AN EXTENSION OF A THEOREM OF HARTOGS

BY

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Abstract. Hartogs proved that every function which is holomorphic on the boundary of the unit ball in $\mathbb{C}^n$, $n > 1$, can be extended to a function holomorphic on the ball itself. It is conjectured that a real $k$-dimensional $\mathcal{C}^{\infty}$ compact submanifold of $\mathbb{C}^n$, $k > n$, is extendible over a manifold of real dimension $(k + 1)$. This is known for hypersurfaces (i.e., $k = 2n - 1$) and submanifolds of real codimension 2. It is the purpose of this paper to prove this conjecture and to show that we actually get $C^r$ extendibility.

1. Introduction. Let $M^k$ be a real $k$-dimensional compact $\mathcal{C}^{\infty}$ manifold embedded in $\mathbb{C}^n$, $k, n \geq 2$. Hartogs proved that every function holomorphic in an open neighborhood of $M^{2n-1}$ can be extended to a function holomorphic in some open subset of $\mathbb{C}^n$. Bochner proved a similar theorem for functions which satisfy the induced Cauchy-Riemann equations on $M^{2n-1}$. It has been conjectured that any real $k$-dimensional compact $\mathcal{C}^{\infty}$ submanifold of $\mathbb{C}^n$ is extendible to a manifold of real dimension $(k + 1)$ if $k > n$. This has been proved for real-analytic submanifolds of $\mathbb{C}^n$ in [3] and generic $C^r$ submanifolds in [2]. It is the purpose of this paper to prove the conjecture with extendibility being replaced by $C^r$-extendibility.

The early work for the higher codimensional study was done by Bishop [1], Wells [6] and Greenfield [2]. A recent article due to Nirenberg [4] led to the results in this paper.

2. Definitions. Let $M^k$ be a real $k$-dimensional compact $\mathcal{C}^{\infty}$ manifold embedded in $\mathbb{C}^n$, $k, n \geq 2$. Suppose $T(M^k)$ is the tangent bundle to $M^k$, and $J$ denotes the almost complex tensor $J: T(\mathbb{C}^n) \rightarrow T(\mathbb{C}^n)$, with $J^2 = -I$. Then we define

$$H_p(M^k) = T_p(M^k) \cap JT_p(M^k),$$

the vector space of holomorphic tangent vectors to $M^k$ at $p$. Then $H_p(M^k)$ is the maximal complex subspace of $T_p(\mathbb{C}^n)$ which is contained in $T_p(M^k)$. It is well known that

$$\max (k - n, 0) \leq \dim H_p(M^k) \leq \lfloor k/2 \rfloor.$$
There is another way of examining almost complex structures which we shall use. Let \( f \) denote the embedding of \( M^k \) into \( C^n \), and let \( J(f) \) be the complex Jacobian of \( f \). If \( q = \min (n, k) \), a point \( p \) in \( M^k \) is said to be an exceptional point of order \( l \), \( 0 \leq l \leq \lceil k/2 \rceil - \max (k - n, 0) \), if the complex rank of \( J(f)|_p \) is equal to \( q - l \).

A point \( p \) in \( M^k \) is generic if \( p \) is an exceptional point of order \( 0 \). The manifold \( M^k \) is locally generic at \( p \) if every point in some open neighborhood of \( p \) is generic, and is locally C-R at \( p \) if every point in some open neighborhood of \( p \) is an exceptional point of the same order.

Suppose \( M^k \) is locally C-R at \( p \) and \( H_p(M^k) \) is nonempty. Then we define the Levi form at any \( x \) near \( p \)

\[
L_x(M^k): H_x(M^k) \rightarrow (T_x(M^k) \otimes C)/C(\pi_x H_x(M^k) \otimes C)
\]

by \( L_x(M^k)(t) = \pi_x([Y, Y]_x) \), where \( Y \) is a local section of the fiber bundle \( H(M^k) \) (with fiber \( H_x(M^k) \)) such that \( Y_x = t \), \([Y, Y]_x \) is the Lie bracket evaluated at \( x \), and \( \pi_x: T_x(M^k) \otimes C \rightarrow (T_x(M^k) \otimes C)/C(\pi_x H_x(M^k) \otimes C) \) is the projection.

