SLICING THEOREMS FOR n-SPHERES IN EUCLIDEAN (n+1)-SPACE

BY

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Abstract. This paper describes conditions on the intersection of an n-sphere Σ in Euclidean (n+1)-space \( E^{n+1} \) with the horizontal hyperplanes of \( E^{n+1} \) sufficient to determine that the sphere be nicely embedded. The results generally are pointed towards showing that the complement of Σ is 1-ULC (uniformly locally 1-connected) rather than towards establishing the stronger property that Σ is locally flat. For instance, the main theorem indicates that \( E^{n+1} - \Sigma \) is 1-ULC provided each non-degenerate intersection of Σ and a horizontal hyperplane be an \( (n-1) \)-sphere bicollared both in that hyperplane and in Σ itself \( (n \neq 4) \).

1. Introduction. Much of the literature that treats the problem of determining properties of the embedding of an object \( \Sigma \) in \( E^n \) from information about the intersection of \( \Sigma \) with the horizontal hyperplanes of \( E^n \) focuses on the case \( n = 3 \). Such a problem first arose in this dimension when J. W. Alexander [1] suggested that a 2-sphere in \( E^3 \) might be embedded just as a round sphere if each of its intersections with the horizontal planes were either a point or a simple closed curve. Recently Eaton [11] and Hosay [12] showed this to be true. After generalizations by Love-land [15] and Jensen [13], Cannon proved that the same property is held by any 2-sphere \( \Sigma \) in \( E^3 \) such that no intersection of \( \Sigma \) with a horizontal plane has a degenerate component [7].

For higher dimensions Bryant has proved that a \( k \)-dimensional compact set \( X \) in \( E^n \), where \( n - k \geq 3 \), has a 1-ULC complement if the complement of \( X \) with respect to each member of some dense subset of the horizontal hyperplanes of \( E^n \) is 1-ULC [5].

The main results of this paper are found in §5, where it is shown that a closed \( n \)-manifold \( \Sigma \) topologically embedded in \( E^{n+1} \) \( (n \neq 4) \) is nice (meaning, \( E^{n+1} - \Sigma \) is 1-ULC) if each nondegenerate intersection \( \Sigma_i \) of \( \Sigma \) and a horizontal, \( n \)-dimensional hyperplane in \( E^{n+1} \) is a PL \( (n-1) \)-manifold bicollared in that hyperplane and nice in \( \Sigma \) \( (\Sigma - \Sigma_i \text{ is 1-ULC}) \). Most of the techniques required are consigned to §4. In §6 other generalizations to the results mentioned in the first paragraph are given, the methods for which are taken from [7] and [12]. In addition, we describe

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in §3 some methods, similar to but weaker than those of the three-dimensional case (see [3]), for improving mappings of a disk into the closure of a complementary domain of an \( n \)-manifold in an \((n + 1)\)-manifold.

2. Definitions and notation. An \( n \)-manifold is a separable metric space which is locally homeomorphic to \( \mathbb{E}^n \); thus, the term manifold is reserved for manifolds without boundary. For simplicity we shall assume all manifolds to be connected, but they need not be compact or triangulated. A manifold that is compact (and without boundary) is said to be closed.

A subset \( S \) of a metric space is called an \( \varepsilon \)-subset if and only if the diameter of \( S \), written \( \text{diam} \ S \), is less than \( \varepsilon \).

Suppose \( f \) and \( g \) are maps of a space \( X \) into a space \( Y \) that has a metric \( \rho \). The symbol \( \rho(f, g) < \varepsilon \) means that \( \rho(f(x), g(x)) < \varepsilon \) for each \( x \) in \( X \). The maps \( f \) and \( g \) are said to be \( \varepsilon \)-homotopic (\( \varepsilon \)-isotopic) if and only if there exists a homotopy (isotopy) \( h_t \) sending \( X \) into \( Y \) such that \( h_0 = f, h_1 = g \) and \( \rho(h_s, h_t) < \varepsilon \) for all \( s, t \) in \([0, 1]\).

A map \( f \) of the metric space \( Y \) into a subset \( A \) is an \( \varepsilon \)-map if and only if \( \rho(y, f(y)) < \varepsilon \) for each \( y \in Y \).

The symbol \( \Delta^2 \) denotes a 2-simplex fixed throughout this paper and \( \partial \Delta^2 \) denotes its boundary. Given a triangulation \( R \) of \( \Delta^2 \), we use \( R^i \) to denote the \( i \)-skeleton of \( R \) (\( i = 0, 1 \)).

For any point \( p \) in a metric space \( S \) and any positive number \( \delta \), \( N_\delta(p) \) denotes the set of points in \( S \) whose distance from \( p \) is less than \( \delta \).

Let \( A \) denote a subset of a metric space \( X \) and \( p \) a limit point of \( A \). We say that \( A \) is locally simply connected at \( p \), written \( 1-\text{LC} \) at \( p \), if and only if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that each map of \( \partial \Delta^2 \) into \( A \cap N_\delta(p) \) can be extended to a map of \( \Delta^2 \) into \( A \cap N_\varepsilon(p) \). Furthermore, we say that \( A \) is uniformly locally simply connected, written \( 1-\text{ULC} \), if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that each map of \( \partial \Delta^2 \) into a \( \delta \)-subset of \( A \) can be extended to a map of \( \Delta^2 \) into an \( \varepsilon \)-subset of \( A \).

