a similar example can be constructed on $C(S)$ for any uncountable compact metric space S .

A minor error, which has repeatedly arisen in connection with almost diffuse operators, concerns the validity of the equation

$$(1) \quad \|I+T\| = 1 + \|T\|$$

which holds for I the identity operator and for all almost diffuse operators T on $C(S)$ [6, Theorem 3, p. 445]. The relation (1) was first proved for compact operators T on $C[0, 1]$ by Daugavet; the review in the Mathematical Reviews indicated that (1) was probably true for compact T on $C(S)$ [3]. Again, see a remark given on p. 446 of [6]. Actually, their Theorem 3 is correct and the map given as a counter-example is not almost diffuse. In connection with his study of another class of operators, Krasnosel'skiĭ claimed that (1) holds for all compact T on $C(S)$, S any compact metric space [9]. If S is a compact Hausdorff space with an isolated point s_0 , then the map $Kf = -f(s_0)\chi_{s_0}$ on $C(S)$ is compact but $\|I+K\| = 1$. The remarks above are correct when S has no isolated points and need a simple modification otherwise.

We introduce a subclass of the countably almost diffuse operators distinguished by a basic property.

DEFINITION 1. Let S be a compact Hausdorff space, X a Banach space and $T: C(S) \rightarrow X$ a bounded linear operator. Then T is a c_0 operator if the nonzero values in $\{\omega(T, s) : s \text{ in } S\}$ are finite or a sequence converging to zero.

The set of c_0 operators is a closed subspace. A c_0 operator when multiplied on the left by a bounded linear operator is still a c_0 operator. As we will see in Example 1 the c_0 operators on $C(S)$ do not form a two-sided ideal.

The following type of operator was studied by Krasnosel'skiĭ [9]: for S a compact metric space and T a bounded linear operator on $C(S)$ (or from $C(S)$ into X), T is ramming if $\|Tf_n\| \rightarrow 0$ for each uniformly bounded sequence of functions $\{f_n\}$ in $C(S)$ with nonoverlapping supports converging to a point in S .

THEOREM 1. Let $T: C(S) \rightarrow X$ be a bounded linear operator.

(a) If T is weakly compact, then T is a c_0 operator.

(b) If S is a compact metric space and T is a ramming operator, then T is a c_0 operator. In general a c_0 operator is not ramming, but if $\omega(T, s) = 0$ for all non-isolated points s , then T is ramming.

Proof. Suppose T is weakly compact and assume it is not a c_0 operator. Then we can find a sequence $\{s_n\}$ of distinct points of S and a pairwise disjoint sequence $\{O_n\}$ of open sets with s_n in O_n and $\omega(T, s_n) > \epsilon > 0$ for all n . Since $\omega(T, s_n) > \epsilon$, there is a function f_n in $C(S)$ of norm one which vanishes outside of O_n and which satisfies $\|Tf_n\| > \epsilon$. Consider the subspace $M = \overline{\text{sp}}(f_n)$. The map $A: c_0 \rightarrow M$ by

$A(\{\alpha_n\}) = \sum \alpha_n f_n$ is easily shown to be a linear isometry of c_0 onto M . The restriction $T|_M$ of T to M is weakly compact, M is isomorphic to c_0 , and thus $T|_M$ is compact [10, 10 Corollary]; the sequence $\{f_n\}$ converges weakly to zero, hence $\{Tf_n\}$ converges to zero in norm. This contradiction shows that T is a c_0 operator.

Suppose that $T: C(S) \rightarrow X$, S a compact metric space, is ramming and assume that it is not a c_0 operator. There is a sequence $\{s_n\}$ of distinct points of S with $\omega(T, s_n) > \epsilon > 0$, for all n , and by passing to a subsequence we may suppose that $\{s_n\}$ converges to a point s_0 . By passing to a further subsequence we can find pairwise disjoint neighborhoods $S(s_n, r_n)$ with $r_n \rightarrow 0$. By assumption there is a function f_n of norm one in $C(S)$ with support in $S(s_n, r_n)$ for which we have $\|Tf_n\| > \epsilon$. Because T is ramming we must have $\|Tf_n\| \rightarrow 0$, a contradiction.

