

A NEW CHARACTERIZATION OF THE F SET OF A RATIONAL FUNCTION⁽¹⁾

BY

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Abstract. In the early part of this century G. Julia and P. Fatou extensively studied the iteration of functions on the complex plane. More recently Hans Brolin reopened the investigation. In this paper, we are interested in the F set which is the set of points at which the family of iterates of a given rational function R is not normal and in a measure which is in some sense naturally imposed on the F set by the iterates of R .

We construct a sequence of probability measures via the inverse functions of the iterates of R and almost any starting point. The measure of primary interest is the weak limit of such sequences. These weak limits are supported by F and have certain invariance properties. We establish that this weak limit measure is unique and is ergodic with respect to the transformation R on the F set for a large class of rational functions. In the course of the proof of uniqueness we develop expressions for the logarithmic potential function and for the energy integral of F . We also establish inequalities for the capacity of the F set which become equalities for the polynomial case.

0. **Preliminaries.** Let $R(z) = P(z)/Q(z)$ be a rational function of degree N , N at least two, where P is a monic polynomial and Q a polynomial. We also assume that P and Q have no common roots. The sequence of iterates $\{R_n(z)\}$ is defined by

$$R_0(z) = z, \quad R_1(z) = R(z), \quad R_{n+1}(z) = R(R_n(z)) \quad n = 1, 2, \dots,$$

where z takes on values from the extended complex plane, $\hat{\mathbb{C}}$, with its topology as the compactified Riemann sphere.

If $w = R_n(z)$ we say that w is a successor of z and z is a predecessor of w , in both cases of order n . If $w = R_n(w)$ and $R_p(w) \neq w$ when $p < n$, we say that w is a fixpoint of order n . The derivative $R'_n(w)$ is called the multiplier of w . The successor of a fixpoint of order n is a fixpoint of order n . Furthermore, the set $\{w, R(w), R_2(w), \dots, R_{n-1}(w)\}$ is called a cycle of order n , and all fixpoints of an n -cycle have the same multiplier since $R'_n(w) = \prod_{k=0}^{n-1} R'(R_k(w))$. A fixpoint w (or a cycle) of order n is called attractive, indifferent or repulsive according as $|R'_k(w)| < 1$, $= 1$, or > 1 respectively.

The set F consists of those points at which the sequence $\{R_n(z)\}$ is not normal in the sense of Montel (see Fatou [2]). An alternate characterization for F is the

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closure of the set of repulsive fixpoints. (See Broliin [1, Theorem 4.1, p. 109], or Julia [4, p. 48].) We shall use the first characterization. The reader may refer to [1] for the results of this section and for the monic polynomial case of §III.

One easily sees from the definition of F that F is closed and completely invariant under R , i.e., $R_{-1}(F) = F = R(F)$.

I. The exceptional points. For the rest of the paper we shall assume that the degree of P is greater than the degree of Q . In this case the point at infinity will be an attractive fixpoint if and only if the coefficient of z^{N-1} in Q has absolute value less than one. Since attractive fixpoints are outside the F set it will then be compact; in particular, this is the case when the degree of P exceeds the degree of Q by at least two. In general, the F set is a compact subset of the Riemann sphere.

We shall call a cycle $\{z_0, z_1, \dots, z_m\}$ a maximal cycle if $R(z) = z_i$ has z_{i-1} as its only solution, and $R(z) = z_0$ has z_m as its only solution. We shall see later that $m = 0$ or 1.

By Montel's theorem, in every neighborhood of $w \in F$ the functions R_n omit at most two values. These, if they occur, are called exceptional points. We next characterize the exceptional points as points of maximal cycles. Such points will be of importance in what follows.