In the same context we say that \( A \) is locally arcwise connected at \( p \), or \( 0-\text{LC} \) at \( p \), if and only if for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that any map of \( \partial I \) (where \( I = [0, 1] \)) into \( A \cap N_\delta(p) \) extends to a map of \( I \) into \( A \cap N_\varepsilon(p) \). We define analogously the uniform condition \( 0-\text{ULC} \).

For a subset \( U \) of a space \( S \) we use the symbol \( \text{Cl} \ U \) to denote the closure of \( U \) and \( \text{Bd} \ U \) to denote the (topological) boundary of \( U \) in \( S \).

Let \( \Sigma \) be an \( n \)-manifold embedded in the interior of an \((n + 1)\)-manifold \( M \) as a closed subset, and let \( U \) be an open subset of \( M - \Sigma \). Then \( \Sigma \) is collared from \( U \) if and only if there exists a homeomorphism \( g \) of \( \Sigma \times \{0\} \) into \( \text{Cl} \ U \) such that \( g(s, 0) = s \) for each \( s \) in \( \Sigma \). Similarly, \( \Sigma \) is bicollared in \( M \) if and only if there exists a homeomorphism \( h \) of \( \Sigma \times [-1, 1] \) into \( M \) such that \( h(s, 0) = s \) for each \( s \) in \( \Sigma \). In addition, \( \Sigma \) is locally flat in \( M \) if and only if each point \( s \) of \( \Sigma \) has a neighborhood \( N \) (relative to \( M \)) such that \( N \cap \Sigma \) is bicollared in \( N \).
Let $n$ be a positive integer. For each real number $t$ define $E_t$, the hyperplane of $E^{n+1}$ at the $t$-level, as $\{x_1, \ldots, x_{n+1} \in E^{n+1} \mid x_{n+1} = t\}$. For any subset $\Sigma$ of $E^{n+1}$ we define $E \Sigma$ as $E \cap E_t$, and for any set $C$ of real numbers we define $E(C)$ as $\bigcup \{E_t \mid t \in C\}$; however, for an interval $(a, b)$ we simplify $E((a, b))$ to $E(a, b)$.

3. Altering maps of a disk.

**Lemma 3.1.** Suppose $\Sigma$ is an $n$-manifold embedded in the interior of an $(n + 1)$-manifold $M$ as a closed separating subset, $U$ a component of $M - \Sigma$, $X$ a closed subset of $\Sigma$ such that $\text{Cl} \: U - X$ is 1-LC at each point of $X$, $R$ a triangulation of $\Delta^2$, $T$ a subdivision of $R$, and $F$ a map of $\Delta^2$ into $\text{Cl} \: U$ such that $F([R^{(1)}]) \subseteq U$.

Then for each $\varepsilon > 0$ there exists a map $G$ of $\Delta^2$ into $\text{Cl} \: U$ such that

- (a) $G \mid \partial \Delta^2 = F \mid \partial \Delta^2$,
- (b) $\rho(F, G) < \varepsilon$,
- (c) $G([T^{(1)}]) \subseteq U$,
- (d) $G(\Delta^2) \cap X = \emptyset$.

**Proof.** Note that, since $F(\Delta^2)$ is compact, there exist a neighborhood $V$ of $F(\Delta^2)$ (relative to $\text{Cl} \: U$) and a positive number $\delta$ such that any map of $\partial \Delta^2$ into a $\delta$-subset of $V - X$ can be extended so as to send $\Delta^2$ into an $(\varepsilon/3)$-subset of $\text{Cl} \: U - X$.

Now we simply modify $F$, beginning with the $0$-skeleton of $T$ and working up. In case $v \in T^{(0)}$ and $F(v) \in \Sigma$, define $G(v)$ as a point of $U \cap V$ very close to $F(v)$; when $F(v) \notin \Sigma$, define $G(v) = F(v)$. Since $U$ is 0-LC at each point of $\Sigma$ [*17, Theorem II.5.35*], then, for each 1-simplex $\sigma$ of $T$, $G$ can be extended along $\sigma$ in such a way that $G(\sigma) \subseteq U \cap V$ and $\rho(G|\sigma, F|\sigma) < \varepsilon/3$; in case $\sigma \subseteq [R^{(1)}]$, define $G|\sigma = F|\sigma$. Because first we could have subdivided $T$ (if necessary), we can assume that, for each 2-simplex $\tau$ of $T$, $\text{Diam} \: F(\tau) < \varepsilon/3$ and $\text{Diam} \: G(\partial \tau) < \delta$. According to the previous paragraph, $G$ can be extended over $\tau$ into an $\varepsilon/3$-subset of $\text{Cl} \: U - X$. It follows easily that $\rho(F, G) < \varepsilon$.

**Theorem 3.2.** Suppose $\Sigma$ is an $n$-manifold embedded in the interior of an $(n + 1)$-manifold $M$ as a closed separating subset, $U$ a component of $M - \Sigma$, and $f$ a map of $\Delta^2$ into $\text{Cl} \: U$ with $f(\partial \Delta^2) \subseteq U$. Suppose $\{X^i\}$ is a countable collection of closed subsets of $\Sigma$ such that $\text{Cl} \: U - X^i$ is 1-LC at each point of $X^i$ ($i = 1, 2, \ldots$). Then for each $\varepsilon > 0$ there exists a map $g$ of $\Delta^2$ into $\text{Cl} \: U$ such that

- (1) $g|\partial \Delta^2 = f|\partial \Delta^2$,
- (2) $\rho(f, g) < \varepsilon$,
- (3) $g^{-1}(\Sigma \cap g(\Delta^2))$ is 0-dimensional,
- (4) $g(\Delta^2) \cap X^i = \emptyset$ for $i = 1, 2, \ldots$.