Suppose that T has $\omega(T, s) = 0$ for all nonisolated points s . Let $\{f_n\}$ be a sequence of uniformly bounded functions in $C(S)$ with pairwise disjoint supports $\{U_n\}$ converging to s_0 . Since s_0 is not isolated, given $\epsilon > 0$ there is a neighborhood U of s_0 with $\|Tf\| \leq \epsilon \|f\|$ for all f in $C(S)$ with support in U . The sequence $\{U_n\}$ is eventually in U which tells us that $\|Tf_n\|$ is eventually bounded by $\epsilon \|f_n\|$, i.e. $\|Tf_n\| \rightarrow 0$ and T is ramming. The example given below will show that a c_0 operator is not necessarily ramming. Note that the operator T defined on $C[0, 1]$ by $Tf = f(0)1$ is ramming and has $\omega(T, 0) = 1$.

EXAMPLE 1. Consider the operator C corresponding to $(C, 1)$ summability on c , i.e. $C(\{a_n\}) = \{(a_1 + \dots + a_n)/n\}$. Writing $c = C(S)$, $S = \{\infty, 1, 2, 3, \dots\}$ the one-point compactification of the positive integers, C is a c_0 operator on this $C(S)$ because $\omega(C, n) = \|Ce_n\| = 1/n$, $e_n = \{\delta_{jn}\}_{j=1}^\infty$. Define B on C by

$$B(\{a_n\}) = (a_1, a_2, a_2, a_3, a_3, a_3, a_4, a_4, a_4, a_4, a_4, a_5, \dots).$$

Even though B is an isometry on c , CB is not a c_0 operator since $\|CB e_n\| = \frac{1}{2}$. Consider the sequence of functions in c , $f_n = \{f_n(j)\}$,

$$\begin{aligned} f_n(j) &= 0, & 1 \leq j < 2^{n-2}, \\ &= 1, & 2^{n-2} \leq j \leq 2^{n-1}, \\ &= 0, & 2^{n-1} < j. \end{aligned}$$

The sequence $\{f_n\}$ is a uniformly bounded sequence of functions with pairwise disjoint supports converging to ∞ , but as we have $\|Cf_n\| \geq \frac{1}{2}$ the operator C is not ramming. This idea can easily be extended to operators on, say, $C[0, 1]$. Define T on $C[0, 1]$ by

$$\begin{aligned} Tf(t) &= \frac{1}{n} \sum_{j=1}^n f(1/j), & t = 1/n, \\ &= f(0), & t = 0, \\ &= \text{linear} & \text{otherwise.} \end{aligned}$$

It is easy to show that T is a c_0 operator on $C[0, 1]$ but is not ramming.

A characterization of almost diffuse operators given in [6, Proposition 1, p. 436] can be restated in the form: A bounded linear operator T on $C[0, 1]$ is almost diffuse if and only if given a closed ideal I in $C[0, 1]$ and $\varepsilon > 0$ there is a closed ideal $J \subseteq I$ with $\|T|_J\| \leq \varepsilon$. There is a provocative similarity between this and a characterization of strictly singular operators, keeping in mind that all closed ideals in $C[0, 1]$ are infinite dimensional, namely [7, III.2.1 Theorem, p. 84]: For a bounded linear operator $T: X \rightarrow Y$, between Banach spaces, the following are equivalent: (1) T is strictly singular. (2) For each infinite dimensional closed subspace M of X and $\varepsilon > 0$ there is a closed infinite dimensional subspace $N \subseteq M$ with $\|T|_N\| \leq \varepsilon$. (3) For each infinite dimensional closed subspace M of X there is a closed infinite dimensional subspace $N \subseteq M$ with $T|_N$ compact. The operator of Example 1(a) of [6] is almost diffuse, in fact a c_0 operator, but is not compact when restricted to any closed ideal in $C[0, 1]$, showing that we cannot expect a condition similar to (3), with closed subspaces replaced by closed ideals, to characterize almost diffuse operators. Another possibility is suggested by a class of operators studied by Herman [8]; those operators described by condition (3) above with "compact operator" replaced by "weakly compact operator." Our example below shows that T may be a c_0 operator on $C[0, 1]$ and yet have $T|_I$ not weakly compact for any ideal I in $C[0, 1]$.

