

## A STURMIAN THEOREM FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

BY  
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**Abstract.** A pair of first order partial differential equations is considered. The system is transformed into a single nonlinear scalar equation of the Riccati type from which some Wirtinger type integral inequalities for functions of several variables are derived. A comparison theorem for two such pairs of first order equations is then proved using the Wirtinger inequalities.

Let  $G$  be a bounded domain of  $d$ -dimensional Euclidean space  $R^d$  with boundary  $\partial G$ . We assume that  $G$  can be approximated from within by a sequence  $\{G_n\}$  of bounded domains each having a smooth boundary  $\partial G_n$  such that  $G_n \subset \bar{G}_n \subset G_{n+1} \subset \bar{G}_{n+1} \subset G$  and  $\bigcup G_n = G$ . Variable points of  $R^d$  will be denoted by  $x = (x_1, \dots, x_d)$  and differentiation with respect to  $x_i$  is denoted by  $D_i$ . Real scalar quantities will be represented by lower case Latin letters  $u, v, w$ , etc.; vectors in  $R^d$  will be denoted by small Greek letters  $\alpha, \beta, \gamma$ , etc.; and matrices will be denoted by capital Latin letters  $A, B, C$ , etc. The inner product between two vectors  $\alpha$  and  $\beta$  will be written as  $\alpha \cdot \beta$  while the length of  $\alpha$  is denoted by  $\|\alpha\|$ . We consider the pair of first order partial differential equations

$$(1) \quad \nabla u = u\alpha(x) + B(x)\zeta, \quad \nabla \cdot \zeta = -p(x)u + \beta(x) \cdot \zeta$$

where  $p, \alpha, \beta$ , and  $B$  are continuous in  $G$  and  $B$  is symmetric and positive definite there. The following is immediate.

LEMMA 1. *Let  $(u, \zeta)$  be a solution of (1) such that  $u(x) \neq 0$  in  $G$ . Define*

$$(2) \quad \varphi(x) = u^{-1}(x)\zeta(x).$$

*Then  $\varphi$  satisfies the generalized Riccati equation*

$$(3) \quad \nabla \cdot \varphi + B\varphi \cdot \varphi + (\alpha - \beta) \cdot \varphi + p = 0.$$

Since  $B$  is symmetric and positive definite,  $B^{-1}$  exists and is symmetric and positive definite in  $G$ . We introduce the functionals  $M(w; G_n)$  and  $Q(w; G_n)$  defined respectively by

$$(4) \quad M(w; G_n) = \int_{G_n} B^{-1}(\nabla w - wB\varphi) \cdot (\nabla w - wB\varphi) \, dx$$

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and

$$(5) \quad Q(w; G_n) = \int_{G_n} (B^{-1} \nabla w \cdot \nabla w - p w^2) dx,$$

where  $w \in C^1(G)$  and  $\varphi$  is any  $C^1$ -solution of (3).

**LEMMA 2.** *Let  $\varphi$  be any  $C^1$ -solution of (3) in  $G$  and let  $G_n \in \{G_n\}$ . Then for every  $w \in C^1(G)$ ,*

$$(6) \quad \int_{\partial G_n} w^2(\varphi \cdot \eta) dS \leq Q(w; G_n) + \int_{G_n} w^2(\beta - \alpha) \cdot \varphi dx,$$

where  $\eta = (\eta_1, \dots, \eta_d)$  denotes the outward pointing unit normal on  $\partial G_n$ . Moreover, equality holds in (6) if, and only if,  $w$  satisfies

$$(7) \quad \nabla w \equiv w(B\varphi).$$

**Proof.** Since  $w \in C^1(G)$  and  $\bar{G}_n \subset G$ , the integral (4) is well defined. Expanding (4) one gets

$$(8) \quad M(w; G_n) = \int_{G_n} \{B^{-1} \nabla w \cdot \nabla w + w^2(B\varphi \cdot \varphi) - 2w \nabla w \cdot \varphi\} dx.$$

Since  $\nabla \cdot (w^2 \varphi) = 2w \nabla w \cdot \varphi + w^2 \nabla \cdot \varphi$ , the last integral in (8) above can be integrated by parts once by means of Green's formula:

$$-2 \int_{G_n} (w \nabla w \cdot \varphi) dx = \int_{G_n} w^2 \nabla \cdot \varphi dx - \int_{\partial G_n} w^2(\varphi \cdot \eta) dS.$$