Denote by \( \mathcal{O}_C^n = \mathcal{O} \) the sheaf of germs of holomorphic functions on \( C^n \). Let \( K \) be a compact subset of \( C^n \) and \( V \) an open subset of \( C^n \) containing \( K \). We set

\[
\mathcal{O}(K) = \text{ind lim}_{V \supset K} \mathcal{O}(V),
\]

where \( \mathcal{O}(V) \) is the Fréchet algebra of holomorphic functions on \( V \). We say that \( K \) is extendible to a connected set \( K' \supset K \) if the map \( r: \mathcal{O}(K') \rightarrow \mathcal{O}(K) \) is onto.

Suppose \( f \in \mathcal{O}^\omega(M^k) \). We say \( f \) is a C-R function at \( p \in M^k \) if \( \bar{X}f(y) = 0 \), for \( y \) near \( p \) and \( X \) any section of \( H(M^k) \). If \( M^k \) is locally C-R at \( p \) it suffices to verify the equality just for \( X \) in a local basis for \( H(M^k) \) at \( p \). We note that our manifold need not be globally C-R. Thus we may have points which are not locally C-R. But obviously, the set of such points is nowhere dense in \( M^k \).

**Definition 2.1.** Let \( f \in \mathcal{O}^\omega(M^k) \). Then \( f \) is a C-R function on \( M^k \) if \( f \) is a C-R function at each point of \( M^k \). The C-R functions are denoted by CR \((M^k)\).

We say that \( M^k \) is C-R extendible to a connected set \( K = M^k \cup K' \), where \( K' \neq \emptyset \), if for every \( f \in \text{CR}(M^k) \) there exists an \( F: M^k \cup K' \rightarrow C \) continuous so that \( F|_{M^k} = f \) and \( F|_{K'} \in \mathcal{O}(K') \). We observe that C-R extendibility implies extendibility.

Let \( K \) be a compact subset of \( C^n \). We shall call a point \( x \in K \) a holomorphic peak point if there exists a function \( f \in \mathcal{O}(K) \) such that, for any \( y \in K - \{x\} \), we have \(|f(y)| < |f(x)|\).
3. Local equations and the Levi form. Again let $M^k$ be a real $k$-dimensional $C^\infty$ manifold embedded in $C^n$, $k, n \geq 2$. Suppose $M^k$ is locally C-R at $p$, and $p$ is an exceptional point of order $l$. If $k > n$ the local equations of $M^k$ in a neighborhood of $p$ are (after a suitable choice of coordinates)

\[ z_1 = x_1 + i h_1(x_1, \ldots, x_2(n-l)-k, w_1, \ldots, w_{k-n+1}) \]
\[ \vdots \]
\[ z_{2(n-l)-k} = x_{2(n-l)-k} + i h_{2(n-l)-k}(x_1, \ldots, x_2(n-l)-k, w_1, \ldots, w_{k-n+1}) \]
\[ z_{2(n-l)-k+1} = u_1 + i v_1 = w_1 \]
\[ \vdots \]
\[ z_{n-l} = u_{k-n+l} + i v_{k-n+l} = w_{k-n+l} \]
\[ z_{n-l+1} = g_1(x_1, \ldots, x_2(n-l)-k, w_1, \ldots, w_{k-n+l}) \]
\[ \vdots \]
\[ z_n = g_l(x_1, \ldots, x_2(n-l)-k, w_1, \ldots, w_{k-n+l}), \]