EXAMPLE 2. For a in $(0, 1)$ define the operator T_a on $C[0, 1]$ by

$$\begin{aligned} T_a f(t) &= f(a), & 0 \leq t \leq a, \\ &= \frac{1}{t-a} \int_a^t f(x) dx, & a < t \leq 1. \end{aligned}$$

The operator obtained by setting $a=0$ is discussed in [6, Example 2, p. 443]. The oscillation $\omega(T_a, s)$ is zero for $s \neq a$, so T_a is a c_0 operator. Let c and d be given satisfying $0 < c < a < d < 1$ and define the functions $f(a, n, \cdot)$, for n with $a+1/n < (a+d)/2$, by

$$\begin{aligned} f(a, n, t) &= 0, & 0 \leq t \leq a, \\ &= 1, & t = a+1/n, \\ &= 1, & t = (a+d)/2, \\ &= 0, & d \leq t \leq 1, \\ &= \text{linear} & \text{otherwise.} \end{aligned}$$

We compute

$$\begin{aligned} T_a f(a, n, t) &= 0, & 0 \leq t \leq a, \\ &= 1 - 1/2n(t-a), & a+1/n \leq t \leq (a+d)/2. \end{aligned}$$

For $a < t \leq (a+d)/2$ we have $T_a f(a, n, t) \rightarrow 1$ as $n \rightarrow \infty$, but $T_a f(a, n, a) = 0$. Thus $\{T_a f(a, n, \cdot)\}$ cannot have a weakly convergent subsequence and T_a is not weakly compact.

For our example we choose $\{a_n\}$ a sequence of distinct points dense in $(0, 1)$ and define

$$T = \sum \frac{1}{2^n} T_{a_n},$$

a c_0 operator since it is the limit of sums of c_0 operators. Let I be a given closed nonzero ideal in $C[0, 1]$; $I = \{f \text{ in } C[0, 1] : f(F) = 0\}$, F a closed subset of $[0, 1]$. For some index j , a_j does not belong to F . Let $\delta = 1/2^{j+2}$ and choose N so that $\sum_{n=1}^{\infty} 1/2^n < \delta$. There is a subinterval $[c, d]$ of $[0, 1]$ which is disjoint from F and contains a_j in its interior but does not contain a_n for $n \neq j$, $1 \leq n \leq N$. The functions $g_m(\cdot) = f(a_j, m, \cdot)$, defined for m satisfying $a_j + 1/m < \frac{1}{2}(a_j + d)$, belong to I . Let $J = \{f \text{ in } C[0, 1] : f([0, 1] - (c, d)) = 0\}$. For $n \neq j$ and $1 \leq n \leq N$ the operator $T_{a_n}|_J$ is compact because the Volterra operator $g(t) \rightarrow \int_{a_n}^t g(x) dx$ is compact on J and the operator $g(t) \rightarrow g(t)/(t - a_n)$ is a bounded linear operator on J . Thus the sequence $\{T_{a_n} \cdot g_m(\cdot)\}_{m=1}^{\infty}$ is an equicontinuous family of functions [5, Theorem 7, p. 266]. Hence for t close enough to a_j ,

$$|T_{a_n} g_m(t) - T_{a_n} g_m(a_j)| < \delta / \left(\sum_{j \neq p \leq N} 2^p \right)$$

holds uniformly for all g_m and for $n \neq j$, $1 \leq n \leq N$. Then

$$\begin{aligned} & \left| \frac{1}{2^j} T_{a_j} g_m(t) - \frac{1}{2^j} T_{a_j} g_m(a_j) - (T g_m(t) - T g_m(a_j)) \right| \\ & \leq 2 \sum_{N+1}^{\infty} 1/2^n + \sum_{j \neq n \leq N} 1/2^n |T_{a_n} g_m(t) - T_{a_n} g_m(a_j)| \leq 3\delta; \end{aligned}$$

thus

$$(2) \quad |T g_m(t) - T g_m(a_j)| \geq \left| \frac{1}{2^j} T_{a_j} g_m(t) - \frac{1}{2^j} T_{a_j} g_m(a_j) \right| - 3\delta.$$

For $a_j < t \leq \frac{1}{2}(a_j + d)$ the right-hand side of (2) tends to $1/2^j - 3\delta = 1/2^{j+2}$ as m tends to infinity. We see that no subsequence of $\{T g_m(\cdot)\}$ can converge weakly to a function continuous on $[0, 1]$ and conclude that $T|_I$ is not weakly compact.

