A METHOD FOR SHRINKING DECOMPOSITIONS OF CERTAIN MANIFOLDS

BY
ROBERT D. EDWARDS(1) AND LESLIE C. GLASER(2),(3)

Abstract. A general problem in the theory of decompositions of topological manifolds is to find sufficient conditions for the associated decomposition space to be a manifold. In this paper we examine a certain class of decompositions and show that the nondegenerate elements in any one of these decompositions can be shrunk to points via a pseudo-isotopy. It follows then that the decomposition space is a manifold homeomorphic to the original one. As corollaries we obtain some results about suspensions of homotopy cells and spheres, including a new proof that the double suspension of a Poincaré 3-sphere is a real topological 5-sphere.

1. Introduction. In this paper we examine a special case of the following question: if \( \mathcal{G} \) is a decomposition of a space \( X \), where \( X \times E^k \) is a manifold for some \( k \), then is the decomposition space \( X/\mathcal{G} \) crossed with \( E^k \) a manifold, and if so, is it homeomorphic to \( X \times E^k \)? This question assumes special significance when one knows that \( X \) is a manifold, but \( X/\mathcal{G} \) is not. The first important result in this context was Bing’s paper [2] in which he showed that his dogbone space (which is noneuclidean) crossed with \( E^1 \) is \( E^4 \); indeed, some of the techniques introduced there are used in this paper.

We are interested in the special case given by the following.

**Hypothesis.** \((X^n, S, E^k)\): \( X \) is a compact metric space with closed subset \( S \) such that (i) \( X \) is contractible, (ii) \( S \) is collared in \( X \) and is simply connected, and (iii) \((X - S) \times E^k \) is an open \((n+k)\)-manifold.

For example, \( X \) may be a fake \( n \)-cell (for \( n = 3 \) or 4), that is, a contractible combinatorial \( n \)-mainfold with \((n-1)\)-sphere boundary \( S \). Given the Hypothesis, we present an elementary proof, using a version of radial engulfing, that if \( n+k \geq 5 \), then for any topological \( k \)-manifold \( M^k \) without boundary, \( X \times M \) is homeomorphic to \((v \ast S) \times M \) by a homeomorphism which is bounded as small as desired in the \( M \) coordinate and is the identity on \( S \times M \). A pleasant corollary is a simple proof of the known fact that the double suspension of a fake 3-cell and the single
suspension of a fake 4-cell are each topologically homeomorphic to $I^5$ (see below and Corollary 7).

If in the Hypothesis $S$ is assumed to be an $(n-1)$-sphere, then the results of this paper follow from work of Siebenmann [21]. However, the generality of the Hypothesis is necessitated in part by the requirements of the authors of [11], where $X$ is merely assumed to be a contractible polyhedron.

Concerning fake cells and the fact that their suspensions are real topological cells, the original result in this direction appeared in [8]. There it was shown that the double suspension of a homotopy 3-sphere which bounds a contractible 4-manifold is topologically $S^5$. The first proof of this fact without the assumption about bounding was given by Siebenmann in [20], and since then several other proofs of varying degrees of simplicity have been discovered. For example, Glaser [9], following a suggestion of Kirby, gave a proof using the topological $h$-cobordism theorem and the local contractibility of the homeomorphism group of a manifold. Kirby and Siebenmann [13] announced a proof using the topological $h$-cobordism theorem and an infinite meshing technique of Černavskii. Glaser [10] generalized his original theorem mentioned above to get the full result. Mindful of possible unnecessary repetition, we present our proof in the belief that it represents a useful and elementary alternative.

It should be noted that in [18], Rosen considers a situation resembling that of the Hypothesis. Namely, he assumes $k=1$ and $S$ is a suspension $n$-sphere (that is, $\Sigma S \approx S^n$), and using an elementary decomposition argument, he shows that $\Sigma X \approx I^{n+1}$.

2. Preliminary results. If $Y$ is a metric space with metric $d$ and $\epsilon > 0$, then we denote the open $\epsilon$-neighborhood of a subset $D$ of $Y$ by $N_\epsilon(D)$. The metric on a product $X \times Y$ of metric spaces will always be taken to be the usual cartesian product metric $d_{X \times Y} = (d_X^2 + d_Y^2)^{1/2}$. The following theorem is the cornerstone of the paper.

Theorem 1. Suppose we have Hypothesis $(X^n, S, E^k)$ and $g : S \times [0, 1) \to X$ is a collar for $S$ in $X$. If $n + k \geq 5$, then given any $\epsilon > 0$ and $A \subset E^k$, $A$ compact, there exists an isotopy $f_t$, $t \in [0, 1]$, of $X \times E^k$ onto itself such that

1. $f_0 =$ identity,
2. $f_t = -$identity on $X \times E^k - N_\epsilon (D \times A)$ for each $t$, where $D = X - g(S \times [0, 1])$,
3. if $w \in E^k$, then $f_t(X \times w) \subset X \times N_\epsilon(w)$ for each $t$, and
4. if $w \in A$, then $\text{diam } f_t(D \times w) < \epsilon$.