where $x_1, \ldots, x_{2(n-l)-k}, u_1, v_1, \ldots, u_{k-n+l}, v_{k-n+l}$ are local coordinates for $M^k$ in a neighborhood of $p$ vanishing at $p$, and $z_1, \ldots, z_n$ are coordinates for $C^n$ vanishing at $p$. The real-valued functions $h_1, \ldots, h_{2(n-l)-k}$ as well as the complex-valued functions $g_1, \ldots, g_l$ vanish to order 2 at $p$. Because $M^k$ is locally C-R at $p$, the functions $g_1, \ldots, g_l$ must be complex-analytic functions of $w_1, \ldots, w_{k-n+l}$ (see [3]).

Letting $g_j = g_j^1 + ig_j^2, j = 1, \ldots, l$, we find from [5] that the Levi form vanishes at $p$ if and only if the complex Hessians at $p$ of each of the functions $h_1, \ldots, h_{2(n-l)-k}$, $g_1^1, g_1^2, \ldots, g_l^1, g_l^2$ with respect to the variables $w_1, \ldots, w_{k-n+l}$ all have zero eigenvalues.

Fix $x_1, \ldots, x_{2(n-l)-k}$ and expand each $g_j$ in a Taylor series in $w_1, \ldots, w_{k-n+l}$,

\[ g_j = \sum a_{j,\alpha} w^\alpha, \]

where $w=(w_1, \ldots, w_{k-n+l})$ and $\alpha=(\alpha_1, \ldots, \alpha_{k-n+l})$. Replacing $z_{n-l+j}$ by $z_{n-l+j} - \sum a_{j,\alpha} w^\alpha$, we have that $z_{n-l+j} = 0, \ldots, z_n = 0$ in our new local equations. Thus the Levi form vanishes at $p$ if and only if the complex Hessians at $p$ of each of the functions $h_1, \ldots, h_{2(n-l)-k}$ are all zero matrices.

Suppose $M^k$ is compact in $C^n$. It is shown in [5] that there exists an open set of holomorphic peak points on $M^k$ which is nonempty. By the remarks before Definition 2.1, we can find a holomorphic peak point $p \in M^k$ such that $p$ is an exceptional point of some order $l$, and $M^k$ is locally C-R at $p$. Assume $p=0$ and $M^k$ near $p$ is given by the equations in (1). Wells proves that through $p$ we can put a hyperplane which intersects $M^k$ at only the point $p$. If $z_j = x_j + iy_j, j=1, \ldots, 2(n-l)-k, n-l+1, \ldots, n$, we can assume the hyperplane is defined by $y_1 = 0$ (the information about the $g_j$'s in this section forces our arbitrary choice to $y_1, \ldots, y_{2(n-l)-k}$).

Let $Q$ denote the 1-dimensional real subspace of $T_0(C^n)$ generated by $\partial/\partial y_1$. Set
\( W = Q \oplus T_0(M^k) \) and let \( \pi \) be the projection from \( C^n \) to \( W \). Under this projection the manifold \( M^k \) projects to a manifold with local equations

\[
\begin{align*}
z_1 &= x_1 + i\bar{h}_1(x_1, \ldots, x_{2(n-i)-k}, w_1, \ldots, w_{k-n+1}) \\
&= x_1 \\
&\quad \vdots \\
z_{2(n-i)-k} &= x_{2(n-i)-k} \\
z_{2(n-i)-k+1} &= u_1 + iv_1 = w_1 \\
&\quad \vdots \\
z_{n-i} &= u_{k-n+1} + iv_{k-n+1} = w_{k-n+1}.
\end{align*}
\]

