Abstract. Let $G$ be a compact abelian group and $\varphi$ a complex-valued function defined on the dual $\Gamma$. The main result of this paper is that $\varphi$ is a compact multiplier of type $(p, q)$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$, if and only if it satisfies the following condition: Given $\epsilon > 0$ there corresponds a finite set $K \subseteq \Gamma$ such that $|\sum a_y b_y \varphi(y)| < \epsilon$ whenever $P = \sum a_y \varphi(y)$ and $Q = \sum b_y \varphi(y)$ are trigonometric polynomials satisfying $\|P\|_p \leq 1$, $\|Q\|_q \leq 1$ ($q'$ the conjugate index of $q$) and $b_y = 0$ for $y \in K$. Using the above characterization we obtain the following necessary and sufficient condition for $\varphi$ to be the Fourier transform of a continuous complex-valued function on $G$: Given $\epsilon > 0$ there corresponds a finite set $K \subseteq \Gamma$ such that $|\sum b_y \varphi(y)| < \epsilon$ whenever $Q = \sum b_y \varphi(y)$ is a trigonometric polynomial satisfying $\|Q\|_1 \leq 1$ and $b_y = 0$ for $y \in K$.

Throughout the paper $G$ is a compact abelian group, $\varphi$ a complex-valued function defined on the dual $\Gamma$ and $L^p(G)$ ($1 \leq p \leq \infty$) the usual Lebesgue space of index $p$ formed with respect to Haar measure on $G$. Let $M(G)$ denote the convolution algebra of complex-valued regular measures which are bounded on $G$, and $C(G)$ the class of all continuous complex-valued functions defined on $G$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(G)$ is defined by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma) \, dx \quad (\gamma \in \Gamma)$$

and the Fourier-Stieltjes transform $\hat{\mu}$ of a measure $\mu \in M(G)$ by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma) \, d\mu(x) \quad (\gamma \in \Gamma).$$

The function $\varphi$ is said to be a multiplier of type $(p, q)$ if given $f \in L^p(G)$ there corresponds a $g \in L^q(G)$ such that $\varphi f = \hat{g}$.

Now any $(p, q)$ multiplier induces a bounded linear operator from $L^p(G)$ into $L^q(G)$, $T_\varphi$, where $(T_\varphi f)_{-\gamma} = \varphi f_{\gamma}$ and $T_\varphi$ commutes with translation. Conversely for $p < \infty$, there corresponds to any such bounded linear operator $T$ mapping $L^p(G)$ into $L^q(G)$, a unique multiplier $\varphi$ of type $(p, q)$ such that $T = T_\varphi$ (see [5, pp. 249–250]). We say that $\varphi$ is a compact multiplier if $T_\varphi$ is a compact operator. Let $M_\varphi^p(\Gamma)$ denote the set of all multipliers of type $(p, q)$ and $m_\varphi^p(\Gamma)$ the set of all $\varphi \in M_\varphi^p(\Gamma)$ which are compact. Then $M_\varphi^p(\Gamma)$ is a Banach space where the norm $\|\cdot\|_{(p, q)}$ of the multiplier $\varphi$ is defined to be the norm of the multiplier operator $T_\varphi$. Let $\mathcal{M}(G)$

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denote the set of all trigonometric polynomials on $G$ and $\mathcal{F}(G)$ the set of all functions on $\Gamma$ which are Fourier transforms of functions in $\mathcal{F}(G)$. The following lemma is important in the sequel.

**Lemma.** Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then the closure of $\mathcal{F}(G)$ in $M^p_\mathcal{F}(\Gamma)$ is precisely $m^p_\mathcal{F}(\Gamma)$.

**Proof.** If $P \in \mathcal{F}(G)$ then the convolution product $P \ast L^p(G)$ is finite dimensional. Therefore convolution by $P$ is an operator of finite rank. Thus the closure of $\mathcal{F}(G)$ is contained in $m^p_\mathcal{F}(\Gamma)$.

On the other hand, if $\varphi \in m^p_\mathcal{F}(\Gamma)$, let $\varphi_a = \varphi \hat{e}_a$, where $e_a$ is a bounded approximate identity in $L^1(G)$ consisting of trigonometric polynomials. Then $\varphi_a \in \mathcal{F}(G)$ and $\|\varphi_a - \varphi\|_{(p,q)} \to 0$. (See Gaudry [8] or Bachelis and Gilbert [1] for details.)

