AN EXTENSION OF A THEOREM OF HARTOGS

BY

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Abstract. Hartogs proved that every function which is holomorphic on the boundary of the unit ball in $C^n$, $n > 1$, can be extended to a function holomorphic on the ball itself. It is conjectured that a real $k$-dimensional $C^\infty$ compact submanifold of $C^n$, $k > n$, is extendible over a manifold of real dimension $(k+1)$. This is known for hypersurfaces (i.e., $k = 2n-1$) and submanifolds of real codimension 2. It is the purpose of this paper to prove this conjecture and to show that we actually get C-R extendibility.

1. Introduction. Let $M^k$ be a real $k$-dimensional compact $C^\infty$ manifold embedded in $C^n$, $k, n \geq 2$. Hartogs proved that every function holomorphic in an open neighborhood of $M^{2n-1}$ can be extended to a function holomorphic in some open subset of $C^n$. Bochner proved a similar theorem for functions which satisfy the induced Cauchy-Riemann equations on $M^{2n-1}$. It has been conjectured that any real $k$-dimensional compact $C^\infty$ submanifold of $C^n$ is extendible to a manifold of real dimension $(k+1)$ if $k > n$. This has been proved for real-analytic submanifolds of $C^n$ in [3] and generic C-R submanifolds in [2]. It is the purpose of this paper to prove the conjecture with extendibility being replaced by C-R extendibility.

The early work for the higher codimensional study was done by Bishop [1], Wells [6] and Greenfield [2]. A recent article due to Nirenberg [4] led to the results in this paper.

2. Definitions. Let $M^k$ be a real $k$-dimensional compact $C^\infty$ manifold embedded in $C^n$, $k, n \geq 2$. Suppose $T(M^k)$ is the tangent bundle to $M^k$, and $J$ denotes the almost complex tensor $J : T(C^n) \to T(C^n)$, with $J^2 = -I$. Then we define

$$H_p(M^k) = T_p(M^k) \cap JT_p(M^k),$$

the vector space of holomorphic tangent vectors to $M^k$ at $p$. Then $H_p(M^k)$ is the maximal complex subspace of $T_p(C^n)$ which is contained in $T_p(M^k)$. It is well known that

$$\max (k-n, 0) \leq \dim H_p(M^k) \leq [k/2].$$

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There is another way of examining almost complex structures which we shall use. Let $f$ denote the embedding of $M^k$ into $C^n$, and let $J(f)$ be the complex Jacobian of $f$. If $q = \min(n, k)$, a point $p$ in $M^k$ is said to be an exceptional point of order $l$, $0 \leq l \leq [k/2] - \max(k - n, 0)$ if the complex rank of $J(f)|_p$ is equal to $q - l$.

A point $p$ in $M^k$ is generic if $p$ is an exceptional point of order 0. The manifold $M^k$ is locally generic at $p$ if every point in some open neighborhood of $p$ is generic, and is locally C-R at $p$ if every point in some open neighborhood of $p$ is an exceptional point of the same order.

Suppose $M^k$ is locally C-R at $p$ and $H_p(M^k)$ is nonempty. Then we define the Levi form at any $x$ near $p$

$$L_x(M^k): H_x(M^k) \to (T_x(M^k) \otimes C)/(H_x(M^k) \otimes C)$$

by $L_x(M^k)(t) = \pi_x([Y, Y], x)$, where $Y$ is a local section of the fiber bundle $H(M^k)$ (with fiber $H_x(M^k)$) such that $Y_x = t$, $[Y, Y]_x$ is the Lie bracket evaluated at $x$, and

$$\pi_x: T_x(M^k) \otimes C \to (T_x(M^k) \otimes C)/(H_x(M^k) \otimes C)$$

is the projection.

Denote by $\mathcal{O}_C^n = \emptyset$ the sheaf of germs of holomorphic functions on $C^n$. Let $K$ be a compact subset of $C^n$ and $V$ an open subset of $C^n$ containing $K$. We set

$$\mathcal{O}(K) = \ind \lim_{V \ni K} \mathcal{O}(V),$$

where $\mathcal{O}(V)$ is the Fréchet algebra of holomorphic functions on $V$. We say that $K$ is extendible to a connected set $K' \supseteq K$ if the map $r: \mathcal{O}(K') \to \mathcal{O}(K)$ is onto.

Suppose $f \in \mathcal{O}^\infty(M^k)$. We say $f$ is a C-R function at $p \in M^k$ if $\overline{f}(y) = 0$, for $y$ near $p$ and $X$ any section of $H(M^k)$. If $M^k$ is locally C-R at $p$ it suffices to verify the equality just for $X$ in a local basis for $H(M^k)$ at $p$. We note that our manifold need not be globally C-R. Thus we may have points which are not locally C-R. But obviously, the set of such points is nowhere dense in $M^k$.