THEOREM 1. *A point belongs to a maximal cycle if and only if it is an exceptional point. Such points are attractive fixpoints and consequently not in F .*

Proof. Points in maximal cycles are attractive fixpoints, in fact, $R'(z_0) = 0$ if z_0 belongs to a maximal cycle. Let z_0 and z_1 be consecutive points in a maximal cycle; z_0 is not necessarily different from z_1 . Then the equation $R(z) - z_1 = 0$ leads us to the equation $P(z) = (z - z_0)^N - z_1 Q(z)$. We have obtained

$$R(z) = (z - z_0)^N / Q(z) - z_1$$

and

$$R'(z) = [N(z - z_0)^{N-1} Q(z) - (z - z_0)^N Q'(z)] / Q^2(z),$$

so $|R'(z_0)| = 0$. In order for ∞ to belong to a maximal cycle, $R(z)$ must either be a polynomial or $R(z) = 1/cz^N$. In the first case ∞ is an attractive fixpoint of order one; the second case is outside the class under consideration. Hence any point of a maximal cycle is an attractive fixpoint, and thus belongs to the complement of F .

Now suppose z_0 belongs to a maximal cycle $\{z_0, z_1, \dots, z_m\}$. Let w be any point in F , and let U be any neighborhood of w not intersecting the maximal cycle, then $\{R_n(z)\} \cap \{z_0, z_1, \dots, z_m\} = \emptyset$ for any n and any z in U by the nature of a maximal cycle. Hence, z_0 and, in fact, all the points of the maximal cycle are exceptional points.

On the other hand, let z_0 be an exceptional point. z_0 may be the only exceptional point and consequently its only predecessor and successor, in which case z_0 is a maximal one-cycle. The other possibility is the existence of two exceptional points z_0 and z_1 , so that z_0 may have the point z_1 as a predecessor.

In this latter case, consideration of the equations $R(z) - z_0 = 0$ and $R(z) - z_1 = 0$ where $R(z) = P(z)/Q(z)$ leads to

$$\begin{aligned} P(z) - z_0 Q(z) &= (z - z_0)^n (z - z_1)^{N-n}, \\ P(z) - z_1 Q(z) &= (z - z_0)^m (z - z_1)^{N-m}. \end{aligned}$$

Now suppose $n < m$, then subtracting the two equations above,

$$Q(z) = (z_1 - z_0)^{-1} [(z - z_0)^n (z - z_1)^{N-m}] [(z - z_1)^{m-n} - (z - z_0)^{m-n}].$$

Multiplying the first equation by z_1 and the second by z_0 , then subtracting, we obtain

$$P(z) = (z_1 - z_0)^{-1} [(z - z_0)^n (z - z_1)^{N-m}] [z_1 (z - z_1)^{m-n} - z_0 (z - z_0)^{m-n}].$$

Since P and Q have no common factors, these equations imply that $n=0$ and $m=N$, thus $P(z) - z_0 Q(z) = (z - z_1)^N$. In this instance $\{z_0, z_1\}$ must be a maximal two-cycle.

Supposing instead that $m < n$, a similar computation shows that $\{z_0\}$ is a maximal one-cycle.

The assumption $n=m$ leads us to $z_0 = z_1$ or the first case where z_0 is the only exceptional point and hence a maximal one-cycle. We conclude that the exceptional points are points of maximal cycles.

COROLLARY. *A maximal cycle is at most of length two.*

THEOREM 2. *We have $R(z) = P(z)/Q(z)$ where P and Q are polynomials, P with leading coefficient one. Suppose that the degree of $P = N >$ degree of Q and $N \geq 2$. We can characterize the occurrence of maximal cycles, i.e., exceptional points, as follows:*

(i) *degree of $Q = 0$: In this case $\{\infty\}$ is a maximal one-cycle. There may be no other one-cycles or one other finite one-cycle, say $\{z_0\}$. In the latter case $P(z) = (z - z_0)^N + z_0 Q$.*

(ii) *degree of $Q > 0$: There may be one finite maximal one-cycle $\{z_0\}$ in which event $P(z) = (z - z_0)^N + z_0 Q(z)$. Secondly, we may have two finite maximal one-cycles $\{z_0\}$ and $\{z_1\}$, then*

$$\begin{aligned} P(z) &= (z_1 - z_0)^{-1} [z_1 (z - z_0)^N - z_0 (z - z_1)^N], \\ Q(z) &= (z_1 - z_0)^{-1} [(z - z_0)^N - (z - z_1)^N]. \end{aligned}$$