The family of those operators on $C(S)$ which have a finite dimensional null manifold and a closed range (i.e. those T with index $\mathcal{K}(T) < \infty$, sometimes called φ_+ operators) is stable under perturbation by weakly compact operators since, on $C(S)$, the weakly compact operators and strictly singular operators coincide [7], [13]. This will not be true if we enlarge our perturbing class to the class of c_0 operators because a c_0 operator may have uncountably many eigenvalues; see Example 2 of [6] and Theorem V.1.8 of [7]. We introduce a set of operators which contain the φ_+ operators and which are stable under perturbation by c_0 operators.

DEFINITION 2. Let S be a compact Hausdorff space, X a Banach space and $T: C(S) \rightarrow X$ a bounded linear operator. The operator T is *concentrated* if $\omega(T, s)$ has only finitely many zeros and is otherwise bounded away from zero.

THEOREM 2. *A bounded linear operator $T: C(S) \rightarrow X$ has a finite dimensional null manifold and a closed range if and only if there are a finite number of points s_1, \dots, s_n in S with the property that, for $J = \{f \text{ in } C(S) : f(s_1) = \dots = f(s_n) = 0\}$, $T|_J$ has a bounded inverse. If T has a finite dimensional null manifold and a closed range, then T is concentrated.*

Proof. Let J be as above and suppose that $T|_J$ has a bounded inverse. Then the range of T is closed [7, V.1.5 Lemma, p. 112], and the dimension of the null manifold of T cannot exceed n .

Conversely suppose that T has a null manifold N of dimension n and a closed range. Choose any f_1 in N with $1 = \|f_1\| = f_1(s_1)$. Any g in N can be written as $h + g(s_1)f_1$, $h(s_1) = 0$, and so we can choose f_2 in N with $f_2(s_1) = 0$, $1 = \|f_2\| = f_2(s_2)$. Any g in N can be written as $h + g(s_1)f_1 + [g(s_2) - g(s_1)f_1(s_2)]f_2$, $h(s_1) = h(s_2) = 0$. Continue, obtaining f_1, \dots, f_n in $C(S)$, $N = \text{sp}(f_1 \cdots f_n)$ and s_1, \dots, s_n in S , $f_i(s_j) = \delta_{ij}$ for $j \leq i$. Let $J = \{f \text{ in } C(S) : f(s_1) = \dots = f(s_n) = 0\}$. Since the map $f \rightarrow (f(s_1), \dots, f(s_n))$ is a one-to-one linear map of N into the set of all n -tuples of scalars with the maximum norm, it has a bounded inverse and there is a constant $a > 0$ with $a\|f\| \leq \max |f(s_i)|$ for all f in N . For g in J and f in N , $\|g - f\| \geq \max |g(s_i) - f(s_i)| \geq a\|f\|$; also, $\|g - f\| \geq \|g\| - \|f\|$. Thus

$$\|g - f\| \geq \max(a\|f\|, \|g\| - \|f\|) \geq (a/a + 1)\|g\|.$$

For g in J , using IV.1.6 Theorem, p. 98 of [7],

$$\|Tg\| \geq \gamma(T) d(g, N) \geq (a\gamma(T)/a + 1)\|g\|.$$

It follows that $T|_J$ has a bounded inverse. Under these circumstances we have seen that for $s \notin \{s_1, \dots, s_n\}$ and U any neighborhood of s disjoint from $\{s_1, \dots, s_n\}$, $\|Tf\| \geq (a\gamma(T)/a + 1)\|f\|$ whenever the support of f is contained in U ; $\omega(T, s) \geq a\gamma(T)/a + 1$ and T is concentrated.

