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EXISTENCE THEOREMS FOR INFINITE PARTICLE SYSTEMS

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Abstract. Sufficient conditions are given for a countable sum of bounded generators of semigroups of contractions on a Banach space to be a generator. This result is then applied to obtain existence theorems for two classes of models of infinite particle systems. The first is a model of a dynamic lattice gas, while the second describes a lattice spin system.

1. **Introduction.** Several models of infinite particle systems with interactions have been introduced recently by Spitzer [8]. They describe the behavior of infinitely many indistinguishable particles which move on a countable set S in such a way that the movement of each particle at any particular time is influenced by the state of the entire system at that time. These models were motivated by the desire to study Markov processes which have as invariant measures some of the classical measures of statistical mechanics.

Given an intuitive description of the behavior of the particles, it is often not clear whether or not there exists a Markov process which corresponds to that description. Therefore it is important to find conditions under which infinite particle systems exist. At least two approaches to this problem have been used. Holley [4] applied the Hille-Yosida theorem to construct the required semigroups of operators from intuitively reasonable generators for several interesting classes of interactions on $S=Z^1$ (the integer lattice). The main difficulty here was in verifying the assumption $\mathscr{R}(I-\lambda\Omega)=C(K)$ in the Hille-Yosida theorem, and it was in that verification that the restriction $S=Z^1$ was necessary. Harris [3], on the other hand, was able to give a rather direct probabilistic construction of the process in the case that $S=Z^d$ and each particle is affected only by neighboring particles. This approach has the advantage that it is successful in higher dimensions and that it permits the particles to be distinguishable, so one can follow the behavior of any particular particle. However, it seems to be limited to the nearest neighbor case, or at least to the case in which only particles which are within some fixed distance of each other can interact. In this paper, we will follow the semigroup approach to

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obtain an existence theorem for general countable S and for several general types of interactions. One of the important models we will consider is that of a dynamic lattice gas. A description of it follows.

Let S be a countable set, and $K=\{0, 1\}^S$ with the product topology, so that K is compact. K will be the state space for the system, and $\eta \in K$ will have the interpretation that $u \in S$ is occupied if $\eta(u)=1$ and unoccupied if $\eta(u)=0$. The interaction of the particles will be described via (i) a nonnegative "speed" function $c(u, \eta)$ on $S \times K$, and (ii) a probability transition function $p(x, y)$ on $S \times S$. The intuitive description of the process in terms of these functions is that if at a certain time the system is in state η, a particle at x will attempt a transition during the next small interval of time Δt with probability $c(x, \eta)\Delta t + o(\Delta t)$, and that if it attempts a transition, it will go to y with probability $p(x, y)$ if $\eta(y)=0$ and will remain at x otherwise. In other words, transitions to occupied states are suppressed in keeping with our requirement that each state be occupied by at most one particle at a time. Of course if S is finite, the system reduces to a finite state continuous time Markov chain, and the existence problem is trivial. We will adopt the convention that $c(x, \eta)=0$ whenever $\eta(x)=0$.

In order to define the generator, the following notation will be useful. If $\eta \in K$ and $u, v \in S$, let η_u and $\eta_{u,v}$ be those elements of K defined by

$$\begin{aligned} \eta_{u,v}(x) &= \eta(x) \quad \text{if } x \neq u, v, \\ &= \eta(u) \quad \text{if } x = v, \\ &= \eta(v) \quad \text{if } x = u; \end{aligned} \qquad \begin{aligned} \eta_u(x) &= \eta(x) \quad \text{if } x \neq u, \\ &= 1-\eta(x) \quad \text{if } x = u. \end{aligned}$$

Then $\eta_{u,v}=\eta$ if u and v are both vacant or both occupied, while otherwise $\eta_{u,v}$ represents a transition from one site to another. Let $C(K)$ be the space of continuous real valued functions on K (with the supremum norm), and $\mathscr{F}$ be the set of those elements of $C(K)$ which depend only on finitely many coordinates. Of course $\mathscr{F}$ is dense in $C(K)$. The intuitively reasonable choice for the generator of the process we have in mind is then

$$\Omega f(\eta) = \sum_{x,y \in S} c(x, \eta)p(x, y)[f(\eta_{x,y})-f(\eta)]. \tag{1.1}$$

Note that nonzero contributions to this sum occur only if $\eta(x)=1$ and $\eta(y)=0$, which corresponds to a transition from x to y. The above expression will of course not make sense for all $f \in C(K)$, but it should at least be well defined for $f \in \mathscr{F}$. The restrictions we will place on the functions $c(x, \eta)$ and $p(x, y)$ will guarantee this.