The proof follows from routine applications of Lemma 3.1.

This result yields the following corollary, which can be regarded as a very weak version of [*3, Theorem 4.2*].

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Corollary 3.3. Suppose \( \Sigma \) is an \( n \)-manifold embedded in the interior of an \( (n + 1) \)-manifold \( M \) as a closed separating subset, \( U \) a component of \( M - \Sigma \), and \( f \) a map of \( \Delta^2 \) into \( \text{Cl} U \) such that \( f(\partial \Delta^2) \subseteq U \). Then for each \( \varepsilon > 0 \) there exists a map \( g \) of \( \Delta^2 \) into \( \text{Cl} U \) such that (i) \( g|\partial \Delta^2 = f|\partial \Delta^2 \), (ii) \( \rho(f, g) < \varepsilon \), and (iii) \( g^{-1}(\Sigma \cap g(\Delta^2)) \) is 0-dimensional.

4. Embedding Cartesian products in \( E^{n+1} \). Let \( n \) denote a fixed positive integer. Obviously there exists a countable collection \( \mathcal{M} \) of closed, PL \( (n-1) \)-manifolds such that any closed, PL \( (n-1) \)-manifold is homeomorphic to some member of \( \mathcal{M} \). (In fact, this holds even without the PL hypothesis \([8]\).)

In this section \( \Sigma \) will denote an \( n \)-manifold embedded in \( E^{n+1} \) as a closed subset, \( U \) a component of \( E^{n+1} - U \), and \( (a, b) \) an interval of real numbers such that, for each \( t \in (a, b) \), \( \Sigma_t \) is homeomorphic to some member of \( \mathcal{M} \) and is collared from \( U_t \). Let \( (a, b)_M \) be the set of all \( t \) in \( (a, b) \) such that \( \Sigma_t \) is homeomorphic to \( M \), where \( M \in \mathcal{M} \). For each such \( M \) and each \( t \in (a, b)_M \) define an embedding of \( M \times [-1, 1] \) into \( \text{Cl} U_t \) such that \( \lambda_t(M \times \{-1\}) = \Sigma_t \). Topologize the set \( \{\Sigma_t | t \in (a, b)_M\} \) by the sup-norm metric in \( E^{n+1} \), producing a separable metric space.

As suggested by Bryant \([4]\) (and in the unpublished work of Bing \([2]\)), one can easily establish the following lemma.

Lemma 4.1. There exists a countable subset \( \mathcal{D} \) of \( (a, b) \) such that to each \( t \) in \( (a, b) - \mathcal{D} \) there correspond two sequences \( \{s(i)\} \) and \( \{u(i)\} \) of real numbers, with \( a < s(i) < t < u(i) < b \), such that each of the associated sequences \( \{\lambda_{s(i)}\} \) and \( \{\lambda_{u(i)}\} \) converges (homeomorphically) to \( \lambda_t \).

Throughout the rest of §§4 and 5, \( \mathcal{D} \) will denote the subset of \( (a, b) \) described in Lemma 4.1, and \( n \) will denote a fixed integer other than 4.

The following result can be established by adding simple epsilonics to the proof of Borsuk's theorem (see \([10, \text{Theorem 10.2}]\)).

Lemma 4.2. Let \( A \) be an ANR embedded as a closed subset of a metric space \( X \), \( \varepsilon > 0 \), and \( f: X \to A \) an \( \varepsilon \)-map such that \( f|A \) is \( \varepsilon \)-homotopic (in \( A \)) to the identity map. Then there exists a 2\( \varepsilon \)-retraction of \( X \) onto \( A \).

Lemma 4.3. A. If \( t \in (a, b)_M - \mathcal{D} \), then there exists a homeomorphism \( h \) of \( M \times [-1, 1] \) onto a subset \( A_t \) of \( \text{Cl} U \) such that

\[
\Sigma \cap A_t = h(M \times \{-1\}) \cup h(M \times \{1\}) = \Sigma_s \cup \Sigma_u,
\]

where \( a < s < t < u < b \) and \( A_t \cap E_z = \emptyset \) for \( z \) not in \([s, u]\).

B. For any such \( A_t \), let \( X_t \) denote the closure of the component of \( \text{Cl} U - A_t \) containing \( \Sigma_t \). Then, for each \( \varepsilon > 0 \), \( A_t \) can be obtained so that there exists an \( \varepsilon \)-retraction of \( X_t \) onto \( A_t \).

Proof. Fix a point \( t \) of \( (a, b)_M - \mathcal{D} \). By restricting \([-1, 1]\) to a subinterval containing \(-1\), if necessary, we may assume that \( \text{diam} \lambda_t([p] \times [-1, 1]) \) is less than...
e/18 for each p in M. Since λ(M × {0}) is an ANR, the obvious retraction of λ(M × [-1, 1]) onto λ(M × {0}) can be extended over a neighborhood N of λ(M × [-1, 1]) to an e/18-retraction R of N onto λ(M × {0}). Furthermore, N can be chosen as a product N' × (s', u') ⊂ E^n × E^1 where N' is a bounded open subset of E^n and N ∩ Σ = Σ(s', u').