Before proving the theorem, we describe a refined version of an engulfing theorem that was presented by Bing in [4] and subsequently sharpened to handle the codimension 3 case by Wright [24]. Let $M^n$ be a piecewise linear manifold without boundary, and let $V \subset U$ be two open subsets of $M$ and $\{X_a\}$ a collection of subsets of $M$. Then (generalizing Bing’s definition slightly) the statement that
finite r-complexes in \( M \) can be pulled into \( U \) rel \( V \) along \( \{X_a\} \) means the following: suppose \( P \) is a closed polyhedron in \( M \) with closed subpolyhedron \( Q \) in \( V \), and suppose that \( R = \text{cl} (P - Q) \) is compact. Then, if \( \dim R \leq r \), there is a homotopy \( H_t: P \to M, \ t \in [0, 1] \), such that \( H_0 = \text{identity}, H_t = \text{identity on } Q, H_t(P) \subseteq U \) and for each point \( x \in P \), the path \( H_t(x), \ t \in [0, 1] \), lies in some element of \( \{X_a\} \).

**Engulfing Lemma (Cf. \([4]\) and \([24]\) which this generalizes slightly).** Suppose \( M^n \) is a piecewise linear manifold without boundary and suppose \( U_0 \supseteq U_1 \supseteq \cdots \supseteq U_{r+1} \) is a collection of open subsets of \( M \) and \( \{X_a\} \) is a collection of subsets of \( M \) such that for each \( i, \ 0 \leq i < r+1 \), finite i-complexes in \( M \) can be pulled into \( U_i \) rel \( U_{i+1} \) along \( \{X_a\} \). Suppose \( P^n \) is a closed polyhedron in \( M \) with closed subpolyhedron \( Q \) in \( U_{r+1} \) such that \( R = \text{cl} (P - Q) \) is compact. Suppose \( p \leq n-3 \) and \( \dim R \leq r \). Then, for each \( \varepsilon > 0 \), there is an engulfing isotopy \( h_t: M \to M, \ t \in [0, 1] \), such that \( h_0 = \text{identity}, h_t = \text{identity on } Q \cup (M - C) \), where \( C \) is some compact subset of \( M \), \( h_1(U_0) \supseteq P \) and for each \( x \in M \) there are \( r+1 \) elements of \( \{X_a\} \) \( [r+2 \text{ if } r = n-3] \) such that the path \( h_t(x), \ t \in [0, 1] \), lies in the \( \varepsilon \)-neighborhood of the union of these \( r+1 \) [or \( r+2 \)] elements.

The proof, which proceeds by induction on \( r \), is essentially given in the references already mentioned and requires only trivial modifications to handle our slightly more general hypothesis.

**Proof of Theorem 1.** The proof uses the above Engulfing Lemma and a dual skeleton argument due to Stallings \([22]\). First, note that it follows as a corollary to Connell's argument in \([6]\) that \((X - S) \times E^k\) is euclidean \((n+k)\)-space, since it is a contractible open topological manifold which has a 1-connected open collar neighborhood of \( \infty \). Alternatively, this also follows from the more general result given in \([19]\).

Given \( \lambda > 0 \), let \( M' = X - g(S \times [0, 1 - \lambda]) \), \( V' = g(S \times (1 - \lambda, 1)) \) and let \( U'_0 \supseteq U'_1 \supseteq \cdots \supseteq U'_{n+k-2} \) be a collection of open subsets of \( M' \) such that \( \text{diam } U'_0 < \varepsilon/2 \), and, for each \( i, \ 0 \leq i < n+k-2 \), \( M' \) can be deformed (that is, homotoped) into \( U'_i \) keeping \( U'_{i+1} \) fixed (in addition to the contractibility of \( M' \), this also uses the fact that \( M' \) is an ANR, which follows since \( M' \) is a retract of the manifold \( M' \times E^k \); see \([12]\)). Let \( M = M' \times N_\lambda(A), \ V = V' \times N_\lambda(A) \) and \( U_i = U'_i \times N_\lambda(A) \) for each \( i \). Then \( M \subseteq (X - S) \times E^k \) is a piecewise linear manifold and if \( \lambda \) is sufficiently small, then \( M \subseteq N_\lambda(D \times A) \). Let \( \{X_a\} \) be the collection of subsets of \( M \) given by

\[
\{M' \times w \mid w \in N_\lambda(A)\}.
\]

Then finite 2-complexes in \( M \) can be pulled into \( V \) rel \( V \) along \( \{X_a\} \), since \( S \) is 1-connected and hence \((M, V)\) is 2-connected. Also, for each \( i, \ 0 \leq i < n+k-2 \), finite \((n+k)\)-complexes in \( M \) can be pulled into \( U_i \) rel \( U_{i+1} \) along \( \{X_a\} \).

Let \( P^{n+k} \) be a closed subpolyhedron of \( M \) such that \( P - V \) is compact and \( P \supseteq M' \times N_\lambda(A) \) for some small \( \eta > 0 \). Let \( \delta > 0 \), and let \( T \) be a triangulation of \( M \).
of mesh < \delta such that \( P \) is a subcomplex of \( T \). Let \( T_{(2)} \) denote the dual 2-skeleton of \( T \), that is, the subcomplex of the barycentric first derived subdivision \( T' \) of \( T \) given by \( T_{(2)} = \{ \alpha \cdots \gamma | \alpha < \cdots < \gamma \in T \ \text{and} \ \dim \alpha \geq n+k-2 \} \).