EXAMPLE 3. A concentrated operator need not have a closed range or a finite dimensional null manifold. Define A on c by the infinite matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & \dots \\ \vdots & & & & & & & & & \end{bmatrix}.$$

We again regard c as $C(S)$, $S = \{\infty, 1, 2, 3, \dots\}$ the one point compactification of the integers. It is easy to show that A has closed range (it is onto c) and the null manifold of A is infinite dimensional; A is concentrated since $\|Ae_n\| = 1$ for the

vector $e_n = \{\delta_{jn}\}$. Define B on c by the infinite matrix

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & \dots \\ \vdots & & & & & & & & \end{bmatrix}.$$

The operator B is one-to-one and its range is not closed (the vectors $x^n = (1/n, 2/n, \dots, (n-1)/n, 1, (n-1)/n, \dots, 2/n, 1/n, 0, 0, 0, \dots)$ have $\|x^n\| = 1$, but $\|Bx^n\| = 2/n$); B is concentrated since $\|Be_n\| = 1$. The product BA is a concentrated operator with infinite dimensional null manifold and nonclosed range.

THEOREM 3. *If $T: C(S) \rightarrow X$ is a concentrated operator and $K: C(S) \rightarrow X$ is a c_0 operator, then $T+K$ is a concentrated operator.*

Proof. Suppose that $T+K$ is not concentrated and let $\epsilon > 0$ be given. There is a sequence $\{s_n\}$ of distinct points of S with $\omega(T+K, s_n) < \epsilon/2$ and by passing to a subsequence we may also suppose that $\omega(K, s_n) < \epsilon/2$. There is a neighborhood U_n of s_n with $\|(T+K)f\| \leq \epsilon/2\|f\|$ for f with support in U_n and there is a neighborhood V_n of s_n with $\|Kf\| \leq \epsilon/2\|f\|$ for f with support in V_n . For f in $C(S)$ with support in $V_n \cap U_n$,

$$\|Tf\| \leq \|(T+K)f\| + \|Kf\| \leq \epsilon\|f\|.$$

Hence $\omega(T, s_n) \leq \epsilon$. That is, for each $\epsilon > 0$ there are infinitely many points s_n with $\omega(T, s_n) \leq \epsilon$, a contradiction since T is concentrated.

Recall that a summability method is an infinite matrix $A = (a_{ij})$ of scalars. For a sequence $\{b_j\}$ of scalars, $A(\{b_j\}) = \{\sum_{j=1}^{\infty} a_{ij}b_j\}$ is defined whenever the sums $\sum_{j=1}^{\infty} a_{ij}b_j$ all converge. A matrix A sums a sequence $\{b_j\}$ if $A(\{b_j\})$ is convergent, i.e. belongs to c , and is conservative if it sums every sequence in c . By the Silverman-Toeplitz conditions [15, p. 118] a conservative matrix, when restricted to c , is a bounded linear operator with $\|A\| = \sup_i \sum_{j=1}^{\infty} |a_{ij}|$ finite. As we have done previously, we will identify c with $C(S)$, $S = \{\infty, 1, 2, \dots\}$ the one point compactification of the positive integers, and will set $e_n = \{\delta_{jn}\}$. A conservative summability matrix A can be regarded as a bounded linear map on a space of type $C(S)$ and there has an oscillation $\omega(A, \cdot)$ which we will show is connected with its summability properties.

For $T: c \rightarrow X$ a bounded linear operator, $\omega(T, n) = \|Te_n\|$ and $\omega(T, \infty) = \lim_{n \rightarrow \infty} \|T|_{M_n}\|$ where $M_n = \{f \text{ in } c : f(j) = 0 \text{ for } 1 \leq j \leq n\}$. The oscillation is most interesting at infinity.

Theorem 4 is unusual in that it characterizes those summability methods which are coercive, i.e. which sum every bounded sequence, among all the bounded linear operators on c not just among the conservative summability methods.