It is sufficient to describe the homeomorphism h of M × [-1, 1] onto some (as yet undefined) A_t subject to the following conditions:

1. X_t ⊂ N,
2. diam h((p) × [-1, 1]) < e/6 for each p in M,
3. h(p, 0) = λ_t(p, 0) for each p in M.

Let g denote the map of A_t to h(M × {0}) sending each h((p) × [-1, 1]) to h(p, 0).

The product structure on A_t will provide the natural guide for defining an e/6-homotopy G_s: A_t → A_t between g and the identity map. Note that condition (2) above and the definition of R imply that

\[\text{diam } RH((p) × [-1, 1]) < e/3\]

for each p in M. Thus, RG_s will be an e/3-homotopy between Rg = g and R. This means that R|A_t will be e/2-homotopic in A_t to the identity map, and part B of this lemma will be a consequence of Lemma 4.2.

Let δ_t denote the distance from λ_t(M × {0}) to Σ ∩ (E^{n+1} − N). It follows from [18] (see also [16, Lemma 5]) in case n > 4 and from [14, Lemma 4] or [9, Theorem 8.2] in case n = 3 that there exists a δ > 0 such that any locally flat n-manifold in \( E_t \) homeomorphically within \( d_t \) of \( \Sigma_t \) is \( \delta \)-isotopic to \( \Sigma_t \) in \( E_t \). According to Lemma 4.1 there exist real numbers s and u, with \( s' < s < t < u < u' \), such that \( λ_s \) and \( λ_u \) are homeomorphically within \( d_t \) of \( λ_t \). If \( p_t \) denotes the map projecting \( E^n × \{t\} \) onto \( E^n × \{t\} \), then \( p_t λ_s(M × \{0\}) \) and \( p_t λ_u(M × \{0\}) \) are each \( δ \)-isotopic to \( λ_t(M × \{0\}) \) in \( E_t \). By lifting this isotopy through the levels \( E_s \) (\( s ≤ r ≤ u \)), we construct an embedding h of \( M × [-\frac{1}{2}, \frac{1}{2}] \) into \( U ∩ N \) such that condition (3), as well as the following conditions, holds:

4. \( h(M × \{-\frac{1}{2}\}) = λ_s(M × \{0\}) \),
5. \( h(M × \{\frac{1}{2}\}) = λ_u(M × \{0\}) \),
6. for each \( w ∈ [-\frac{1}{2}, \frac{1}{2}] \), there exists a distinct \( z ∈ [s, u] \) such that \( h(M × \{w\}) ⊂ E_z \).
7. \( \text{diam } h((p) × [-\frac{1}{2}, \frac{1}{2}]) < e/18 \) for each p in M.

Since both \( λ_s \) and \( λ_u \) are homeomorphically close to \( λ_t \), we may assume s and u were chosen so that

8. \( \text{diam } λ_s((p) × [-1, 0]) < e/18 \) for each p in M,
9. \( \text{diam } λ_u((p) × [-1, 0]) < e/18 \) for each p in M.

Now \( h(M × [-\frac{1}{2}, \frac{1}{2}] \) can be extended to a homeomorphism h of \( M × [-1, 1] \) onto

\[A_t = λ_s(M × [-1, 0]) ∪ h(M × [-\frac{1}{2}, \frac{1}{2}]) ∪ λ_u(M × [-1, 0]).\]
To verify that \( X_t \subset N \), observe that \( N \) has connected boundary, as does \( X_t \) (where \( \text{Bd} \ X_t \) is taken in \( E^{n+1} \), not in \( \text{Cl} U \)). Since \( \text{Bd} \ X_t \subset N \) by construction and since both \( X_t \) and \( N \) are bounded, \( X_t \) must be a subset of \( N \). The only unverified requirement on the construction, condition (2), follows easily from (7)–(9).

**Addendum.** Suppose \( \varepsilon > 0 \), \( X_t \) and \( A_t \) satisfy the conclusions of Lemma 4.3B, and \( Z \) is a compact subset of \( X_t \). Then there exists an \( \varepsilon \)-map of \( Z \) into \( A_t \) such that \( f(Z) \cap \text{Bd} A_t \subset Z \) and \( f|_{Z \cap A_t} = \text{identity} \).

**Proof.** Follow the \( \varepsilon \)-retraction of \( X_t \) onto \( A_t \) by a small homeomorphism \( g \) of \( A_t \) into \((Z \cap \text{Bd} A_t) \cup \text{Int} A_t \) such that \( g|Z \cap A_t = \text{identity} \).

5. **Submanifolds of** \( E^{n+1} \)** whose levels are twice bicollared. The basic result in this section is the following application of Lemma 4.3.

**Theorem 5.1.** Let \( \Sigma \) denote an \( n \)-manifold embedded in \( E^{n+1} \) (\( n \neq 4 \)) as a closed subset, \( U \) a component of \( E^{n+1} - \Sigma \), and \( (a, b) \) an interval such that for each \( t \in (a, b) \) (i) \( \Sigma_t \) is a closed, PL \( (n - 1) \)-manifold that is collared from \( U_t \) and (ii) \( \text{Cl} U - \Sigma_t \) is 1-LC at each point of \( \Sigma_t \). Then \( U \) is 1-LC at each point of \( \Sigma(a, b) \).