Let \( g_t : M \to M, t \in [0, 1] \), be an engulfing isotopy such that \( g_0 = \text{identity}, g_t \) is the identity off of some compact subset of \( M \), \( g_t(V) \supseteq P \cap T_{(2)} \) and for each \( x \in M \), there are 4 (or fewer) elements of \( \{ X_\alpha \} \) such that the path \( g_t(x), t \in [0, 1] \), lies in the \( \delta \)-neighborhood of the union of these elements. If \( \delta \) is sufficiently small, then \( g_t(D \times A) \) remains in \( M' \times N_\epsilon(A) \), and therefore \( g_1(D \times A) \cap T_{(2)} = \emptyset \). Let \( R \) be a finite subcomplex of \( T \) such that \( g_1(D \times A) \subseteq R \). By a second application of the engulfing lemma, there is an engulfing isotopy \( h_t \) of \( M, t \in [0, 1] \), such that \( h_0 = \text{identity}, h_t \) is the identity off of some compact subset of \( M \), \( h_1(U_0) \supseteq R_{n+k-3} = \text{the (n+k-3)-skeleton of } R \) and for each \( x \in M \), there are \( n+k-1 \) elements of \( \{ X_\alpha \} \) such that the path \( h_t(x), t \in [0, 1] \), lies in the \( \delta \)-neighborhood of the union of these elements. Let \( \theta_t : M \to M, t \in [0, 1] \) be an isotopy with compact support such that \( \theta_0 = \text{identity}, \theta_t(U_0) \supseteq g_1(D \times A) \) and each \( \theta_t \) moves points less than \( \delta \). To get the desired isotopy of the theorem, let \( f_t \) be the isotopy of \( M \) defined by taking the isotopy \( g_t \) on the first third of the interval \([0, 1]\), and then \( \theta_t^{-1}g_1 \) on the second third of the interval, and finally \( h_t^{-1}\theta_t^{-1}g_1 \) on the final third. Note that \( f_t \) is the identity off a compact subset of \( M \), and so can be extended via the identity to an isotopy of \( X \times E^k \). It is a trivial matter to verify that if \( \delta \) is chosen small enough, then the isotopy \( f_t \) satisfies the conditions of the theorem, completing the proof.

The following theorem generalizes Theorem 1 in two useful directions, replacing the \( E^k \) factor by an arbitrary topological manifold \( M^k \) without boundary (where manifold means a separable metric space which is locally euclidean) and replacing the compact subset \( A \) by a closed subset. The constant \( \epsilon \) of Theorem 1 is replaced by a map \( \epsilon : M \to (0, \infty) \).

If \( Y \) is a metric space and \( \epsilon : Y \to (0, \infty) \) is a map, then the \( \epsilon \)-neighborhood of a subset \( D \) of \( Y \) is the set \( N_\epsilon(D) = \bigcup_{x \in D} N_\epsilon(x) \).

**Theorem 2.** Suppose we have Hypothesis \((X^*, S, E^k)\), with \( n+k \geq 5 \), and \( g : S \times [0, 1] \to X \) is a collar for \( S \) in \( X \), and suppose that \( M^k \) is a topological manifold without boundary.

If \( A \) is a closed subset of \( M \), then given any map \( \epsilon : M \to (0, \infty) \), there exists an isotopy \( f_t, t \in [0, 1] \), of \( X \times M \) onto itself such that

1. \( f_0 = \text{identity}, \)
2. \( f_t = \text{identity on } X \times M - N_\epsilon(D \times A) \) for each \( t \), where \( D = X - g(S \times [0, 1]) \) and \( \epsilon(x, w) = \epsilon(w) \),
3. \( \text{if } w \in M, \text{ then } f_t(X \times w) \subseteq X \times N_\epsilon(w) \) for each \( t \), and
4. \( \text{if } w \in A, \text{ then } \text{diam } f_t(D \times w) < \epsilon(w) \).

**Proof.** The proof is broken up into two cases.
Case (1). A compact. For this case we may as well assume that \( \varepsilon \) is constant. Suppose \( A \subseteq \bigcup_{i=1}^{s} h_i(E^k) \), where \( h_i: E^k \to M \), \( 1 \leq i \leq s \), are coordinate homeomorphisms of \( M \), and suppose the theorem is true for compact subsets of \( M \) which lie in the union of \( s-1 \) or fewer coordinate neighborhoods. Express \( A \) as a union of two compact subsets, \( A = A_1 \cup A_2 \), such that \( A_1 \subseteq \bigcup_{i=1}^{s-1} h_i(E^k) \) and \( A_2 \subseteq h_s(E^k) \). By the induction hypothesis, there is an isotopy \( g_t \) of \( X \times M \) such that conditions (1) through (4) of the theorem hold with \( A \) replaced by \( A_1 \) and \( \varepsilon \) by \( \varepsilon/3 \). Now \( g_t \) has compact support (assuming \( \varepsilon \) is sufficiently small) and is therefore uniformly continuous, so there is a \( \delta, 0 < \delta < \varepsilon/3 \), such that

\[
(*) \quad \text{If } B \text{ is a subset of } X \times M \text{ with diam } B < 2\delta, \text{ then diam } g_t(B) < \varepsilon/3 \text{ for each } t.
\]

This implies that \( g_t \) satisfies the following conditions:

(3') If \( w \in M \), then \( g_t(X \times N_\delta(w)) \subseteq X \times N_\varepsilon(w) \) for each \( t \), and

(4') if \( w \in A_1 \), then \( \text{diam } g_t(N_\delta(D \times w)) < \varepsilon \).