Putting this into (8) and using the fact that  $\varphi$  satisfies (3), we arrive at

$$(8') \quad 0 \leq M(w; G_n) = Q(w; G_n) + \int_{G_n} w^2(\beta - \alpha) \cdot \varphi dx - \int_{\partial G_n} w^2(\varphi \cdot \eta) dS$$

from which (6) follows. It is clear that equality will hold if, and only if,  $M(w; G_n) = 0$  for every  $n$ , i.e.,  $\nabla w \equiv w B\varphi$ .

Lemma 2 can be stated as a Wirtinger type inequality for certain classes of functions of several variables. To see this we suppose  $\varphi$  is a given solution of (3). Denote by  $\Omega$  all those functions  $w \in C^1(G)$  for which  $M(w) = \lim_n M(w; G_n)$ ,  $Q(w) = \lim_n Q(w; G_n)$ , and

$$\int_{\partial G} w^2(\varphi \cdot \eta) dS = \lim_n \int_{\partial G_n} w^2(\varphi \cdot \eta) dS$$

all exist.

**THEOREM 1.** *Let  $\varphi$  be a solution of (3) such that  $(\alpha - \beta) \cdot \varphi \geq 0$  in  $G$ . Then for every  $w \in \Omega$ ,*

$$(9) \quad \int_{\partial G} w^2(\varphi \cdot \eta) dS \leq Q(w).$$

Moreover, if  $(\alpha - \beta) \cdot \varphi \equiv 0$  in  $G$ , then equality will hold in (9) if, and only if,  $w$  is a solution of (7).

Combining this with Lemma 1 we have

**COROLLARY 1.1.** *Let  $(u, \zeta)$  be a solution of (1) such that  $u(x) \neq 0$  and  $u^{-1}(\alpha - \beta) \cdot \zeta \geq 0$  in  $G$ . Then for every  $w \in \Omega$ ,*

$$(10) \quad \int_{\partial G} w^2 u^{-1} (\zeta \cdot \eta) dS \leq Q(w).$$

Moreover, if  $(\alpha - \beta) \cdot \zeta \equiv 0$  in  $G$ , then equality holds in (10) if, and only if,  $w$  is a solution of

$$(11) \quad \nabla(u^{-1}w) = (u^{-1}w)\alpha.$$

**COROLLARY 1.2.** *Suppose in addition  $\alpha \in C^1(G)$  such that  $D_i a_j \equiv D_j a_i$ , where  $\alpha = (a_1, \dots, a_d)$ . Let  $(u, \zeta)$  be a solution of (1) such that  $u(x) \neq 0$  and  $(\alpha - \beta) \cdot \zeta \equiv 0$  in  $G$ . Then, for every  $w \in \Omega$ , inequality (10) is valid. Moreover, equality holds if, and only if,  $w \equiv u \exp f$  for some  $f \in C^1(G)$ .*

We need only verify the last statement of the corollary. To do this we set  $u^{-1}w = f$ . Then equation (11) becomes

$$(12) \quad \nabla f = \alpha.$$

According to the theorem of Frobenius [4], when  $\alpha \in C^1(G)$ , a necessary and sufficient condition for (12) to be solvable is that  $D_i a_j \equiv D_j a_i$ . This proves the assertion.

For the case where  $\alpha = \beta \equiv 0$ , (1) has the form

$$(1') \quad \nabla u = B\zeta, \quad \nabla \cdot \zeta = -pu.$$

In this case equality will hold in (10) if, and only if,  $w \equiv ku$ , where  $k$  is a constant.

We remark that the Wirtinger inequality (9) is valid even though the coefficient functions in (3) may have singularities on all or parts of the boundary  $\partial G$ . The family  $\Omega$  of admissible functions must of course be so chosen that the integrals appearing in (6) have finite limits. In the one dimensional case such integral inequalities are well known, cf., [1] and [3, Theorem 253]. In the example below we shall give one such inequality for a plane rectangular domain.