**Proof.** Suppose \( f: \Delta^2 \to \text{Cl} U \) is a map such that \( f(\partial \Delta^2) \subset U \) and \( f(\Delta^2) \cap \Sigma \subset \Sigma(a, b) \). Let \( \varepsilon \) be a positive number less than both \( \rho(f(\Delta^2), \Sigma_a \cup \Sigma_b) \) and \( \rho(f(\partial \Delta^2), \Sigma) \), and let \( D \) denote the countable subset of \( (a, b) \) described in Lemma 4.1. Then by Theorem 3.2 there exists a map \( f_0: \Delta^2 \to \text{Cl} U \) such that

1. \( f_0|\partial \Delta^2 = f|\partial \Delta^2 \),
2. \( \rho(f, f_0) < \varepsilon/3 \),
3. \( f_0(\Delta^2) \cap \Sigma_d = \emptyset \) for each \( d \) in \( D \).

For each \( t \in (a, b) - D \), application of Lemma 4.3 yields an \( \varepsilon/3 \)-retraction \( r_t \) of \( X_t \) onto the \( A_t \) associated (by Lemma 4.3B) with \( t \) and \( \varepsilon/3 \), where \( \Sigma \cap A_t = \Sigma_{s(t)} \cup \Sigma_{u(t)} \). Let \( \pi \) denote the projection of \( E^n \times E^1 \) onto the second factor. Then \( \pi(\Sigma \cap f_0(\Delta^2)) \), a subset of \( (a, b) - D \), is covered by the collection \( \mathcal{C} \) of open intervals \( \mathcal{C} = \{(s(t), u(t)) \mid t \in \pi(\Sigma \cap f_0(\Delta^2))\} \), from which we can extract a finite sub-covering \( \mathcal{F} \) of \( \pi(\Sigma \cap f_0(\Delta^2)) \). After eliminating unnecessary elements of \( \mathcal{F} \), we can regard \( \mathcal{F} \) as the union of two (finite) collections \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) such that for \( i = 1, 2 \) no two elements of \( \mathcal{F}_i \) intersect. Let \( F_1 \) and \( F_2 \) denote the underlying point sets of \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) respectively.

In case \((s(t), u(t)) \in \mathcal{F}_1 \), the addendum to Lemma 4.3 implies the existence of an \( \varepsilon/3 \) map \( R_t \) of \( X_t \cap f_0(\Delta^2) \) into \( A_t \) such that

4. \( R_t(X_t \cap f_0(\Delta^2)) \cap \text{Bd} A_t \subset X_t \cap f_0(\Delta^2) \),
5. \( R_t|_{f_0(\Delta^2) \cap A_t = \text{identity} \).

Define a map \( f_1: \Delta^2 \to \text{Cl} U \) by the rule

\[
f_1(x) = \begin{cases} R_t f_0(x), & \text{if } f_0(x) \in X_t \text{ and } (s(t), u(t)) \in \mathcal{F}_1, \\ f_0(x), & \text{otherwise.} \end{cases}
\]
It follows from (5) and the fact that the $X_t$'s considered are pairwise disjoint that $f_t$ is well defined and continuous. Note that

$$(6) f_t|\partial \Delta^2 = f_0|\partial \Delta^2,$$

$$(7) \rho(f_0, f_t) < \epsilon/3,$$

$$(8) \pi(\Sigma \cap f_t(\Delta^2)) \subset F_0.$$ 

Similarly, in case $(s(t), u(t)) \in \mathcal{F}_0$, the addendum to Lemma 4.3 implies the existence of an $\epsilon/3$-map $R_t$ of $X_t \cap f_t(\Delta^2)$ into $A_t$ such that

$$(9) R_t(X_t \cap f_t(\Delta^2)) \cap \text{Bd} \ A_t = \varnothing,$$

$$(10) R_t|f_t(\Delta^2) \cap A_t = \text{identity}.$$ 

Define $f_2 : \Delta^2 \to \text{Cl} \ U$ by the rule

$$f_2(x) = \begin{cases} R_t f_t(x), & \text{if } (s(t), u(t)) \in \mathcal{F}_0, \\ f_t(x), & \text{otherwise.} \end{cases}$$ 

As before, $f_2$ is a continuous function satisfying

$$(11) f_2|\partial \Delta^2 = f_1|\partial \Delta^2,$$

$$(12) \rho(f_1, f_2) < \epsilon/3,$$

$$(13) f_2(\Delta^2) \cap \Sigma = \varnothing.$$ 

Since $f_2|\partial \Delta^2 = f_1|\partial \Delta^2$ and $\rho(f_1, f_2) < \epsilon$, it follows immediately that $U$ is 1-LC at each point of $\Sigma(a,b)$.

REMARKS. Although condition (ii) clearly is a necessary hypothesis for Theorem 5.1, one questions whether it might be superfluous. Even without this condition it follows from [4], by means of the trick emerging here in Lemma 4.1, that $\text{Cl} \ U - \Sigma_t$ is 1-LC at each point of $\Sigma_t$ for all but at most countably many points $t$ in $(a, b)$, but this fact, obviously, is no help. By attacking the problem differently in the next section, we shall prove, under the hypotheses of Theorem 5.1 without condition (ii), that $U$ is 1-LC at many points of $\Sigma$ (see Corollary 6.2).