By Theorem 1, there is an isotopy \( h_t, t \in [0, 1] \), of \( X \times M \) such that conditions (1) through (4) hold with \( f_t \) replaced by \( h_t \), \( A \) replaced by \( A_2 \) and \( \varepsilon \) by \( \delta \). Define an isotopy \( f_t, t \in [0, 1] \), of \( X \times M \) by

\[
\begin{align*}
   f_t &= h_{2t}, \quad t \in [0, \frac{1}{2}], \\
   &= g_{2t-1}h_1, \quad t \in [\frac{1}{2}, 1].
\end{align*}
\]

Case (2). The General Case. Since \( M \) is a locally compact separable metric space, there are two collections \( \mathcal{C} \) and \( \mathcal{D} \) of compact subsets of \( M \) such that \( M = \bigcup (\mathcal{C} \cup \mathcal{D}) \), and each collection is countable and discrete, where discrete means that each point of \( M \) has a neighborhood which intersects at most one member of the collection (for a short proof of this fact, see the paragraph on pp. 165, 166 of [17]). Let \( C_1, C_2, \ldots \) and \( D_1, D_2, \ldots \) be the intersections of the members of these collections with \( A \). Then by infinitely many simultaneous applications of Case (1), the theorem holds for the closed set \( A_1 = \bigcup_{i=1}^{s-1} C_i \), and likewise for \( A_2 = \bigcup_{i=1}^{s-1} D_i \). From this point on the argument of Case (1) applies, since \( A \) can be written as the union of two closed subsets, \( A = A_1 \cup A_2 \), each of which satisfies the theorem. The \( \varepsilon \) and \( \delta \) are now assumed to be maps, and \( (*) \) should be replaced by

\[
(*)' \quad \text{If } B \text{ is a subset of } X \times N_{2\delta}(w) \text{ with diam } B < 2\delta(w), \text{ then diam } g_t(B) < \varepsilon(w)/3 \text{ for each } t.
\]

3. Main results. A pseudo-isotopy of a space \( Y \) is a homotopy \( h_t: Y \to Y \), \( t \in [0, 1] \), such that for each \( t \in [0, 1] \), \( h_t \) is a homeomorphism. A pseudo-isotopy is a desirable means of shrinking an upper semicontinuous decomposition of a manifold, for if the limiting map \( h_t \) is a closed surjection, then the decomposition space is a manifold homeomorphic to the original one (see Corollary 6). In this section, we construct a pseudo-isotopy by generalizing a theorem of Bing [2, Theorem 3] to our situation.
First we establish some notation. Suppose we have Hypothesis \((X^n, S, E^k)\) and 
\(g: S \times [0, 1) \to X\) is a fixed collar for \(S\) in \(X\). Let \(D = X - g(S \times [0, 1))\). If \(M\) is a topological manifold and \(A\) is a closed subset of \(M\), let \(\mathcal{G}(X \times M, D \times A)\) denote the decomposition of \(X \times M\) having nondegenerate elements only of the form \(D \times w, w \in A\); that is,
\[
\mathcal{G}(X \times M, D \times A) = \{G \subseteq X \times M | G = D \times w, w \in A \text{ or } G = (x, w) \in X \times M - D \times A\}.
\]

Let \(X \times M/\{(D \times w | w \in A)\}\) denote the decomposition space given by
\[
X \times M/\mathcal{G}(X \times M, D \times A).
\]

**Theorem 3.** Suppose we have Hypothesis \((X^n, S, E^k)\), with \(n + k \geq 5\), and 
\(g: S \times [0, 1) \to X\) is a collar for \(S\) in \(X\), and suppose that \(M^k\) is a topological manifold without boundary.

If \(A\) is a closed subset of \(M\), then given any map \(\varepsilon: M \to (0, \infty)\), there exists a pseudo-isotopy \(h_t\) of \(X \times M\) such that
1. \(h_0 = \text{identity}\),
2. \(h_t = \text{identity on } X \times M - N_\varepsilon(D \times A)\) for each \(t\),
3. if \(w \in M\), then \(h_t(X \times w) \subseteq X \times N_\varepsilon(w)\) for each \(t\), and
4. \(h_1\) is a closed map taking \(X \times M\) onto itself and each element of \(\mathcal{G}(X \times M, D \times A)\) onto a distinct element of \(X \times M\).