It is clear, of course, that some restrictions are needed. Suppose, for example, that $c(x, \eta)=1$ whenever x is occupied and $p(x, y)$ is a transition function on S which satisfies $\sum_x p(x, y)=\infty$ for some $y \in S$. Then Ωf will not be well defined for $f(\eta)=\eta(y)$. This is not merely a technical problem, but rather reflects in a real sense the fact that no process exists which corresponds to our intuitive prescription.

In this case, for example, if we start the process in the state η given by $\eta(x)=1$ for $x\neq y$ and $\eta(y)=0$ our intuitive description of the process would lead y to become occupied by the "first" particle which tried to jump to y, after which no further transitions would take place. But it is easy to check that there is no such "first" attempt.

A simplified version of our main existence theorem is the following:

THEOREM 1.2. *Suppose that $c(x, \eta)$ and $p(x, y)$ satisfy*

$$p(x, y) \geqq 0, \qquad \sum_y p(x, y) \leqq 1, \tag{1.3}$$

$$\sup_y \sum_x p(x, y) < \infty, \tag{1.4}$$

$$c(x, \eta) \geqq 0, \qquad \sup_{x,\eta} c(x, \eta) < \infty, \tag{1.5}$$

$$\sup_x \sum_u \sup_\eta |c(x, \eta) - c(x, \eta_u)| < \infty. \tag{1.6}$$

Then there is a unique strongly continuous semigroup of positive contractions on $C(K)$ whose generator Ω is given by (1.1) *for $f \in \mathscr{F}$. Hence there is a unique strong Markov process on K with generator Ω.*

All the conditions of the above theorem except (1.4) and (1.6) are quite natural. Both (1.4) and (1.6) express in a general way the requirement that particles which are far apart should not affect each other very much. Condition (1.4) is made reasonable by the example given above, and is satisfied in most cases of interest. For example, it is automatic if $p(x, y)$ is symmetric, or if S is a discrete Abelian group and $p(x, y)=p(0, y-x)$. Condition (1.6) is a type of uniform Lipschitz condition on the functions $c(x, \eta)$ in the following sense. If $f \in C(K)$ satisfies a Lipschitz condition with respect to a metric on K of the form

$$d(\eta, \zeta) = \sum_{x\in S} \alpha(x)|\eta(x) - \zeta(x)|$$

where $\alpha(x) > 0$ and $\sum \alpha(x) < \infty$, then

$$\sum_u \sup_\eta |f(\eta) - f(\eta_u)| < \infty. \tag{1.7}$$

Conversely, if f satisfies (1.7), then for some choice of $\alpha(x)$, f satisfies a Lipschitz condition with respect to d. Condition (1.6) is certainly satisfied, for example, in the case that the speed function has finite range in the sense that there is an N so that, for each $x \in S$, $c(x, \eta)$ depends on η only through at most N coordinates. In the case which is of interest in statistical mechanics, $c(x, \eta)$ is of the form

$$c(x, \eta) = \exp\left\{\sum_{\eta(y)=1} V(x, y)\right\}$$

where $V(x, y)$ is a (potential) function on $S \times S$ which satisfies $\sup_x \sum_y |V(x, y)| < \infty$. Assumptions (1.5) and (1.6) are automatic here.

The next section is devoted to studying the general question of when an infinite sum of bounded generators of semigroups is again the generator of a semigroup. Our sufficient conditions involve the degree to which the summands commute with each other. In §3 we apply this abstract result to prove a somewhat more general version of Theorem 1.2 in which we allow the transition probabilities to depend on the state η of the system in ways other than the simple exclusion model discussed above. The final section derives an existence theorem for a somewhat different type of system from the results of §2. The model considered there describes the behavior of a lattice spin system. While this is an important model in its own right, our point of view here is that it illustrates the wide range of applicability of Theorem 2.8.

Our results can be applied to other models which we have not described in detail. For example, when Theorem 2.8 is applied to the "zero range interaction" model discussed in [4] and [8], the results obtained are somewhat better than those known previously.