In this example  $J_p(t)$  will denote as usual the Bessel function of the first kind of order  $p$ . Let  $H = \{(x_1, x_2) \in R^2 : 0 < x_1 < t^*, 0 < x_2 < 2\pi/\sqrt{3}\}$ , where  $t^*$  denotes the first zero of  $J'_{3/4}(t)$  to the right of the origin  $t = 0$ . The boundary  $\partial H$  consists of four edges  $\gamma_i$ ,  $i = 1, \dots, 4$ , where

$$\gamma_1 = \{(x_1, x_2) \in R^2 : 0 \leq x_1 < t^*, x_2 = 2\pi/\sqrt{3}\},$$

$$\gamma_2 = \{(x_1, x_2) \in R^2 : x_1 = 0, 0 \leq x_2 < 2\pi/\sqrt{3}\},$$

$$\gamma_3 = \{(x_1, x_2) \in R^2 : 0 < x_1 \leq t^*, x_2 = 0\},$$

and

$$\gamma_4 = \{(x_1, x_2) \in R^2 : x_1 = t^*, 0 < x_2 \leq 2\pi/\sqrt{3}\}.$$

We consider the linear second order equation

$$(13) \quad \Delta_2 u + (x_1)^{-1} D_1 u + (x_1)^{-2} u = 0$$

subject to the boundary conditions

$$(14) \quad u = 0 \text{ on } \gamma_1 \cup \gamma_2 \cup \gamma_3, \quad D_1 u = 0 \text{ on } \gamma_4.$$

Equation (13) can be put into the form of (1) by setting  $\nabla u = \zeta$ ,  $\alpha \equiv 0$ ,  $\beta = (-1/x_1, 0)$ ,  $p(x_1, x_2) = 1/x_1^2$ , and  $B = I_2$ , the  $2 \times 2$  identity matrix. Thus the coefficients  $p$  and  $\beta$  are singular on the left edge  $\gamma_2$ . It is a simple matter to verify that

$$(15) \quad u(x_1, x_2) = J_{3/4}(x_1) \sin(\sqrt{3}x_2/2)$$

is a solution for (13) and (14). Moreover,  $u(x) > 0$  and  $(\alpha - \beta) \cdot \varphi = -\beta \cdot (u^{-1} \nabla u) = D_1 u / u > 0$  in  $H$ . Finally,  $u$  has a simple zero on  $\gamma_1 \cup \gamma_3$ , a zero of order  $3/4$  on  $\gamma_1$ , and  $D_1 u = \nabla u \cdot \eta = 0$  on  $\gamma_4$ . If we take as admissible class  $\Omega_H$  all those functions  $w \in C^1(H)$  for which  $w$  has a zero of order  $r > 1/2$  on  $\gamma_1 \cup \gamma_2 \cup \gamma_3$ , then we have the following special example of Theorem 1.

COROLLARY 1.3. *Let  $H$  and  $\Omega_H$  be as defined above. Then for every  $w \in \Omega_H$*

$$\iint_H \|\nabla w\|^2 dx_1 dx_2 > \iint_H (x_1)^{-2} w^2 dx_1 dx_2$$

unless  $w \equiv 0$ .

As an application of the Wirtinger type inequality (10) we shall prove a comparison theorem between two first order systems of type (1). To simplify the formulation of such a result we shall let  $\Gamma_0$  be an arc of  $\partial G$  containing a subarc  $\gamma_0$  on which the coefficient functions may have singularities. Note that the possibility of  $\gamma_0 = \Gamma_0 = \partial G$  is not excluded. Denote by  $\Gamma^* = \partial G \setminus \Gamma_0$ . The equations to be compared are

$$(16) \quad \begin{aligned} \nabla u &= u\alpha_1 + B_1 \zeta, & x \in G; & & u &= 0, & x \in \Gamma_0, \\ \nabla \cdot \zeta &= -p_1 u + \alpha_1 \cdot \zeta, & x \in G; & & \zeta \cdot \eta &= g_1(x)u, & x \in \Gamma^*, \end{aligned}$$

and

$$(17) \quad \begin{aligned} \nabla w &= w\alpha_2 + B_2 \xi, & x \in G; & & w &= 0, & x \in \Gamma_0, \\ \nabla \cdot \xi &= -p_2 w + \alpha_2 \cdot \xi, & x \in G; & & \xi \cdot \eta &= g_2(x)w, & x \in \Gamma^*. \end{aligned}$$

We make the following assumptions:

H1.  $B_1$  and  $B_2$  are symmetric positive definite matrices of class  $C(\bar{G})$ .