**Corollary 4.** Under the assumptions of Theorem 3, there exists a homeomorphism \(f: X \times M \to X \times M/\{(D \times w | w \in A)\}\), with \(f = \text{identity on } S \times M\), such that
\[
d(w, p_2f(x, w)) < \varepsilon(w)
\]
for each \((x, w) \in X \times M\), where \(p_2: X \times M/\{(D \times w | w \in A)\} \to M\) is the natural projection and \(d\) is the metric on \(M\).

Theorem 3 easily generalizes to the broader context of upper semicontinuous decomposition theory, and it seems to be more transparent and useful in this setting. Thus we present Theorem 5 and Corollary 6 below, which include Theorem 3 and Corollary 4 as special cases (upon application of Theorem 2 and Remark 1 below). Theorem 5 presents a useful general statement of a result that has been applied many times in more specific situations (e.g. \([1], [5], [14], \text{ and } [25]\)).

Let \(M\) be a metric space and let \(\mathcal{G} = \{G_a\}\) be an upper semicontinuous decomposition of \(M\) into compact subsets, that is, a collection of disjoint compact subsets of \(M\), whose union is all of \(M\), such that the quotient map \(\rho: M \to M/\mathcal{G}\) of \(M\) onto the decomposition space is closed. Following McAuley \([15]\) we say that such a \(\mathcal{G}\) is shrinkable if given any map \(\varepsilon: M \to (0, \infty)\) and any saturated open cover \(\mathcal{U}\) of \(M\) (where saturated means that for any \(U \in \mathcal{U}\) and \(G_a \in \mathcal{G}\), either \(G_a \cap U = \emptyset\)
or \( G_a \subseteq U \), there is an isotopy \( f_t: M \rightarrow M \), \( t \in [0, 1] \), such that \( f_0 = \text{identity} \) and, for each \( G_a \in \mathcal{G} \),

(i) there is a \( U \in \mathcal{U} \) such that \( U \supset G_a \cup f_t(G_a) \) for all \( t \in [0, 1] \), and

(ii) \( \text{diam} f_t(G_a) < \inf \varepsilon(G_a) \).

The significance of this notation is demonstrated by the following result.

**Theorem 5** (Cf. [2, Theorem 3] and [15, Theorem 2]). Suppose that \( \mathcal{G} \) is an upper semicontinuous decomposition into compact subsets of the metric space \( M \), and suppose that \( \mathcal{G} \) is shrinkable and \( M \) is complete. Then given any saturated open cover \( \mathcal{V} \) of \( M \), there is a pseudo-isotopy \( h_t: M \rightarrow M \), \( t \in [0, 1] \), such that

1. for each \( G_a \in \mathcal{G} \), there is a \( U \in \mathcal{U} \) such that \( U \supset G_a \cup h_t(G_a) \) for all \( t \in [0, 1] \), and
2. \( h_t \) is a closed map taking \( M \) onto itself and each element of \( \mathcal{G} \) onto a distinct element of \( M \).

**Corollary 6.** Given the hypotheses of Theorem 5, then the decomposition space \( M/\mathcal{G} \) is homeomorphic to \( M \).

**Proof of Corollary.** The relation \( f = h_t \beta^{-1} \) gives a well-defined homeomorphism from \( M/\mathcal{G} \) to \( M \).

**Remark 1.** If in addition in the definition of shrinkable we assume that each isotopy \( f_t \) can be chosen to be the identity on some given subset \( D \) of \( M \) (for example, the complement of the \( \varepsilon \)-neighborhood of the union of all nondegenerate elements of \( \mathcal{G} \)), then we can further assume in Theorem 5 that the pseudo-isotopy \( h_t \) is also the identity on \( D \) for each \( t \). We shall not dwell here on this modification (which of course we need for Theorem 3), inasmuch as it is trivial.

**Remark 2.** It is useful to note that the notion of shrinkability is independent of the metric chosen for \( M \). This can be seen by showing that an equivalent definition is the following: given any two open covers \( \mathcal{U} \), \( \mathcal{V} \) of \( M \), with \( \mathcal{U} \) saturated, there is an isotopy \( f_t: M \rightarrow M \), \( t \in [0, 1] \), such that for each \( G_a \in \mathcal{G} \),

1. there is a \( U \in \mathcal{U} \) such that \( U \supset G_a \cup f_t(G_a) \) for all \( t \), and
2. there is a \( V \in \mathcal{V} \) such that \( f_t(G_a) \subseteq V \).

Thus, in the proof that follows, we are justified in assuming that the given metric \( d \) on \( M \) is complete, thereby simplifying things considerably.

**Remark 3.** Theorem 5 shows that the shrinking criterion is a sufficient condition for \( M/\mathcal{G} \) to be homeomorphic to \( M \). It is by no means a necessary condition; see Bing’s well-known figure eight example in [3, p. 7]. However, Siebenmann [21], generalizing some results of Armentrout for three dimensions, has shown that if \( M \) is a manifold without boundary (\( \dim M > 4 \)) and the elements of \( \mathcal{G} \) are cell-like (that is, cellular as subsets of some euclidean space), then if \( M/\mathcal{G} \) is locally euclidean, then \( M/\mathcal{G} \approx M \) and the decomposition \( \mathcal{G} \) is shrinkable.