Our main result is the following characterization of $m^p_\mathcal{F}(\Gamma)$:

**Theorem.** Let $\varphi$ be a complex-valued function defined on $\Gamma$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. The following statements are equivalent:

(i) $\varphi \in m^p_\mathcal{F}(\Gamma)$;

(ii) Given $\varepsilon > 0$, there corresponds a finite subset $K \subseteq \Gamma$ such that $|\sum a_i b_i \varphi(y)| < \varepsilon$ whenever $P = \sum a_i \gamma$ and $Q = \sum b_i \gamma$ are trigonometric polynomials satisfying $\|P\|_p \leq 1$, $\|Q\|_{q'} \leq 1$ ($q'$ the conjugate index of $q$) and $b_i = 0$ for $\gamma \in K$.

If $p = 1$, then both (i) and (ii) are equivalent to

(iii) Given $\varepsilon > 0$, there corresponds a finite subset $K \subseteq \Gamma$ such that $|\sum b_i \varphi(y)| < \varepsilon$ whenever $Q = \sum b_i \gamma$ is a trigonometric polynomial with $\|Q\|_{q'} \leq 1$ and $b_i = 0$ for $\gamma \in K$.

**Proof.** (i) $\Rightarrow$ (ii). Let $\varepsilon > 0$ be given. There corresponds by the preceding lemma a trigonometric polynomial $L$ such that $\|L - T_\varphi\|_{(p,q)} < \varepsilon$.

Let $K$ be the finite support of $L$ and let $P$ and $Q$ be as in (ii). Then $Q \ast L = 0$, so

$$\left|\sum \hat{P}(\gamma) \hat{Q}(\gamma) \varphi(y)\right| = |T_\varphi(P) \ast Q(0)| = |T_\varphi(P) \ast Q(0) - L \ast L \ast Q(0)| \\ \leq \|T_\varphi(P) - L \ast L \ast Q\|_{q'} \leq \|T_\varphi - L\|_{(p,q)} \|P\|_p \|Q\|_{q'} < \varepsilon.$$

(ii) $\Rightarrow$ (i). The function $\varphi$ induces a linear mapping of $\mathcal{F}(G)$ into $\mathcal{F}(G)$ as follows:

$$T(P) = \sum \hat{P}(\gamma) \varphi(\gamma) \gamma \quad (P \in \mathcal{F}(G)).$$

Let $\varepsilon > 0$. We claim it is enough to show that there exists a trigonometric polynomial $N$ such that

$$(*) \quad \|T(P) - N \ast P\|_q < \varepsilon$$

for all trigonometric polynomials $P$ such that $\|P\|_p \leq 1$. Since $p < \infty$, this implies that $T$ has a continuous extension $\tilde{T}$ to $L^p(G)$ which necessarily is compact and commutes with translation. Hence, $\tilde{T} = T_\varphi$ for some $\psi$ in $m^p_\mathcal{F}(\Gamma)$. Since $\tilde{T}(\gamma) \varphi(\gamma) = \varphi(\gamma)$ we may conclude $\psi = \varphi$. 

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So let $K$ be as in (ii) corresponding to $\varepsilon/3$. Now choose a trigonometric polynomial $R$ such that $\|R\|_1 \leq 2$ and $\hat{R}|K|=1$; see [10, p. 53]. Put $N=\sum \hat{R}(\gamma)\varphi(\gamma)y$. To show (*) it suffices to prove that 

$$|(T(P)-N*P)*Q(0)| < \varepsilon$$

for all trigonometric polynomials $Q$ such that $\|Q\|_{q'} \leq 1$. Given such a $Q$ let $Q_1 = \frac{1}{4}(Q-Q*R)$. Then 

$$\|Q_1\|_{q'} \leq \frac{1}{4}(\|Q\|_{q'}+\|Q\|_{q'}\|R\|_1) \leq 1 \quad \text{and} \quad \hat{Q}_1|K|=0.$$

Thus by the choice of $K$, $|\sum \hat{P}(\gamma)\hat{Q}_1(\gamma)\varphi(\gamma)| < \varepsilon/3$. But 

$$\sum \hat{P}(\gamma)\hat{Q}_1(\gamma)\varphi(\gamma) = \frac{1}{4}[\sum \hat{P}(\gamma)\hat{Q}(\gamma)\varphi(\gamma)-\sum \hat{P}(\gamma)\hat{Q}(\gamma)\hat{R}(\gamma)\varphi(\gamma)]$$

$$= \frac{1}{4}[T(P)*Q(0)-N*P*Q(0)]$$

which proves (**).