H2.  $p_i, \alpha_i \in C(G)$ ,  $i = 1, 2$ , and they can all be extended continuously to  $(G \cup \Gamma^*)$ .

H3. If  $\bar{x} \in \gamma_0$  and  $\{y_n\}$  is a sequence of points in  $G$  such that  $y_n \in G_n$  and  $\lim_n \|y_n - \bar{x}\| = 0$ , then the functions  $p_i$  are of order  $O(\|y_n - \bar{x}\|^{-2})$  and  $\alpha_i$  are of order  $O(\|y_n - \bar{x}\|^{-1})$ .

**THEOREM 2.** *Suppose H1, H2 and H3. Let  $(w, \xi)$  be a solution of (17) such that  $w \neq 0$  in  $G$  and that  $w$  has a zero of order  $r > 1/2$  on  $\Gamma_0$ . If  $(u, \zeta)$  is a solution of (16) and if*

$$(18) \quad \int_G \{ \nabla w \cdot (B_1^{-1} - B_2^{-1}) \nabla w + (p_2 + \alpha_2 \cdot B_2^{-1} \alpha_2 - p_1) w^2 \} dx \leq \int_{\Gamma^*} w^2 (g_1 - g_2) dS,$$

*then  $u$  must have a zero in  $G$  unless  $u$  and  $w$  are related by (11).*

**Proof.** We first note that under the given assumptions it is not difficult to verify that all the integrals appearing in (10) and (18) exist. Suppose the contrary conclusion and let  $(u, \zeta)$  be a solution of (16) such that  $u(x) \neq 0$  in  $G$ . If we multiply the second equation in (17) by  $w$  and integrate by parts once, we have

$$\begin{aligned} \int_G (w \alpha_2 \cdot \xi - p_2 w^2) dx &= \int_G w \nabla \cdot \xi dx \\ &= \int_{\partial G} w (\xi \cdot \eta) dS - \int_G \nabla w \cdot \xi dx. \end{aligned}$$

Using the first equation in (17) and the boundary conditions together with H1, this may be rewritten as

$$(19) \quad \int_G \{ \nabla w \cdot B_2^{-1} \nabla w - (p_2 + \alpha_2 \cdot B_2^{-1} \alpha_2) w^2 \} dx = \int_{\Gamma^*} w^2 g_2 dS.$$

Now  $u(x) \neq 0$  in  $G$  implies (10) holds so that

$$\int_G (\nabla w \cdot B_1^{-1} \nabla w - p_1 w^2) dx \geq \int_{\Gamma^*} w^2 g_1 dS.$$

Combining this with (19) we arrive at a contradiction to (18) unless we have equality. According to Corollary 1.1, this latter occurs if, and only if,  $u$  and  $w$  are related by (11). This proves the theorem.

We remark that this result contains as special cases the theorems of Clark and Swanson [2] and Kreith [5]. We also note that the coefficients in (16) and (17) may depend on  $(u, \zeta)$  and  $(w, \xi)$  respectively as well as on  $x$  so that the systems are quasilinear. Kreith [6] has recently proved a comparison theorem for two such systems using a generalized Picone-type identity. The order relation assumed in H3 may be replaced by a somewhat weaker one, but we must then assume a corresponding change in the order  $r$  of zeros of  $w$  on  $\Gamma_0$ .

Another comparison theorem between two systems of type (1) will now be derived. To do this we shall first establish another inequality similar to (9) in which a different assumption is used in place of the requirement  $(\alpha - \beta) \cdot \varphi \geq 0$ . To this end we let  $\lambda$  be the  $d$ -vector  $\nabla w - w B \varphi$  and denote by  $\lambda^*$  the  $(d+1)$ -vector  $\lambda^* = (\lambda, w)$ . Let  $\theta = B^{-1}(\beta - \alpha)/2 = (t_1, \dots, t_d)$  and let  $E$  be the  $(d+1) \times (d+1)$  matrix

$$E = \begin{pmatrix} I_d & \theta \\ \theta' & g \end{pmatrix},$$

where  $g$  is some scalar valued function and  $\theta'$  is the transpose of  $\theta$ . Denote by  $q(\psi^*)$  the quadratic form in  $(d+1)$  variables  $\psi^* = (f_1, \dots, f_{d+1})$ ,

$$(20) \quad q(\psi^*) = \psi^* \cdot E\psi^*,$$

and by  $t_i^*$  the cofactor of  $t_i$  in  $E$ .