THEOREM 4. *Suppose that $T: c \rightarrow c$ is a bounded linear operator. Then $\omega(T, \infty) = 0$ if and only if T is a summability method which sums every bounded sequence, i.e. if and only if T is a conservative summability method which is a compact operator on c .*

Proof. We begin by showing that a bounded linear operator $L: c \rightarrow c$ is compact if and only if $\lim \|L|_{N_n}\| = 0$, where $N_n = \{x \in c : x(j) = 0, 1 \leq j \leq n, \text{ and } x(\infty) = 0\}$. Suppose that L is compact and assume that $\|L|_{N_n}\|$ does not converge to zero. There is an $\varepsilon > 0$ and a sequence z_{n_i} with $\|z_{n_i}\| = 1$, z_{n_i} belongs to N_{n_i} and $\|Lz_{n_i}\| \geq \varepsilon$. But the sequence $\{z_{n_i}\}$ converges weakly to zero and, since L is compact, $\{Lz_{n_i}\}$ must converge to zero in norm, which is a contradiction. Conversely suppose that $\lim \|L|_{N_n}\| = 0$. Define P_n on c by $P_n\{b_j\} = (\lim b_j)(1, 1, 1, \dots) + \sum_{j=1}^n (b_j - \lim b_j)e_j$. Then $\|L - LP_n\| \leq \|I - P_n\| \|L|_{N_n}\| \leq 2\|L|_{N_n}\|$. The bounded linear operators LP_n are compact because their ranges are finite dimensional and the limit L of these compact operators is compact.

Let $T: c \rightarrow c$ be a bounded linear operator on c with $\omega(T, \infty) = 0$. The general form of an operator on c is well known [14], to T there corresponds a sequence of continuous linear functionals on c , $x_n^* = \{a_{nj}\}_{j=0}^\infty$ in l , converging in the c -topology on l , with $Tx = \{x_n^*(x)\}$ for each x in c . (Recall that $x_n^*(\{b_j\}) = a_{n0} \lim b_j + \sum_{j=1}^\infty a_{nj} b_j$.) Define y_m in c by

$$\begin{aligned} y_m(j) &= 0, & 1 \leq j \leq m, \\ &= 1, & m < j. \end{aligned}$$

Since $\omega(T, \infty) = 0$, $\|Ty_m\| \rightarrow 0$ and thus $a_{n0} = 0$ for all n , which is to say that T is a (conservative) summability method. Because T is a summability method, $\|T|_{N_n}\| = \|T|_{M_n}\|$, $M_n = \{x \in c : x(j) = 0, 1 \leq j \leq n\}$. Hence if T has $\omega(T, \infty) = 0$, then T is a summability method with $0 = \lim \|T|_{M_n}\| = \lim \|T|_{N_n}\|$ and T is therefore compact. Conversely if T is a compact summability method then $0 = \lim \|T|_{N_n}\| = \lim \|T|_{M_n}\| = \omega(T, \infty)$.

To complete the proof we need to show that a summability method is coercive, i.e. sums every bounded sequence, if and only if it is compact. This is probably well known and is given in [15, p. 243, Problem 43] for a special type of matrix. We indicate a proof. Suppose that $T: c \rightarrow c$ is a compact summability method. Then T has the form described above, $Tx = \{x_n^*(x)\}$, and the sequence $\{x_n^*\}$ converges in norm to the functional x^* [14]. Then for any bounded sequence $\{b_j\}$, $\lim x_n^*(\{b_j\}) = x^*(\{b_j\})$ and T is coercive. On the other hand suppose that T is a coercive summability matrix. The Silverman-Toeplitz conditions show that $T: m \rightarrow c$ is a bounded linear operator (with $\|T\| = \|T|_c\| = \sup \|x_n^*\|$). Let $\{y_n^*\}$ be a bounded sequence in c^* ; it contains a subsequence $\{y_{n_i}^*\}$ converging to y^* in the weak* topology [5, Theorem 1, p. 426]. The sequence $\{T^*y_{n_i}^*\}$ is then a weak* convergent sequence in m^* and so it converges weakly [4, p. 109]. Hence T^* is a weakly compact operator and therefore T is also weakly compact. But then as a map on c , T is weakly compact and $T|_c$ is therefore compact [10, 10 Corollary].

Albert Wilansky has pointed out that the matrix A of Example 3 occurs in his paper *Topological divisors of zero and Tauberian theorems*, Trans. Amer. Math. Soc. **113** (1964), 240–251, and that there it is also mentioned that coercive and compact are equivalent (see Theorem 4), with a reference to A. Wilansky and K. Zeller, *Banach algebra and summability*, Illinois J. Math. **2** (1958), 378–385. Wilansky credits this result to E. K. Dorff in his unpublished Ph.D. thesis (Lehigh University, 1958).

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