**Question.** Is there an explicit expression or formulation for a metric on \( M/\mathcal{G} \), given a metric on \( M \)? If \( \mathcal{G} \) has only one nondegenerate element \( G \), then such a
metric is \( d^*(x, y) = \min\{d(x, y), d(x, G) + d(G, y)\} \) for \( x, y \in \mathcal{G} \), where \( d \) is the given metric on \( M \). That \( M//G \) is in fact a metric space is proved in [23, Theorem 1], see also [7, p. 235].

In what follows, we make extensive use of the fact that \( M//G \) is paracompact. For this and other useful properties of paracompact spaces, we refer the reader to [7].

**Proof of Theorem 5.** Let \( \varepsilon_1, \varepsilon_2, \ldots \) be any sequence of positive numbers such that \( \sum_{i=1}^{\infty} \varepsilon_i < \infty \). Inductively we construct a sequence \( \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots \) of successively finer saturated open covers of \( M \) such that \( \{ U \mid U \in \mathcal{U}_i \} \) refines \( \mathcal{U} \) and \( \mathcal{U}_i \) refines \( \{ N_{\varepsilon_i}(G_a) \mid G_a \in \mathcal{G} \} \) for each \( i > 1 \), and a sequence of isotopies of \( M \),

\[ h(x, t), t \in [0, \frac{1}{i}); \quad h(x, t), t \in [\frac{1}{i}, \frac{1}{i+1}); \ldots \]

such that \( h(x, 0) = x \), any two adjacent \( h(x, t) \)'s agree on their common end, and for each \( i \geq 1 \),

(a) for each \( G_a \in \mathcal{G} \), there is a \( U_i \in \mathcal{U}_i \) and a \( \lambda > 0 \) such that \( h(U_i, (i-1)/i) \supseteq h(N_{\varepsilon_i}(U_{i+1}), t) \) for any \( U_{i+1} \in \mathcal{U}_{i+1} \) containing \( G_a \) and any \( t \in [(i-1)/i, i/(i+1)] \), and

(b) for each \( U_{i+1} \in \mathcal{U}_{i+1} \), \( \text{diam } h(U_{i+1}, i/(i+1)) < 2\varepsilon_i \).

From these conditions it follows that for each \( i \geq 1 \),

(c) each point \( h(x, t) \in M \) moves less than \( 2\varepsilon_{i-1} \) during \( t \in [(i-1)/i, i/(i+1)] \), and

(d) for each \( G_a \in \mathcal{G} \), there is a \( U_i \in \mathcal{U}_i \) such that \( h(U_i, (i-1)/i) \supseteq h(G_a, t) \) for all \( t \in [(i-1)/i, i) \).

Taking the limit of such a sequence of isotopies provides the desired pseudo-isotopy (the only nontrivial verification is that the limit map is surjective).

To start the construction of the \( h \)'s and the \( \mathcal{U} \)'s, let \( h(x, 0) = \text{identity} \) and let \( \mathcal{U}_1 \) be any saturated open cover such that \( \{ U \mid U \in \mathcal{U}_1 \} \) refines \( \mathcal{U} \). In general, for \( i \geq 1 \), given \( h(x, (i-1)/i) \) and \( \mathcal{U}_i \), first choose a map \( \delta : M \to (0, \infty) \) so small that

(8) for each \( G_a \in \mathcal{G} \), if \( S \) is a subset of \( N_{\delta}(G_a) \) of diameter \( < \sup \delta(N_{\delta}(G_a)) \), then \( \text{diam } h(S, (i-1)/i) < \varepsilon_i \).

The construction of \( \delta \) uses the paracompactness of \( M//G \). Next, let \( \mathcal{W} \) be a saturated open cover of \( M \) which refines \( \{ N_{\delta}(G_a) \mid G_a \in \mathcal{G} \} \), and let \( f(x, t), t \in [(i-1)/i, i/(i+1)] \), be an isotopy of \( M \) given by the assumption of shrinkability, reparametrized as indicated, using the cover \( \{ W \cap U_i \mid W \in \mathcal{W}, U_i \in \mathcal{U}_i \} \) and the map \( \delta \). To extend the sequence of \( h(x, t) \)'s one more step, define \( h(x, t) = f(x, t), (i-1)/i \) for \( t \in [(i-1)/i, i/(i+1)] \). Then for each \( G_a \in \mathcal{G} \),

(a') there is a \( U_i \in \mathcal{U}_i \) such that for all \( t \in [(i-1)/i, i/(i+1)] \), \( h(U_i, (i-1)/i) \supseteq h(G_a, t) \) and

(b') \( \text{diam } h(G_a, i/(i+1)) < \varepsilon_i \).