**Definition 2.1.** Let $f \in \mathcal{O}^\infty(M^k)$. Then $f$ is a C-R function on $M^k$ if $f$ is a C-R function at each point of $M^k$. The C-R functions are denoted by $\mathrm{CR}(M^k)$.

We say that $M^k$ is C-R extendible to a connected set $K = M^k \cup K'$, where $K' \neq \emptyset$, if for every $f \in \mathrm{CR}(M^k)$ there exists an $F: M^k \cup K' \to C$ continuous so that $F|_{M^k} = f$ and $F|_{K'} \in \mathcal{O}(K')$. We observe that C-R extendibility implies extendibility.

Let $K$ be a compact subset of $C^n$. We shall call a point $x \in K$ a holomorphic peak point if there exists a function $f \in \mathcal{O}(K)$ such that, for any $y \in K - \{x\}$, we have $|f(y)| < |f(x)|$. 

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3. Local equations and the Levi form. Again let $M^k$ be a real $k$-dimensional $C^\infty$ manifold embedded in $C^n$, $k, n \geq 2$. Suppose $M^k$ is locally C-R at $p$, and $p$ is an exceptional point of order $l$. If $k > n$ the local equations of $M^k$ in a neighborhood of $p$ are (after a suitable choice of coordinates)

$$
egin{align*}
0 &= x_1 + ix_2 + \cdots + x_{2(n-l)-k} + iw_1 + \cdots + w_{k-n+l} \\
0 &= x_1 + ix_2 + \cdots + x_{2(n-l)-k} + iw_1 + \cdots + w_{k-n+l} \\
0 &= u_1 + iv_1 = w_1 \\
0 &= u_{k-n+l} + iv_{k-n+l} = w_{k-n+l} \\
0 &= g_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
0 &= g_l(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
0 &= g_l(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l})
\end{align*}
$$

where $x_1, \ldots, x_{2(n-l)-k}, u_1, v_1, \ldots, u_{k-n+l}, v_{k-n+l}$ are local coordinates for $M^k$ in a neighborhood of $p$ vanishing at $p$, and $z_1, \ldots, z_n$ are coordinates for $C^n$ vanishing at $p$. The real-valued functions $h_1, \ldots, h_{2(n-l)-k}$ as well as the complex-valued functions $g_1, \ldots, g_l$ vanish to order 2 at $p$. Because $M^k$ is locally C-R at $p$, the functions $g_1, \ldots, g_l$ must be complex-analytic functions of $w_1, \ldots, w_{k-n+l}$ (see [3]).

Letting $g_j = g_j + ig_j^*$, $j = 1, \ldots, l$, we find from [5] that the Levi form vanishes at $p$ if and only if the complex Hessians at $p$ of each of the functions $h_1, \ldots, h_{2(n-l)-k}$, $g_1, g_1^*, \ldots, g_l, g_l^*$ with respect to the variables $w_1, \ldots, w_{k-n+l}$ all have zero eigenvalues.

Fix $x_1, \ldots, x_{2(n-l)-k}$ and expand each $g_j$ in a Taylor series in $w_1, \ldots, w_{k-n+l}$,

$$
g_j = \sum_a a_{j,a} w^a,
$$

where $w=(w_1, \ldots, w_{k-n+l})$ and $\alpha=(\alpha_1, \ldots, \alpha_{k-n+l})$. Replacing $z_{n-l+j}$ by $z_{n-l+j} - \sum_a a_{j,a} w^a$, we have that $z_{n-l+1} = 0, \ldots, z_n = 0$ in our new local equations. Thus the Levi form vanishes at $p$ if and only if the complex Hessians at $p$ of each of the functions $h_1, \ldots, h_{2(n-l)-k}$ are all zero matrices.

Suppose $M^k$ is compact in $C^n$. It is shown in [5] that there exists an open set of holomorphic peak points on $M^k$ which is nonempty. By the remarks before Definition 2.1, we can find a holomorphic peak point $p \in M^k$ such that $p$ is an exceptional point of some order $l$, and $M^k$ is locally C-R at $p$. Assume $p=0$ and $M^k$ near $p$ is given by the equations in (1). Wells proves that through $p$ we can put a hyperplane which intersects $M^k$ at only the point $p$. If $z_j = x_j + iy_j$, $j = 1, \ldots, 2(n-l)-k$, $n-l+1, \ldots, n$, we can assume the hyperplane is defined by $y_1 = 0$ (the information about the $g_j$'s in this section forces our arbitrary choice to $y_1, \ldots, y_{2(n-l)-k}$).