2. **The Hille-Yosida Theorem for sums of bounded generators.** We recall that a (possibly unbounded) linear operator A on a Banach space X is called dissipative if $f-\lambda Af=g$ implies that $\|f\|\leqq\|g\|$ whenever $f\in\mathscr{D}(A)$ and $\lambda>0$. A is closed if the graph of A is a closed subset of $X\times X$. Consider two sequences $\{M_n\}$ and $\{U_n\}$ of bounded linear operators on X with the property that $\Omega_n=\sum_{k=1}^n M_kU_k$ is dissipative for each n. We wish to find conditions under which the "limit" of the sequence Ω_n satisfies the assumptions of the Hille-Yosida theorem.

Let μ_k be a sequence of positive numbers such that $\|M_k\|\leqq\mu_k$, and define

$$\mathscr{C}=\left\{f\in X\;\middle|\;\sum_{k=1}^{\infty}\|U_kf\|\mu_k<\infty\right\}.$$

Then $\mathscr{C}$ is a linear space, and for $f\in\mathscr{C}$ we can define

$$\Omega_0f=\lim_{n\to\infty}\Omega_nf.$$

Since Ω_n is dissipative for each n, Ω_0 is also. Let Ω be the closure of the graph of Ω_0 in $X\times X$. Then Ω is dissipative, and since Ω is also closed, $R(I-\lambda\Omega)$ is closed in X for $\lambda>0$.

A slight technical problem arises here because Ω is not necessarily the graph of a linear operator. A useful condition which guarantees this (i.e., that Ω_0 have a minimal closed extension which is a linear operator), is that $\mathscr{C}$ be dense in X (see, for example, Lemma 3.3 of [7]). This assumption is made in Theorem 2.8, and is satisfied thereafter. However, Theorem 2.2 below is valid even if Ω is "multivalued."

The restrictions we will place on the sequences $\{M_n\}$ and $\{U_n\}$ will involve the degree to which they commute with each other. To describe this, we introduce the

usual concept of the commutator of two bounded operators A and B on X: $[A, B] = AB - BA$. Also, put

$$\gamma(A, B) = \sup_f \frac{\|[A, B]f\|}{\|Af\| + \|Bf\|} \tag{2.1}$$

where the supremum is taken over all $f \in X$ for which the denominator is not zero. Note that $\gamma(A, B) = 0$ if A and B commute and that $\gamma(A, B) \leqq \max(\|A\|, \|B\|)$.

THEOREM 2.2. *Suppose there is a constant L so that for all n*

$$\sum_{k=1}^{\infty} \mu_k \gamma(U_k, U_n) \leqq L \tag{2.3}$$

and

$$\sum_{k=1}^{\infty} \mu_k \|[U_k, M_n]\| \leqq L\mu_n. \tag{2.4}$$

Then $\mathscr{R}(I - \lambda\Omega) \supseteq \mathscr{C}$ for $0 < \lambda < \frac{1}{3}L$.

Proof. Fix $g \in \mathscr{C}$. Since Ω_n is bounded and dissipative, $R(I - \lambda\Omega_n) = X$ for all $\lambda > 0$. Fix λ such that $0 < \lambda < \frac{1}{3}L$ and define $f_n \in X$ by

$$f_n - \lambda\Omega_n f_n = g. \tag{2.5}$$

Apply U_m to both sides of (2.5) to obtain

$$U_m f_n - \lambda \sum_{k=1}^{n} U_m M_k U_k f_n = U_m g.$$

By adding and subtracting several terms this can be rewritten in the form

$$U_m f_n - \lambda\Omega_n U_m f_n = U_m g + \lambda \sum_{k=1}^{n} [U_m, M_k] U_k f_n + \lambda \sum_{k=1}^{n} M_k [U_m, U_k] f_n.$$

Using (2.1) and the fact that Ω_n is dissipative,

$$\begin{aligned} \|U_m f_n\| \leqq \|U_m g\| &+ \lambda \sum_{k=1}^{n} \|[U_m, M_k]\| \, \|U_k f_n\| \\ &+ \lambda \sum_{k=1}^{n} \mu_k \gamma(U_m, U_k)[\|U_k f_n\| + \|U_m f_n\|]. \end{aligned}$$