From these conditions it follows that there exists a saturated open cover \( \mathcal{V} \) of \( M \), refining \( \{ N_{\varepsilon_i+1}(G_a) \mid G_a \in \mathcal{G} \} \), such that for each \( V \in \mathcal{V} \), there exists a \( U_i \in \mathcal{U}_i \) and there exists a \( \lambda > 0 \) such that for all \( t \in [(i-1)/i, i/(i+1)] \), \( h(U_i, (i-1)/i) \supseteq h(N_{\lambda}(V), t) \), and also \( \text{diam } h(V, i/(i+1)) < 2\varepsilon_i \). Let \( \mathcal{U}_{i+1} \) be a saturated open
cover of $M$ which is a barycentric refinement of $\mathcal{V}$, that is, the set $\bigcup \{ U \mid G_a \subset U \in \mathcal{U}_{i+1} \}$, for each $G_a \in \mathcal{G}$, is contained in some element of $\mathcal{V}$ [7, p. 168]. Then $\mathcal{U}_{i+1}$ is the desired cover. This completes the proof.

4. Corollaries and applications. Recall that $D = X - g(S \times [0, 1])$.

**Corollary 7.** Suppose we have Hypothesis $(X^n, S, E^k)$, with $n + k \geq 5$. Then

1. there exists a homeomorphism $f: X \times E^k \to (v \ast S) \times E^k$, bounded as small as desired on the $E^k$ factor, such that $f|S \times E^k = \text{identity}$, and
2. there exists a homeomorphism $f: X \ast S^{k-1} \to (v \ast S) \ast S^{k-1}$ with $f|S \ast S^{k-1} = \text{identity}$, and
3. if $M^k$ is a topological manifold with boundary, then there is a homeomorphism $f: X \times M/(D \times w \mid w \in \partial M) \to (v \ast S) \times M$ which is the "identity" on $(v \ast S) \times \partial M \cup S \times M$.

**Remark 1.** Statement (2) in particular says that $\Sigma^2 F^3 \approx I^5$ and $\Sigma F^4 \approx I^5$, and therefore $\Sigma^2 H^3 \approx S^5$ and $\Sigma H^4 \approx S^5$, where $F$ and $H$ are a homotopy cell and a homotopy sphere, respectively, and $\approx$ means "is homoeomorphic to." Statement (3) provides an interesting fact for homotopy 3-cells, saying that

$$F^3 \times I^3/(D \times w \mid w \in \partial I^3) \approx I^5$$

by a homeomorphism which is the "identity" on the boundary.

**Remark 2.** An important problem in relation to simplicial triangulations of topological manifolds is whether there exists some nonsimply-connected homology $n$-sphere $H$ whose $k$th suspension, for some $k$, is homoeomorphic to $S^{n+k}$. Since $n + k$ is necessarily $\geq 5$, it follows by part (2), using $((\Sigma H) - \text{int } B, \partial B, E^{k-1})$ for $(X^n, S, E^k)$, where $B$ is a locally flat $(n+1)$-cell in $\Sigma H$ missing $H$ and the suspension points, that $\Sigma^k H \approx S^{n+k}$ if and only if $v \ast H \times E^{k-1} = \text{an (n+k)-manifold with boundary.}$

**Remark 3.** An application of Corollary 7, where the $S$ in Hypothesis $(X^n, S, E^k)$ may not be a sphere, is obtained as follows: Suppose $S \subset H$ is a locally flat embedding of a homotopy 3-sphere $S$ in a homotopy 4-sphere $H$. Then $(\Sigma^2 H, \Sigma^2 S) \approx (\Sigma S^5, S^5)$. This is because $S$ separates $H$ into two contractible manifolds $X_1$ and $X_2$, each containing $S$ as a collared subset. Thus by part (1) of the corollary, $X_1 \times E^1 \approx (v_1 \ast S) \times E^1$ and, by part (2), $\Sigma X_i \approx \Sigma (v_i \ast S)$, $i = 1, 2$. Hence $(\Sigma H, \Sigma S)$ $(\Sigma (v_1 \ast S \ast v_2), \Sigma S)$. But then

$$(\Sigma^2 H, \Sigma^2 S) \approx (\Sigma^2 (v_1 \ast S \ast v_2), \Sigma^2 S) = (v_1 \ast (\Sigma^2 S) \ast v_2, \Sigma^2 S)$$

$$\approx \text{[by Remark 1 above]} (v_1 \ast S^5 \ast v_2, S^5) = (\Sigma S^5, S^5) .$$

**Proof of Corollary 7. Part (1).** This is immediate from Corollary 4.

**Part (2).** The homeomorphism $f$ is induced by the homeomorphism given in part (1) (call it $f_1$) in a natural manner. Simply let $\lambda_1: X \times E^k \to X \ast S^{k-1} - S^{k-1}$ and $\lambda_2: (v \ast S) \times E^k \to (v \ast S) \ast S^{k-1} - S^{k-1}$ be the homeomorphisms which are
naturally induced by a homeomorphism $\lambda: E^k \to (v \ast S^{k-1}) \ast S^{k-1}$ which preserves radial lines. Then $f$ is defined as indicated by the diagram.

$$
\begin{array}{ccc}
X \ast S^{k-1} & \xrightarrow[f]{f} & (v \ast S) \ast S^{k-1} \\
\downarrow \lambda_1 & & \downarrow \lambda_2 \\
X \times E^k & \xrightarrow[f_1]{f_1} & (v \ast S) \times E^k 
\end{array}
$$