Thirdly, $R(z)$ may produce a single maximal two-cycle $\{z_0, z_1\}$ in which case

$$\begin{aligned} P(z) &= (z_1 - z_0)^{-1} [z_1 (z - z_1)^N - z_0 (z - z_0)^N], \\ Q(z) &= (z_1 - z_0)^{-1} [(z - z_1)^N - (z - z_0)^N]. \end{aligned}$$

Proof. (i) $R(z)$ is a polynomial so $\{\infty\}$ is a maximal one-cycle. Suppose we have a finite maximal one-cycle $\{z_0\}$. Since z_0 is the only solution of $R(z) - z_0 = 0$, we have $P(z) - z_0 Q = (z - z_0)^N$.

(ii) We suppose that the degree of $Q > 0$. Again suppose we have only one finite maximal one-cycle $\{z_0\}$. In this case $P(z) - z_0Q(z) = (z - z_0)^N$ so P takes the form $P(z) = (z - z_0)^N + z_0Q(z)$.

Suppose we have two maximal one-cycles $\{z_0\}$ and $\{z_1\}$. Then $R(z) - z_0 = 0$ has one solution z_0 , and $R(z) - z_1 = 0$ has one solution z_1 , so $P(z) - z_0Q(z) = (z - z_0)^N$ and $P(z) - z_1Q(z) = (z - z_1)^N$, or

$$P(z) = (z - z_0)^N + z_0Q(z) \quad \text{and} \quad P(z) = (z - z_1)^N + z_1Q(z).$$

Subtracting, we obtain

$$Q(z) = (z_1 - z_0)^{-1}[(z - z_0)^N - (z - z_1)^N].$$

Thus

$$\begin{aligned} P(z) &= (z - z_0)^N + (z_0/(z_1 - z_0))[(z - z_0)^N - (z - z_1)^N] \\ &= (z_1 - z_0)^{-1}[z_1(z - z_0)^N - z_0(z - z_1)^N]. \end{aligned}$$

Finally, we consider the case of maximal cycles of length two. Let the maximal cycle be $\{z_0, z_1\}$. From the equations $P(z) - z_0Q(z) = (z - z_1)^N$ and $P(z) - z_1Q(z) = (z - z_0)^N$ we obtain

$$\begin{aligned} P(z) &= (z_1 - z_0)^{-1}[z_1(z - z_1)^N - z_0(z - z_0)^N], \\ Q(z) &= (z_1 - z_0)^{-1}[(z - z_1)^N - (z - z_0)^N]. \end{aligned}$$

II. The weak limit measure and its support F . The results of this section extend the results of Brolin [1] which treat the monic polynomial case. This section and the next provide a new characterization of the F set as the support of a particular measure, the weak limit measure.

LEMMA 1. *Given that P and Q have no common roots, then P_k and Q_k also have no common roots for all k . We have the following expressions for the iterates of R :*

$$R = \frac{P}{Q}, \quad R_2 = \frac{Q^N \cdot P \circ R}{Q^N \cdot Q \circ R}, \quad \dots, \quad R_{k+1} = \frac{Q^{N^k} \cdot P_k \circ R}{Q^{N^k} \cdot Q_k \circ R} = \frac{P_{k+1}}{Q_{k+1}}.$$

Proof. By hypothesis P_1 and Q_1 have no common roots. Let us suppose now that P_n and Q_n have no common roots and consider P_{n+1} and Q_{n+1} :

$$P_{n+1} = Q^{N^n} \cdot P_n(P/Q) \quad \text{and} \quad Q_{n+1} = Q^{N^n} \cdot Q_n(P/Q).$$

If w is a common root of P_{n+1} and Q_{n+1} we have

$$Q^{N^n}(w) \cdot P_n(R(w)) = 0 \quad \text{and} \quad Q^{N^n}(w) \cdot Q_n(R(w)) = 0.$$

Since P_n and Q_n cannot have $R(w)$ as a common root, these equations imply that $Q^{N^n}(w) = 0$; i.e., $Q(w) = 0$. But then

$$\begin{aligned} 0 &= Q^{N^n}(w)P_n(P(w)/Q(w)) \\ &= P(w)^{N^n} + aP(w)^{N^n-1}Q(w) + \dots = (P(w))^{N^n} \end{aligned}$$

which contradicts the hypothesis.