**THEOREM 3.** *Let  $\varphi$  be a solution of (3) in  $G$ . Suppose there exists  $g \in C(G)$  such that*

$$(21) \quad g \det B^{-1} \geq \sum_1^d t_i t_i^*.$$

Then for every  $w \in \Omega$  for which  $\lim_n \int_{G_n} gw^2 dx = \int_G gw^2 dx$  exists, we have

$$(22) \quad \int_{\partial G} w^2 \varphi \cdot \eta dS \leq \int_G \{ \nabla w \cdot B^{-1} \nabla w + w \nabla w \cdot B^{-1} (\beta - \alpha) + (g - p) w^2 \} dx.$$

**Proof.** It is known that condition (21) is both necessary and sufficient for the quadratic form (20) to be positive semidefinite, cf. [7]. It follows that

$$\begin{aligned} 0 &\leq \int_{G_n} q(\lambda^*) dx \\ &= \int_{G_n} \{ B^{-1} (\nabla w - w B \varphi) \cdot (\nabla w - w B \varphi) + w \nabla w \cdot B^{-1} (\beta - \alpha) - w^2 (\beta - \alpha) \cdot \varphi + gw^2 \} dx \\ &= M(w; G_n) + \int_{G_n} \{ w \nabla w \cdot B^{-1} (\beta - \alpha) - w^2 (\beta - \alpha) \cdot \varphi + gw^2 \} dx. \end{aligned}$$

Using (8') and taking the limit as  $n$  tends to infinity, we arrive at (22).

We are now ready to state a comparison theorem between the two first order systems

$$(23) \quad \begin{aligned} \nabla u &= u \alpha_1 + B_1 \zeta, & x \in G; & & u &= 0, & x \in \Gamma_0, \\ \nabla \cdot \zeta &= -p_1 u + \beta_1 \cdot \zeta, & x \in G; & & \zeta \cdot \eta &= g_1 u, & x \in \Gamma^*, \end{aligned}$$

and

$$(24) \quad \begin{aligned} \nabla w &= w \alpha_2 + B_2 \xi, & x \in G; & & w &= 0, & x \in \Gamma_0, \\ \nabla \cdot \xi &= -p_2 w + \beta_2 \cdot \xi, & x \in G; & & \xi \cdot \eta &= g_2 w, & x \in \Gamma^*. \end{aligned}$$

In addition to H1, H2 and H3 we also assume that  $\beta_i$  satisfies the same hypotheses as  $\alpha_i$ . Moreover, we suppose the existence of a  $g \in C(G)$  satisfying the same assumptions as  $p_i$  and that (21) holds. Denote by

$$\begin{aligned} V(w) &= \int_G \{ \nabla w \cdot (B_1^{-1} - B_2^{-1}) \nabla w + (g + p_2 + \beta_2 \cdot B_2^{-1} \alpha_2 - p_1) w^2 \} dx \\ &\quad + \int_G w \nabla w \cdot [B_1^{-1} (\beta_1 - \alpha_1) - B_2^{-1} (\beta_2 - \alpha_2)] dx. \end{aligned}$$

THEOREM 4. *Under the assumptions stated above, let  $(w, \xi)$  be a solution of (24) such that  $w \neq 0$  in  $G$  and that  $w$  has a zero of order  $r > \frac{1}{2}$  on  $\Gamma_0$ . If  $(u, \zeta)$  is a solution of (23) and if*

$$V(w) < \int_{\Gamma_0} w^2(g_1 - g_2) dS,$$

*then  $u$  must have a zero in  $G$ .*

Except for the obvious changes the proof is entirely similar to that of Theorem 2 and will therefore be omitted.

We remark that Theorem 4 includes in particular a result of Swanson [7] on nonselfadjoint second order elliptic equations. The technique used here can also be applied to a single elliptic equation of the fourth order [9] as well as to matrix systems of second order elliptic equations [8].

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