This becomes

$$(1 - \lambda L)\|U_m f_n\| \leqq \|U_m g\| + \lambda \sum_{k=1}^{n} \beta_{m,k} \|U_k f_n\| \tag{2.6}$$

where $\beta_{m,k} = \|[U_m, M_k]\| + \mu_k \gamma(U_m, U_k)$. The assumptions of the theorem imply that

$$\sum_{m=1}^{\infty} \mu_m \beta_{m,k} \leqq 2L\mu_k,$$

which says that, if we regard $\{\mu_m\}$ as a measure μ on the positive integers, $\{\beta_{m,k}\}$ acts as a bounded operator B on $L_1(\mu)$ by $(Bu)^{(m)}=\sum_{k=1}^{\infty}\beta_{m,k}u^{(k)}$ with $\|B\|\leqq 2L$. With this in mind, fix an n and define

$$v^{(m)} = \|U_m f_n\| \quad \text{and} \quad w^{(m)} = \|U_m g\|.$$

Since $g\in\mathscr{C}$, $w=\{w^{(m)}\}\in L_1(\mu)$, and hence $v=\{v^{(m)}\}\in L_1(\mu)$ by (2.6). In this notation, (2.6) becomes

$$(1-\lambda L)v \leqq w+\lambda Bv \tag{2.7}$$

where the inequality is to be understood componentwise. Since B is a positive operator and the norm of $(1-\lambda L)^{-1}\lambda B$ is less than 1, we may iterate (2.7) to obtain

$$v \leqq (1/(1-\lambda L))(I-(\lambda/(1-\lambda L))B)^{-1}w.$$

The important thing to note is that the term on the right of this inequality is independent of n and is in $L_1(\mu)$. So, we conclude that there is a sequence $\{u_m\}$ such that $\sum_{m=1}^{\infty}u_m\mu_m<\infty$ and $\|U_m f_n\|\leqq u_m$ for each n. In particular, $f_n\in\mathscr{C}$ so we may define

$$g_n = f_n-\lambda\Omega_0 f_n.$$

Now, recalling (2.5),

$$\|g_n-g\| = \lambda\|\Omega_0 f_n-\Omega_n f_n\| \leqq \lambda\sum_{k=n+1}^{\infty}\|M_k U_k f_n\| \leqq \lambda\sum_{k=n+1}^{\infty}\mu_k u_k \to 0,$$

so $g_n\to g$. Since $g_n\in\mathscr{R}(I-\lambda\Omega)$ which is closed, $g\in\mathscr{R}(I-\lambda\Omega)$. So, we have shown that $\mathscr{R}(I-\lambda\Omega)\supset\mathscr{C}$. To complete the proof we use again the fact that $\mathscr{R}(I-\lambda\Omega)$ is closed.

As an immediate consequence of this result, we have

THEOREM 2.8. *If $\mathscr{C}$ is dense in X and the assumptions of Theorem 2.2 hold, then Ω generates a unique strongly continuous semigroup $S(t)$ of contractions on X. Furthermore, if $S_n(t)$ is the semigroup generated by Ω_n,*

$$\sup_{0\leqq t\leqq t_0}\|S_n(t)f-S(t)f\|\to 0$$

as $n\to\infty$ for each $t_0>0$ and $f\in X$.

Proof. The first statement follows from the Hille-Yosida Theorem, since (i) $\mathscr{D}(\Omega)\supset\mathscr{C}$ which is dense, (ii) Ω is dissipative, and (iii) $\mathscr{R}(I-\lambda\Omega)=X$ for all sufficiently small $\lambda>0$ by Theorem 2.2. The convergence statement is a consequence of Trotter's Theorem (see, for example, Theorem 3 of [6]).

3. **Application to infinite particle systems with speed change and exclusion.** In this section we will obtain an existence theorem for infinite particle systems as a consequence of the main result of §2. The notation S, K, $\mathscr{F}$ and $C(K)$ will be as

in §1. We begin with nonnegative continuous functions $c(x, \eta)$ on $S \times K$ and $p(x, y, \eta)$ on $S \times S \times K$ which satisfy $c(x, \eta) = 0$ if $\eta(x) = 0$, $p(x, y, \eta) = 0$ if $\eta(y) = 1$, and $\sum_{y \in S} p(x, y, \eta) \leqq 1$, for each $x \in S$ and $\eta \in K$. In order to place the existence problem in the context of the previous section, define bounded operators $M_{(x,y)}$ and $U_{(x,y)}$ on $X = C(K)$ for $x, y \in S$ by

$$U_{(x,y)}f(\eta) = f(\eta_{x,y}) - f(\eta), \qquad M_{(x,y)}f(\eta) = c(x, \eta)p(x, y, \eta)f(\eta).$$

LEMMA 3.1. *For any finite subset T of $S \times S$, the operator*

$$\Omega_T = \sum_{(x,y) \in T} M_{(x,y)} U_{(x,y)}$$

is dissipative.