We now wish to construct a measure on the extended complex plane. Let z_0 be a point which does not belong to a maximal cycle. Define $\{\mu_n\}$ to be the following sequence of probability measures on the Borel σ -field, \mathcal{B} .

- μ_0 places weight 1 at z_0 .
- μ_1 places weight N^{-1} at the N predecessors of order 1 of z_0 .
- \vdots
- μ_n places weight N^{-n} at the N^n predecessors of order n of z_0 .

The weights are placed at each predecessor according to its multiplicity as a zero of $R_n(z) - z_0 = 0$ and Lemma 1 assures us that this equation has exactly N^n solutions counted according to their multiplicities.

Consider the following sequence of probability measures:

$$\nu_n = \frac{1}{n+1} \sum_{i=0}^n \mu_i, \quad n = 1, 2, \dots$$

This sequence of probability measures on the compactified Riemann sphere has a subsequence $\{\nu_{n_k}\}$ tending to a weak limit probability measure μ , i.e., $\int f d\nu_{n_k} \rightarrow \int f d\mu$ as $k \rightarrow \infty$ for all continuous f .

For arbitrary f , set $Af(z) = N^{-1} \sum_{\{w:R(w)=z\}} f(w)$.

LEMMA 2. $\int Af d\mu = \int f d\mu$ for any integrable Borel measurable f .

Proof. For continuous f

$$\begin{aligned} \int Af d\mu &= \lim_{k \rightarrow \infty} \frac{1}{n_k+1} \sum_{i=0}^{n_k} \int Af d\mu_i \\ &= \lim_{k \rightarrow \infty} \frac{1}{n_k+1} \sum_{i=0}^{n_k} \frac{1}{N^i} \sum_{R_i(z)=z_0} Af(z) \\ &= \lim_{k \rightarrow \infty} \frac{1}{n_k+1} \sum_{i=0}^{n_k} \int f d\mu_{i+1} = \int f d\mu. \end{aligned}$$

The result is now extended to χ_0 for 0 open by using a sequence f_n of continuous functions with $f_n \uparrow \chi_0$. The sets B for which $\int A\chi_B d\mu = \int \chi_B d\mu$ are easily seen to form a σ -algebra so that all Borel sets have the property. The extension to Borel measurable functions is done in the usual way.

LEMMA 3. If ω is any A -invariant probability measure, f and g are Borel measurable and B is a Borel set, then

- (i) $\int (Af) \cdot g d\omega = \int f(g \circ R) d\omega$,
 - (ii) $\int g d\omega = \int g \circ R d\omega$,
 - (iii) $\omega(R^{-1}(B)) = \omega(B)$,
 - (iv) $\omega(R(B)) \leq N\omega(B)$.
- (i) and (ii) hold whenever the integrals involved exist.)

Proof. An easy computation yields $A(f \cdot g \circ R) = (Af) \cdot g$ and (i), (ii) and (iii) follow from this. Since $A\chi_B(z) \geq N^{-1}\chi_{R(B)}(z)$ we have

$$\omega(B) = \int \chi_B d\omega = \int A(\chi_B) d\omega \geq N^{-1}\omega(R(B)).$$

LEMMA 4. *An A -invariant probability measure ω can have atoms only on points of a maximal cycle.*