Without much extra effort one can prove the following slightly stronger version of Theorem 5.1. Statements of the other results in this section can be altered in a similar manner.

**Theorem 5.2.** Let $\Sigma$ denote an $n$-manifold embedded in $E^{n+1}$ ($n \neq 4$) as a closed subset, $U$ a component of $E^{n+1} - \Sigma$, and $(a, b)$ an interval such that (i) for each $t \in (a, b)$, $\text{Cl} \ U - \Sigma_t$ is 1-LC at each point of $\Sigma_t$ and (ii) for all but countably many points $t$ in $(a, b)$, $\Sigma_t$ is a closed, PL $(n-1)$-manifold that is collared from $U_t$. Then $U$ is 1-LC at each point of $\Sigma(a,b)$.

**Proof.** Simply incorporate those countably many $t$'s of $(a, b)$ that fail to satisfy condition (ii) into the set $\mathcal{D}$ and reapply the proof of Theorem 5.1.

**Theorem 5.3.** Let $\Sigma$ denote an $n$-manifold embedded in $E^{n+1}$ ($n \neq 4$) as a closed subset, $U$ a component of $E^{n+1} - \Sigma$, and $(a, b)$ an interval such that for each $t \in (a, b)$ (i) $\Sigma_t$ is a closed, PL $(n-1)$-manifold that is collared from $U_t$ and (ii) $\Sigma - \Sigma_t$ is 1-LC at each point of $\Sigma_t$. Then $U$ is 1-LC at each point of $\Sigma(a,b)$.
Proof. Let $p$ be a point of $\Sigma$ and $\epsilon > 0$. There exists a $\delta > 0$ such that any loop in $N_\delta(p) \cap (\Sigma - \Sigma_i)$ is contractible in $N_\delta(p) \cap (\Sigma - \Sigma_i)$. Since $U$ is locally 1-connected at $p$ in the homology sense [17, Theorem II, 5.35], it follows from [6, Proposition 3.3] that there exists an $\alpha > 0$ such that any loop in $N_\alpha(p) \cap U$ is contractible in $N_\alpha(p) \cap (E^{n+1} - \Sigma)$. By cutting off such a contraction in $\Sigma - \Sigma_i$ one can show that each loop in $N_\alpha(p) \cap U$ is contractible in $N_\alpha(p) \cap (\Sigma - \Sigma_i)$. Using this property one easily can prove that $\text{Cl} \, U - \Sigma_i$ is 1-CL at $p$. Hence, Theorem 5.1 gives the desired result.

Theorem 5.4. Let $\Sigma$ denote a closed $n$-manifold in $E^{n+1}$ $(n \neq 4)$, $U$ a component of $E^{n+1} - \Sigma$, and $[a, b]$ the interval such that $\Sigma = \Sigma([a, b])$. Suppose that for each $t$ in $(a, b)$, $\Sigma_t$ is a closed PL $(n-1)$-manifold that is collared from $U_t$ and that for each $t$ in $[a, b]$, $\text{Cl} \, U - \Sigma_t$ is 1-ULC. Then $U$ is 1-ULC.

Proof. Obviously $U$ is 1-CL at points of $\Sigma(a, b)$. Furthermore, by hypothesis, any small loop in $U$ near a point of $\Sigma_0$ is contractible in a small subset of $\text{Cl} \, U - \Sigma_0$. According to Theorem 3.2 such a contraction can be modified slightly so that the range of the resulting map is a small subset of $U$. Thus, $U$ is 1-CL at points of $\Sigma_0$. The same argument applies to points of $\Sigma_0$. Consequently, $U$ is 1-ULC.

Using Theorem 5.4 one can extend results of [11] and [12] to higher dimensions in the following ways.

Corollary 5.5. Suppose $\Sigma$ is a closed $n$-manifold in $E^{n+1}$ $(n \neq 4)$ such that (i) $\Sigma = \Sigma([-1, 1])$, (ii) $\Sigma - (\Sigma_{-1} \cup \Sigma_1)$ is 1-ULC, and (iii) for each $t \in (-1, 1)$, $\Sigma_t$ is a closed, PL $(n-1)$-manifold bicollected in $E_t$ and $\Sigma - \Sigma_t$ is 1-ULC. Then $E^{n+1} - \Sigma$ is 1-ULC.

Proof. For either component $U$ of $E^{n+1} - \Sigma$ and each $t \in [-1, 1]$, the proof of Theorem 5.3 indicates that $\text{Cl} \, U - \Sigma_t$ is 1-ULC. Although the hypotheses of Proposition 3.3 of [6] do not apply when $t = \pm 1$, the argument there can be used to establish the property employed in proving Theorem 5.3, namely, for each $\delta > 0$ there exists an $\alpha > 0$ such that each $\alpha$-loop in $U$ is contractible in a $\delta$-subset of $E^{n+1} - \Sigma_t (t \neq \pm 1)$.

Corollary 5.6. Suppose $\Sigma$ is an $n$-sphere in $E^{n+1}$ $(n \neq 4)$ such that both $\Sigma_1$ and $\Sigma_{-1}$ are points and, for each $t \in (-1, 1)$, $\Sigma_t$ is an $(n-1)$-sphere bicollected in $E_t$ and $\Sigma - \Sigma_t$ is 1-ULC. Then $E^{n+1} - \Sigma$ is 1-ULC.