**Part (3).** Consider the subsets $X \times \text{int } M \subseteq X \times M / \{D \times w \mid w \in \partial M\}$ and $(v \ast S) \times \text{int } M \subseteq (v \ast S) \times M$. By Corollary 4, given any $\varepsilon: \text{int } M \to (0, \infty)$, there exists a homeomorphism $f: X \times \text{int } M \to (v \ast S) \times \text{int } M$, $f = \text{identity on } S \times M$, such that $d(w, p_2f(x, w)) < \varepsilon(w)$ for all $w \in \text{int } M$. We can further assume (Theorem 3) that $f$ is the identity off of the $e(x, w) = \varepsilon(w)$ neighborhood of $D \times \text{int } M$ in $X \times \text{int } M$. Then if $\varepsilon$ is sufficiently small, $f$ extends via the identity to the desired homeomorphism of the corollary.

**Remark.** Part (1) (and therefore part (2)) of Corollary 7 holds under the weaker hypothesis that $S \times E^k$ merely be collared in $X \times E^k$. This result follows by attaching a collar to $X$ to get $X_+ = X \cup_{S \times 0} S \times [0, 1]$ and applying part (1) above to get $X_+ \times E^k \simeq (v \ast S) \times E^k$, and then observing that $X_+ \times E^k \simeq X \times E^k$ using the collar of $S \times E^k$ in $X \times E^k$.

This remark provides for a proof of the following fact, originally proved by Siebenmann in [20] under the added assumption that $\partial M$ is a manifold when $m = 5$.

**Corollary 8.** Suppose $M^m$ is a simplicial homotopy $m$-manifold, $m \neq 4$. Then $M - \partial M$ is a topological $m$-manifold without boundary and $\partial M$ is collared in $M$. In particular, then, if $m \neq 4, 5$, or if $m = 5$ and $\partial M$ is a topological manifold, then $M$ is a topological manifold.

**Proof.** Assume $M$ is triangulated with $\partial M$ as a full subcomplex. Then by definition for each $k$-simplex $\sigma \in M$, $lk(\sigma, M)$ is homotopically equivalent to either an $(m-k-1)$-sphere or ball according as $\sigma \in M - \partial M$ or $\sigma \in \partial M$, and if $\sigma \in \partial M$, then $lk(\sigma, \partial M)$ is homotopically equivalent to an $(m-k-2)$-sphere.

First, consider $\text{int } M = |M| - |\partial M|$. Suppose inductively that $\text{int } M - M^{(k)}$ $(M^{(k)} = k$-skeleton of $M$) is an open $m$-manifold and suppose $\sigma$ is a $k$-simplex in $M - \partial M$. The induction assumption implies that $lk(\sigma, M) \times E^{k+1}$ is an open $m$-manifold, since $lk(\sigma, M) \times E^{k+1} \simeq \text{open star } (\sigma, M) - \hat{0} \subseteq \text{int } M - M^{(k)}$, where the open star $(\sigma, M) = \bigcap \{\text{open star } (v, M) \mid v \text{ vertex of } \sigma\}$. We will show that $lk(\sigma, M) \ast S^k \simeq S^m$, which then implies that open star $(\sigma, M)$ is locally euclidean, being homeomorphic to an open subset of $lk(\sigma, M) \ast S^k$.

Without loss assume $k \leq m - 4$ (for $k \geq m - 3$, $lk(\sigma, M)$ is a real PL sphere). Let $B$ be a small collared $(m-k-1)$-cell in $lk(\sigma, M)$ and let $$(X, S) = (lk(\sigma, M) - \text{int } B, \partial B).$$
X is acyclic by duality in $X \times S^q$, large $q$. So by Corollary 7, part (2) we have $(X, S) \ast S^k \approx (B^{m-k-1}, \partial B^{m-k-1}) \ast S^k$ and therefore sewing back in $B \ast S^k$ along $\partial B \ast S^k$ we get that $lk(\sigma, M) \ast S^k \approx S^m$.

Now consider $\partial M$. Inductively assume that $\partial M - M^{(k)}$ is collared in $M - (\partial M \cap M^{(k)})$ and suppose $a = \sigma^k \in \partial M$. Again without loss $k \leq m - 4$. Let

$$(X, S) = (lk(\sigma, M), lk(\sigma, \partial M)).$$

The induction hypothesis implies that $S \times E^{k+1}$ is collared in $X \times E^{k+1}$, and the first part of the corollary implies that $(X - S) \times E^{k+1}$ is an open $m$-manifold. Now applying the preceding remark to the pair $(X, S)$, we have

$$(X, S) \ast S^k \approx (v \ast S, S) \ast S^k = (v \ast (S \ast S^k), S \ast S^k),$$

and therefore $S \ast \delta$ is collared in $X \ast \delta$. Thus the induction can proceed.

**References**

5. J. L. Bryant, Euclidean $n$-space modulo an $(n-1)$-cell (to appear).

**DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540**

**DEPARTMENT OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540**

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024***

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112**

* Current address of Robert D. Edwards.
**Current address of Leslie C. Glaser.