Let $Q$ denote the 1-dimensional real subspace of $T_0(C^n)$ generated by $\partial/\partial y_1$. Set
\( W = Q \oplus T_0(M^k) \) and let \( \pi \) be the projection from \( C^n \) to \( W \). Under this projection the manifold \( M^k \) projects to a manifold with local equations

\[
\begin{align*}
    z_1 &= x_1 + \bar{h}_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
    z_2 &= x_2 \\
    &\vdots \\
    z_{2(n-l)-k} &= x_{2(n-l)-k} \\
    z_{2(n-l)-k+1} &= u_1 + i\bar{v}_1 = w_1 \\
    &\vdots \\
    z_{n-l} &= u_{k-n+l} + i\bar{v}_{k-n+l} = w_{k-n+l}.
\end{align*}
\]

Wells shows that

\[
\begin{align*}
    \frac{\partial^2 h_1}{\partial x_1^2}, \ldots, \frac{\partial^2 h_1}{\partial x_{2(n-l)-k}^2}, \ldots, \frac{\partial^2 h_1}{\partial w_{k-n+l}^2}, \ldots, \frac{\partial^2 h_1}{\partial \bar{w}_{k-n+l}^2}
\end{align*}
\]

are all \( > 0 \) on some open neighborhood \( U \) of \( p \) in \( M^k \). In particular

\[
\begin{align*}
    \frac{\partial^2 h_1}{\partial w_1 \partial \bar{w}_1}, \ldots, \frac{\partial^2 h_1}{\partial w_{k-n+l} \partial \bar{w}_{k-n+l}}
\end{align*}
\]

are positive on the set \( U \). By diagonalizing, we find that the Hessian of \( h_1 \) with respect to \( w_1, \ldots, w_{k-n+l} \) is positive definite.

4. The main result. Assume \( M^k \) is a real \( k \)-dimensional \( \mathcal{C}^\infty \) manifold embedded in \( C^n \), and \( M^k \) is locally C-R at \( p \in M^k \). Suppose at least one of the following conditions is satisfied.

(I) There is a real hypersurface containing \( M^k \) whose Levi form restricted to \( H(M^k) \) has at \( p \) at least one positive and one negative eigenvalue.

(II) There is a real hypersurface containing \( M^k \) whose Levi form restricted to \( H(M^k) \) has at \( p \) all its eigenvalues of the same sign different from zero.

Then we have the following theorem due to Nirenberg [4].

**Theorem 4.1.** Let \( M^k \) be locally C-R at \( p \in M \) and assume either (I) or (II) holds. Then \( M^k \) is locally C-R extendible to a manifold \( \bar{M} \) of real dimension one higher than that of \( M^k \).

We are now able to prove the main result.

**Theorem 4.2.** Let \( M^k \) be a real \( k \)-dimensional compact \( \mathcal{C}^\infty \) manifold embedded in \( C^n \), \( k > n \geq 2 \). Then \( M^k \) is C-R extendible to a real \((k+1)\)-dimensional submanifold of \( C^n \).

**Proof.** We showed in the previous section that there exists a point \( p \in M^k \) such that:

(i) \( M^k \) is locally C-R at \( p \),

(ii) \( M^k \) is given by the local equations (1) near \( p \), and

(iii) the complex Hessian of the function \( h_1 \) with respect to the variables \( w_1, \ldots, w_{k-n+l} \) has all positive eigenvalues at \( p \).
Consider the real hypersurface containing $M^k$ defined by the function $\rho = y_1 - h_1$. The Levi form of this hypersurface restricted to $H(M^k)$ is the negative of the complex Hessian of $h_1$ with respect to the variables $w_1, \ldots, w_{k-n+1}$. Then this hypersurface satisfies condition (II) at the point $p$, and we apply Theorem 4.1. Q.E.D.

**Theorem 4.3.** Let $M^k$ be a real $k$-dimensional compact $C^\infty$ manifold embedded in $C^n$, $k > n \geq 2$. Then $M^k$ is extendible to a real $(k+1)$-dimensional submanifold of $C^n$.

**Remark 1.** The manifold $\tilde{M}$ of Theorem 1 can be taken to have $C^q$ structure, $1 \leq q < \infty$.

**Remark 2.** If $k \leq n$, then there are examples of totally real submanifolds which are always holomorphically convex. Thus, from the standpoint of dimension, Theorems 4.2 and 4.3 are the best possible.

### References


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