Wells shows that

\[
\det \begin{pmatrix}
\frac{\partial^2 h_1}{\partial x_1^2} & \cdots & \frac{\partial^2 h_1}{\partial x_{2(n-i)-k}^2} & \cdots & \frac{\partial^2 h_1}{\partial u_1^2} & \cdots & \frac{\partial^2 h_1}{\partial w_{k-n+1}^2} \\
& \frac{\partial^2 h_1}{\partial x_1 \partial x_{2(n-i)-k}} & \cdots & \frac{\partial^2 h_1}{\partial x_{2(n-i)-k} \partial u_1} & \cdots & \frac{\partial^2 h_1}{\partial x_{2(n-i)-k} \partial w_{k-n+1}} \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& \frac{\partial^2 h_1}{\partial u_1 \partial u_{k-n+1}} & \cdots & \frac{\partial^2 h_1}{\partial u_1 \partial w_{k-n+1}} & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& \frac{\partial^2 h_1}{\partial w_{k-n+1} \partial w_{k-n+1}} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

are all \( > 0 \) on some open neighborhood \( U \) of \( p \) in \( M^k \). In particular

\[
\det \begin{pmatrix}
\frac{\partial^2 h_1}{\partial w_1 \partial \bar{w}_1} & \cdots & \frac{\partial^2 h_1}{\partial w_{k-n+1} \partial \bar{w}_{k-n+1}} \\
\end{pmatrix}
\]

are positive on the set \( U \). By diagonalizing, we find that the Hessian of \( h_1 \) with respect to \( w_1, \ldots, w_{k-n+1} \) is positive definite.

4. **The main result.** Assume \( M^k \) is a real \( k \)-dimensional \( \mathcal{C}^\infty \) manifold embedded in \( C^n \), and \( M^k \) is locally C-R at \( p \in M^k \). Suppose at least one of the following conditions is satisfied.

(I) There is a real hypersurface containing \( M^k \) whose Levi form restricted to \( H(M^k) \) has at \( p \) at least one positive and one negative eigenvalue.

(II) There is a real hypersurface containing \( M^k \) whose Levi form restricted to \( H(M^k) \) has at \( p \) all its eigenvalues of the same sign different from zero.

Then we have the following theorem due to Nirenberg [4].

**Theorem 4.1.** Let \( M^k \) be locally C-R at \( p \in M \) and assume either (I) or (II) holds. Then \( M^k \) is locally C-R extendible to a manifold \( \tilde{M} \) of real dimension one higher than that of \( M^k \).

We are now able to prove the main result.

**Theorem 4.2.** Let \( M^k \) be a real \( k \)-dimensional compact \( \mathcal{C}^\infty \) manifold embedded in \( C^n \), \( k > n \geq 2 \). Then \( M^k \) is C-R extendible to a real \( (k+1) \)-dimensional submanifold of \( C^n \).

**Proof.** We showed in the previous section that there exists a point \( p \in M^k \) such that:

(i) \( M^k \) is locally C-R at \( p \),

(ii) \( M^k \) is given by the local equations (1) near \( p \), and

(iii) the complex Hessian of the function \( h_1 \) with respect to the variables \( w_1, \ldots, w_{k-n+1} \) has all positive eigenvalues at \( p \).
Consider the real hypersurface containing $M^k$ defined by the function $\rho = y_1 - h$. The Levi form of this hypersurface restricted to $H(M^k)$ is the negative of the complex Hessian of $h$ with respect to the variables $w_1, \ldots, w_{k+n+1}$. Then this hypersurface satisfies condition (II) at the point $p$, and we apply Theorem 4.1. Q.E.D.

**Theorem 4.3.** Let $M^k$ be a real $k$-dimensional compact $C^\infty$ manifold embedded in $C^n$, $k > n \geq 2$. Then $M^k$ is extendible to a real $(k+1)$-dimensional submanifold of $C^n$.

**Remark 1.** The manifold $\tilde{M}$ of Theorem 1 can be taken to have $C^q$ structure, $1 \leq q < \infty$.

**Remark 2.** If $k \leq n$, then there are examples of totally real submanifolds which are always holomorphically convex. Thus, from the standpoint of dimension, Theorems 4.2 and 4.3 are the best possible.

**References**


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