From Theorem 9 of [16] we obtain the following flatness conditions.

Theorem 5.7. Suppose $\Sigma$ is a closed PL $n$-manifold in $E^{n+1}$ $(n \geq 5)$ satisfying the hypothesis of Corollary 5.5. Then $\Sigma$ is locally flat if and only if $\Sigma$ can be homeomorphically approximated by locally flat manifolds.

Theorem 5.8. Let $\Sigma$ denote the boundary of an $(n+1)$-cell $B$ in $E^{n+1}$ $(n \geq 5)$ and $U$ the complement of $B$. Suppose (i) $\Sigma = \Sigma([a, b])$, (ii) for each $t \in (a, b)$, $\Sigma_t$ is a PL $(n-1)$-manifold that is collared from $U_t$, and (iii) for each $t \in [a, b]$, $\text{Cl} \, U - \Sigma_t$ is 1-ULC. Then $\Sigma$ is locally flat.
6. Submanifolds of \( E^{n+1} \) whose levels satisfy 1-ULC conditions. In this section we give sufficient conditions, in the spirit of Cannon's work in \( E^3 \) [7], for a complementary domain of a closed \( n \)-manifold in \( E^{n+1} \) to be 1-ULC. In place of the hypothesis typical of the results found in §5 that the \( \Sigma_t \)'s be collared PL manifolds stands the weakened hypothesis that the \( U_t \)'s be 1-ULC, together with strong restrictions on the embeddings of the \( \Sigma_t \)'s in \( \Sigma \). As one advantage of this approach, the case \( n=4 \) need not be excluded.

**Theorem 6.1.** Suppose that \( \Sigma \) is a closed \( n \)-manifold in \( E^{n+1} \), \( Z \) a component of \( E^{n+1} - \Sigma \), and \((a, b)\) an interval such that (i) \( \Sigma_t = \operatorname{Bd} Z_t \) and (ii) \( Z_t \) is both 0-ULC and 1-ULC for each \( t \) in \((a, b)\). Then \((a, b)\) contains a dense \( G_\delta \)-subset \( G \) such that \( Z \) is 1-LC at each point of \( \Sigma(G) \).

**Proof.** Equivalently we shall show that the set \( F \) of levels \( t \) in \((a, b)\) at which \( Z_t \) fails to be 1-LC at \( q \) is a 0-dimensional \( F_\sigma \)-set.

For each positive integer \( n \) let \( X_n \) denote the set of points \( x \) in \( Z_t \) such that for no neighborhood \( V \) of \( x \) is every loop of \( V \cap Z \) contractible in a \((1/n)\)-subset of \( Z \). Then \( X_n \) is a compact subset of \( \Sigma_t \), and therefore \( \pi(X_n) \) is a closed subset of \( E^1 \), where \( \pi \) denotes the map projecting \( E^n \times E^1 \) onto the second factor. Define \( F_n \) as \((a, b) \cap \pi(X_n) \). Obviously \( F = \bigcup F_n \). Hence, we only need show \( F_n \) to be 0-dimensional.

Suppose to the contrary that some \( F_n \) contains a subinterval \([a', b']\) of \((a, b)\). By the Baire Category Theorem \([a', b']\) then contains a subinterval \([c, d]\) such that corresponding to some dense subset \( Y \) of \([c, d]\) there exists a positive number \( \delta \) with the property that each \( \delta \)-loop in \( Z_t \) is null homotopic in a \((1/3n)\)-subset of \( Z_t \) \((t \in Y)\). To reach the required contradiction we shall apply Hosay's argument [12] to prove that each point \( q \) of \( \Sigma(c, d) \) has a neighborhood \( V \) such that every loop in \( V \cap Z \) is null-homotopic in a \((1/n)\)-subset of \( Z \).

Given any such point \( q \) let \( U \) be a round open ball in \( E^{n+1} \) containing \( q \) of diameter less than \( \min \{\delta, 1/3n\} \). Assume further that \( U \) misses \( E_c \) and \( E_d \). Let \( V \) be a neighborhood of \( q \) such that \( V \subseteq U \) and \( V \cap \Sigma \) lies in an \( n \)-cell in \( U \cap (\Sigma - (\Sigma_c \cup \Sigma_d)) \). We must show that any map \( f \) of the boundary of a 2-cell \( D \) into \( V \cap Z \) has an extension \( g \) sending \( D \) into a \((1/n)\)-subset of \( Z \).

Using the notation developed in [12] we trace the argument given on pp. 371–373 there with certain modifications. First, replace part A by the following observation: the hypotheses that \( \Sigma_t = \operatorname{Bd} Z_t \) and \( Z_t \) is 0-ULC imply that there exists an arc in \( U \cap Z_t \) connecting each pair of points of \( h(A_t) \cap f(\partial D) \). Let \( K_t \) denote the union of the (finitely many) arcs obtained in this way. Second, in part B observe that the special levels (the \( r \)'s) can be chosen from \( Y \), since \( Y \) is dense in \([c, d]\). The last two paragraphs of B may be ignored, noting instead that any map of a simple closed curve into

\[
(K_t^1 \cup K_t^{n+1} \cup h(\partial D)) \cap \{ E_t | t_t \leq t \leq t_{t+1} \}
\]
is homotopic, in the intersection of \( Z \) and the \((1/3n)\)-neighborhood of \( U \), to a constant map. Making use of this fact at the end of part D, we can construct the required extension \( g \) sending \( D \) into \( Z \).