Suppose now that $p=1$. We will show that (ii) $\Rightarrow$ (iii). If (ii) holds and $\varepsilon>0$, let $K$ be as given by (ii) corresponding to $\varepsilon/2$.

If $Q$ is a trigonometric polynomial with $\|Q\|_{q'} \leq 1$ and $\hat{Q}|K|=0$, choose a trigonometric polynomial $P$ such that $\|P\|_1 \leq 3/2$ and $P*Q=Q$.

Then 

$$\left|\sum \hat{Q}(\gamma)\varphi(\gamma)\right| = \left|\sum \hat{P}(\gamma)\hat{Q}(\gamma)\varphi(\gamma)\right| < (\varepsilon/2)\|P\|_1 < \varepsilon.$$ 

Therefore (ii) $\Rightarrow$ (iii).

Suppose now that (iii) holds. Given $\varepsilon>0$ let $K$ be as given by (iii). If $P$ and $Q$ are trigonometric polynomials with $\|P\|_1 \leq 1$, $\|Q\|_{q'} \leq 1$, and $\hat{Q}|K|=0$, then 

$$\|P*Q\|_{q'} \leq \|P\|_1\|Q\|_{q'} \leq 1 \quad \text{and} \quad (P*Q)^\dagger|K|=0.$$ 

Thus 

$$\left|\sum \hat{P}(\gamma)\hat{Q}(\gamma)\varphi(\gamma)\right| = \left|\sum (P*Q)^\dagger(\gamma)\varphi(\gamma)\right| < \varepsilon$$

and this concludes the proof.

Applying the above characterization in the special cases $m_1^\wedge(\Gamma)$ and $m_2^\wedge(\Gamma)$ we obtain the following corollary:

**Corollary.** Let $\varphi$ be a complex-valued function defined on $\Gamma$.

(a) The function $\varphi \in L^q(\Gamma)^\wedge$ if and only if it satisfies the following condition: Given $\varepsilon>0$ there corresponds a finite subset $K \subseteq \Gamma$ such that $|\sum b_\gamma \varphi(\gamma)| < \varepsilon$ whenever $Q = \sum b_\gamma$ is a trigonometric polynomial satisfying $\|Q\|_1 \leq 1$ and $b_\gamma=0$ for $\gamma \not\in K$.

(b) The function $\varphi \in C(\Gamma)^\wedge$ if and only if it satisfies the following condition: Given $\varepsilon>0$ there corresponds a finite subset $K \subseteq \Gamma$ such that $|\sum b_\gamma \varphi(\gamma)| < \varepsilon$ whenever $Q = \sum b_\gamma$ is a trigonometric polynomial satisfying $\|Q\|_1 \leq 1$ and $b_\gamma=0$ for $\gamma \not\in K$. 

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Proof. Clearly it is enough to show that

\[(1) \quad m_1^\Gamma (\Gamma') = L^1(G)^\wedge\]

and

\[(2) \quad m_\infty^\Gamma (\Gamma') = C(G)^\wedge.\]

Now \(M_1^\Gamma (\Gamma') = M(G)^\wedge\) and \(M_\infty^\Gamma (\Gamma') = L^\infty(G)^\wedge\) (see [7, p. 368] and [9]), thus (1) and (2) follow from the lemma since the closure of \(\mathcal{F}(G)\) in \(M(G)\) \((L^\infty(G))\) is \(L^1(G)\) \((C(G))\).

Remarks. For compact abelian groups, the above characterization of transforms of absolutely continuous measures is Theorem 2 of Doss [3, pp. 361–362]. Theorem 2 of [3] in the noncompact case may be obtained by simple modifications of the above proofs, which we omit. For an interesting reformulation of Theorem 2 of [3] the reader is referred to [2, p. 114]. For different characterizations of transforms of \(L^1(G)\) and \(C(G)\) functions, see Theorems 3 and 4 of [6, pp. 245–246]. In this connection see also Theorem 2 of [4, p. 78].

References


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