Proof. Suppose $f - \lambda \Omega_T f = g$ with $\lambda > 0$. Choose η so that $f(\eta) = \max\{f(\zeta), \zeta \in K\}$. Then for each $(x, y) \in T$, $U_{(x,y)}f(\eta) \leqq 0$ and so $f(\eta) \leqq g(\eta)$. Applying the same argument to the minimum of f, we see that $\|f\| \leqq \|g\|$.

Throughout the remainder of the section, we will assume that we are given functions $c(x)$ on S and $\rho(x, y)$ on $S \times S$ such that $c(x, \eta) \leqq c(x)$ and $p(x, y, \eta) \leqq \rho(x, y)$ for all $\eta \in K$. The additional assumptions which will be made at various points are

$$\sup_x c(x) \sum_y \rho(x, y) < \infty, \qquad \sup_y \sum_x c(x)\rho(x, y) < \infty, \tag{3.2}$$

$$\sum_x \sup_\eta |c(u, \eta_x) - c(u, \eta)| \leqq c(u), \tag{3.3}$$

$$\sum_x \sup_\eta |p(u, v, \eta_x) - p(u, v, \eta)| \leqq \rho(u, v). \tag{3.4}$$

Put $\mu_{(x,y)} = c(x)\rho(x, y)$ and define Ω as in §2, using any enumeration of the countable set $S \times S$. It is clear that the definition is independent of the enumeration. In order to show later that the semigroup we construct is positive, we need the following simple result.

LEMMA 3.5. *If $f \in \mathscr{D}(\Omega)$, $g \geqq 0$ and $f - \lambda \Omega f = g$ then $f \geqq 0$.*

Proof. Since $f \in \mathscr{D}(\Omega)$, there are $f_n \in \mathscr{C}$ so that $f_n \to f$ and $\Omega f_n \to \Omega f$. Let $g_n = f_n - \lambda \Omega f_n$. Then $\min\{f_n(\eta), \eta \in K\} \geqq \min\{g_n(\eta), \eta \in K\}$ by the argument of Lemma 3.1. So since $g_n \to g \geqq 0$, it follows that $f \geqq 0$.

LEMMA 3.6. *If (3.2) holds, then $\mathscr{F} \subset \mathscr{C}$. Therefore $\mathscr{C}$ is dense in $C(K)$, and for each $f \in \mathscr{F}$,*

$$\Omega f(\eta) = \sum_{x,y} c(x, \eta)p(x, y, \eta)[f(\eta_{x,y}) - f(\eta)].$$

Proof. If $f \in \mathscr{F}$ depends only on the coordinates in the finite subset T of S, $\|U_{(x,y)}f\| = 0$ if $x \notin T$ and $y \notin T$, and $\|U_{(x,y)}f\| \leqq 2\|f\|$ otherwise. So,

$$\sum_{x,y} \|U_{(x,y)}f\| \mu_{(x,y)} \leqq 2\|f\| \left\{ \sum_{x \in T} c(x) \sum_{y \in S} \rho(x, y) + \sum_{y \in T} \sum_{x \in S} c(x)\rho(x, y) \right\} < \infty,$$

which says that $f \in \mathscr{C}$.

We can now derive our main existence theorem from the results of §2.

THEOREM 3.7. *Assume that* (3.2), (3.3) *and* (3.4) *hold. Then* Ω *generates a unique strongly continuous semigroup* $S(t)$ *of positive contractions on* $C(K)$. *Furthermore, if* $S_T(t)$ *is the semigroup generated by* Ω_T *for* T *a finite subset of* S,

$$\lim_{T\uparrow S} \sup_{0\leqq t\leqq t_0} \|S_T(t)f-S(t)f\| = 0$$

for each $t_0>0$ *and* $f\in C(K)$.