**Corollary 6.2.** Suppose that \( \Sigma \) is a closed \( n \)-manifold in \( E^{n+1} \), \( U \) a component of \( E^{n+1} - \Sigma \), and \((a, b)\) an interval such that for each \( t \in (a, b) \), \( \Sigma_t \) is a closed \((n-1)\)-manifold that is collared from \( U \). Then \((a, b)\) contains a dense \( G_\delta \)-subset \( G \) such that \( U \) is 1-LC at each point of \( \Sigma(G) \).

**Theorem 6.3.** Suppose that \( \Sigma \) is a closed \( n \)-manifold in \( E^{n+1} \), \( Z \) a component of \( E^{n+1} - \Sigma \), and \((a, b)\) an interval such that (i) \( Cl \{ Z \} - \Sigma \) is 1-ULC and (ii) \( Z_t \) is 1-ULC for each \( t \) in \((a, b)\). Then \((a, b)\) contains a dense \( G_\delta \)-subset \( G \) such that \( Z \) is 1-LC at each point of \( \Sigma(G) \).

**Proof.** We begin by repeating the first three paragraphs of the proof of Theorem 6.1. Then, with minor modifications similar to those given in the preceding proposition, Cannon's argument [7, Theorem 1] can be applied to complete the proof.

It would be interesting to know whether the hypotheses of either Theorem 6.1 or 6.3 actually imply that \( Z \) is 1-LC at each point of \( \Sigma(a, b) \). One should note that some restrictions on the set \( Z \) are necessary, for it is quite simple to describe a connected open subset \( Z \) of \( E^{n+1} \) \((n \geq 2)\) such that each \( Z_t \) is 1-ULC but \( Z \) fails to be 1-LC at certain points of \( Bd Z \). In particular, a bounded open subset \( Z \) of \( E^{n+1} \) \((n \geq 2)\) need not be 1-ULC even if each \( Z_t \) is 1-ULC.

**Theorem 6.4.** Suppose \( \Sigma \) is a closed \( n \)-manifold in \( E^{n+1} \), \( U \) a component of \( E^{n+1} - \Sigma \), and \((a, b)\) an interval such that (1) for each \( t \in (a, b) \), \( U_t \) is 1-ULC and (2) for each compact 0-dimensional subset \( C \) of \((a, b) \), \( Cl U - \Sigma(C) \) is 1-ULC. Then \( U \) is 1-LC at each point of \( \Sigma(a, b) \).

**Proof.** Let \( f \) be a map of the disk \( \Delta^2 \) into \( Cl U \) such that \( f(\partial \Delta^2) \subseteq U \) and \( f(\Delta^2) \cap \Sigma \subseteq \Sigma(a, b) \). If \( G \) denotes the dense \( G_\delta \)-subset of \((a, b)\) promised by Theorem 6.3, then from hypothesis (2) above and Theorem 3.2 we find that \( f \) can be adjusted slightly, not changing the map on \( \partial \Delta^2 \), such that \( f(\Delta^2) \cap \Sigma \subseteq \Sigma(G) \) and \( f^{-1}(\Sigma \cap f(\Delta^2)) \) is 0-dimensional. As a result, \( U \) is 1-LC at each point of \( \Sigma \cap f(\Delta^2) \), and it is then a simple matter to alter \( f \) further so that \( f(\Delta^2) \) misses \( \Sigma \) entirely. Hence, \( U \) is 1-LC at each point of \( \Sigma(a, b) \).

**Corollary 6.5.** Suppose \( \Sigma \) is a closed \( n \)-manifold in \( E^{n+1} \) such that each component \( U \) of \( E^{n+1} - \Sigma \) satisfies (1) for each \( t \in E^1 \), \( U_t \) is 1-ULC and (2) for each compact, 0-dimensional subset \( C \) of \( E^1 \), \( Cl U - \Sigma(C) \) is 1-ULC. Then \( E^{n+1} - \Sigma \) is 1-ULC.

Corollary 6.5 can be interpreted as a generalization of [7, Corollary 3].

**Remark.** Variations on Theorem 6.4 and Corollary 6.5 can be obtained by exchanging condition (2) in each for the following condition:

(2*) for each compact 0-dimensional subset \( C \) of \((a, b)\), or \( E^1 \), as the context requires, \( \Sigma - \Sigma(C) \) is 1-ULC.
The proof given for Theorem 2 of [7] establishes that to each $\delta > 0$ there corresponds an $\alpha > 0$ such that each $\alpha$-loop in $U$ is contractible in an $\varepsilon$-subset of $E^{n+1} - \Sigma(C)$. Using this we can prove, as in Theorem 5.3, that $Cl U - \Sigma(C)$ is 1-ULC.

Finally, Corollary 6.5 and [16, Theorem 9] can be combined, as in §5, to produce a local flatness criterion.

**Theorem 6.6.** Suppose $\Sigma$ is a closed PL $n$-manifold in $E_{n+1}$ ($n \geq 4$) satisfying the hypotheses of Corollary 6.5. Then $\Sigma$ is locally flat if and only if $\Sigma$ can be homeomorphically approximated by locally flat manifolds.

**References**

6. ———, Euclidean $n$-space modulo an $(n-1)$-cell (to appear).

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