Proof. We must verify the assumptions of Theorem 2.8. $\mathscr{C}$ is dense in $C(K)$ by Lemma 3.6. If $\{x, y\}\cap\{u, v\}=\varnothing$, $U_{(x,y)}$ and $U_{(u,v)}$ commute, while $\|U_{(x,y)}\|\leqq 2$ for all pairs (x, y). So, $\gamma(U_{(x,y)}, U_{(u,v)})=0$ if $\{x, y\}\cap\{u, v\}=\varnothing$ and $\gamma(U_{(x,y)}, U_{(u,v)})\leqq 2$ in any case. It is easy to check then that condition (3.2) implies condition (2.3) of Theorem 2.2 in this case.

The verification of condition (2.4) requires somewhat more computation. First note that

$$\|[U_{(x,y)}, M_{(u,v)}]\| \leqq \sup_{\eta} |c(u, \eta_{x,y})p(u, v, \eta_{x,y})-c(u, \eta)p(u, v, \eta)|.$$

For any function $g\in C(K)$,

$$\sup_{\eta} |g(\eta_{x,y})-g(\eta)| \leqq \sup_{\eta} |g(\eta_x)-g(\eta)|+\sup_{\eta} |g(\eta_y)-g(\eta)|.$$

Using this, we obtain

$$\begin{aligned}\|[U_{(x,y)}, M_{(u,v)}]\| \leqq{}& \rho(u, v)\Big[\sup_{\eta} |c(u, \eta_x)-c(u, \eta)|+\sup_{\eta} |c(u, \eta_y)-c(u, \eta)|\Big]\\ &+c(u)\Big[\sup_{\eta} |p(u, v, \eta_x)-p(u, v, \eta)|+\sup_{\eta} |p(u, v, \eta_y)-p(u, v, \eta)|\Big].\end{aligned}$$

Assumptions (3.3) and (3.4) then lead to

$$\begin{aligned}&\sum_{x,y} c(x)\rho(x, y)\|[U_{(x,y)}, M_{(u,v)}]\|\\ &\qquad\leqq 2c(u)\rho(u, v)\Big\{\sup_{x}\sum_{y} c(x)\rho(x, y)+\sup_{y}\sum_{x} c(x)\rho(x, y)\Big\}.\end{aligned}$$

Using (3.2) again completes the verification of condition (2.4). So Theorem 2.8 applies. The assertion that the semigroup is positive follows from Lemma 3.5, which says that $(I-\lambda\Omega)^{-1}$ leaves the positive cone invariant.

We conclude this section with several remarks. Theorem 3.7 really asserts the existence of a strong Markov process on K with generator Ω, since a standard theorem (e.g. Theorem 9.4 of [1]) gives the existence of the process once the semigroup $S(t)$ is obtained. The convergence part of Theorem 3.7 is often useful in studying properties of the resulting Markov process. For instance, the existence of invariant measures in some models has been obtained by Holley [4] by approxi-

mating the desired measure by measures which are invariant for the processes which correspond to the semigroups $S_T(t)$.

Finally, we should point out that Theorem 1.2 is a special case of Theorem 3.7. To see this, put

$$\begin{aligned} p(x, y, \eta) &= p(x, y) \quad \text{if } \eta(y) = 0, \\ &= 0 \qquad\quad\ \ \text{if } \eta(y) = 1. \end{aligned}$$

To verify the assumptions of Theorem 3.7, take $\rho(x, y) = 2p(x, y)$ and $c(x)$ to be a sufficiently large constant.

4. **Application to lattice spin systems.** In this section, we present a second application of the results of §2 to the existence theory of infinite particle systems. Again let S be a countable set, but now we regard each point of S as being occupied by a particle which can be in any state φ of F, where F is a compact set. The configuration of the entire system can then be described by a point η in $K = F^S$, which we give the product topology. A particle at $x \in S$ waits a random amount of time determined by a "speed" function $c(x, \eta)$ on $S \times K$. Then it changes its state from $\eta(x)$ to another state φ of F according to a transition function $p(\eta(x), d\varphi)$ on F. If $F = \{-1, 1\}$, an interpretation of the process is that the particles, which might be located at the lattice sites of a crystal, can each be spinning up (if $\varphi = +1$) or down (if $\varphi = -1$), and that at certain random times which are influenced by the state of the entire system, the direction of spin of a particle switches.

The problem is to find conditions on $c(x, \eta)$ and $p(\psi, d\varphi)$ under which there exists a strong Markov process on K which behaves according to the intuitive description given above. Some partial existence results were obtained in [2] for a special case. Other properties of this model have been studied in [5].

If $x \in S$ and $\varphi \in F$, let η_x^φ be the element of K defined by

$$\begin{aligned} \eta_x^\varphi(y) &= \eta(y) \quad y \neq x, \\ &= \varphi \qquad\ \ y = x. \end{aligned}$$

Then $f(\eta_x^\varphi)$ is a continuous function of φ whenever $\eta \in K$ and $f \in C(K)$. Throughout this section, we will assume that the transition function $p(\psi, d\varphi)$ satisfies the following regularity condition: the map $\psi \to p(\psi, d\varphi)$ is continuous from F to the space of probability measures on F with the weak* topology. Then the integral $\int_F p(\eta(x), d\varphi) f(\eta_x^\varphi)$ makes sense and is a continuous function of η for each $f \in C(K)$ and $x \in S$. So,

$$(U_x f)(\eta) = \int_F p(\eta(x), d\varphi)[f(\eta_x^\varphi) - f(\eta)]$$

defines a bounded operator on $C(K)$ for each $x \in S$. We assume also that $c(x, \eta) \geqq 0$ is a continuous function of η for each $x \in S$ and that $\sup_{x,\eta} c(x, \eta) < \infty$. Then $(M_x f)(\eta) = c(x, \eta) f(\eta)$ also defines a bounded operator on $C(K)$.

That $\sum_{x\in T} M_x U_x$ is dissipative for any finite subset T of S follows from the argument used in Lemma 3.1. So, define Ω as in §2, taking μ_x to be the constant $\sup_{y,\eta} c(y, \eta)$. Then any $f \in C(K)$ which depends on only finitely many coordinates is in $\mathscr{D}(\Omega)$ and

$$\Omega f = \sum_{x\in S} c(x, \eta) \int_F p(\eta(x), d\varphi)[f(\eta_x^\varphi) - f(\eta)]$$

for such f. The basic assumption we will need for the existence theorem is

$$\sup_y \sum_x \sup_{\eta,\varphi} |c(y, \eta_x^\varphi) - c(y, \eta)| < \infty. \tag{4.1}$$

This is quite similar to condition (1.6) of Theorem 1.2 and again describes the requirement that the speed of a particle at one point should not be affected very much by the behavior of particles at distant sites.

THEOREM 4.2. *If* (4.1) *is satisfied,* Ω *generates a strongly continuous semigroup of positive contractions on* $X = C(K)$. *Therefore there is a unique strong Markov process on* K *with generator* Ω.

Proof. We must verify the assumptions of Theorem 2.8. $\mathscr{C}$ is dense in $C(K)$ since it contains all $f \in C(K)$ which depend only on finitely many coordinates. In this case, U_x and U_y commute, so $\gamma(U_x, U_y) = 0$ and assumption (2.3) is automatic. To obtain condition (2.4) from (4.1), it suffices to note that

$$[U_x, M_y]f(\eta) = \int_F p(\eta(x), d\varphi)[c(y, \eta_x^\varphi) - c(y, \eta)]f(\eta_x^\varphi)$$

so

$$\|[U_x, M_y]\| \leqq \sup_{\varphi,\eta} |c(y, \eta_x^\varphi) - c(y, \eta)|.$$

Finally, the positivity of the semigroup follows again from the fact that $(I - \lambda\Omega)^{-1}$ is a positive operator for $\lambda > 0$.

In order to understand the content of (4.1), it is helpful to consider the special case of this model which is of interest in physics. This is the case which is studied in [5], for example. Take $S = Z^d$ and $F = \{-1, 1\}$. For each finite subset T of Z^d, we are given a number J_T. Define $\sigma_T(\eta) = \prod_{x\in T} \eta(x)$ and

$$c(x, \eta) = \exp\left\{\sum_{T\ni x} J_T \sigma_T(\eta)\right\}.$$

In this context, a sufficient condition for (4.1) to hold is that the numbers J_T should satisfy

$$\sup_{y\in S} \sum_{T\ni y} |J_T| \#(T) < \infty$$

where $\#(T)$ denotes the cardinality of T. This condition is satisfied in [5].

Finally, we note that more complex models could be analyzed using the same techniques. For example, the transition function $p(\psi, d\varphi)$ could be made to depend on the state η. We did not pursue this, since there are many ways of generalizing the model, and we feel that the existence of most models of this type which are of interest can be deduced directly from Theorem 2.8.

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