Proof. Suppose that $\omega(\{\zeta\}) = \alpha_0 > 0$. Using Lemma 2,

$$\begin{aligned} \alpha_0 &= \int \chi_{\{\zeta\}}(z) d\omega(z) = N^{-1} \int \sum_{R(w)=z} \chi_{\{\zeta\}}(w) d\omega(z) \\ &= N^{-1} \int \left[\sum_{R(w)=R(\zeta)} \chi_{\{\zeta\}}(w) \right] \cdot \chi_{\{R(\zeta)\}}(z) d\omega(z) \\ &= k_1 N^{-1} \omega(\{R(\zeta)\}) \end{aligned}$$

where k_1 is the multiplicity of ζ as a solution to $R(w) = R(\zeta)$; $k_1 \leq N$. Hence $\alpha_1 = \omega(\{R(\zeta)\}) = N_1^{-1} \alpha_0 \geq \alpha_0$. Similarly, $\alpha_2 = \omega(\{R_2(\zeta)\}) = N k_2^{-1} \alpha_0 \geq \alpha_1 \geq \alpha_0$ where k_2 is the multiplicity of $R(\zeta)$ as a solution to $R(z) = R_2(\zeta)$, etc.

In order that ω not exceed one in total weight, ζ must be a fixpoint of some order, say n . So we have a cycle $\{\zeta, R(\zeta), \dots, R_{n-1}(\zeta)\}$ with respective weights $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$. Now the monotonicity of the α_i 's forces them all to be equal since $\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_{n-1} \leq \alpha_0$. Consequently, each $k_i = N$, and the cycle is a maximal cycle.

LEMMA 5. *μ is nonatomic.*

Proof. Suppose ζ belongs to a maximal cycle, by Lemma 4, these are the only points at which μ can have an atom. If ζ belongs to a maximal cycle of length one, let U be an open neighborhood of ζ such that $R(U) \subseteq U$ and $z_0 \notin U$. This is possible since ζ is an attractive fixpoint of order one. If ζ and ζ' constitute a maximal two-cycle, each is an attractive fixpoint of order two. Let U' be a small neighborhood of ζ' excluding z_0 . Pick U to be an open set containing ζ such that $R_2(U) \subseteq U$ and $R(U) \subseteq U'$, U also excluding z_0 , then $\bigcup_{n=0}^{\infty} R_n(U) \subseteq U \cup U'$.

In both cases, it is easily seen that $\mu_n(U) = 0$ for all n and hence $\mu(U) = 0$.

THEOREM 3. *F is the support of μ .*

Proof. If $\zeta \in F^c$ there is a neighborhood U of ζ such that $\{R_n\}$ is normal in U . Let \mathcal{O}_ζ be a subset of U which is a relatively compact, open neighborhood of ζ and a μ -continuity set (i.e., $\partial\mathcal{O}_\zeta = 0$) so that $\mu(\mathcal{O}_\zeta) = \lim \nu_{n_k}(\mathcal{O}_\zeta)$. Then if ζ belongs to the support of μ , there is a sequence $\{n_k\}$ such that $\mu_{n_k}(\mathcal{O}_\zeta) = N^{-k}$ (number of zeros of $R_{n_k}(z) - z_0$ in \mathcal{O}_ζ) $\geq \varepsilon > 0$. This implies that the number of zeros of $R_{n_k}(z) - z_0$ in \mathcal{O}_ζ approaches ∞ as $k \rightarrow \infty$.

Since $\{R_n\}$ is normal in U , we may extract a subsequence of $\{n_k\}$ which converges uniformly in compact subsets of U . We shall retain the notation $\{n_k\}$ for such a

subsequence. Then R_{n_k} converges uniformly to some meromorphic L on \mathcal{O}_ζ . If L is not constant there is a small circle about ζ where $L(w) - z_0$ is bounded away from 0. By Rouché's theorem, for large enough k , $L(w) - z_0$ and $R_{n_k}(w) - z_0$ must have the same number of zeros, i.e., $L(w) \equiv z_0$.

Now let f be any function such that f is continuous, $f(z_0) = 1$ and $f \geq 0$. Then

$$\int f d\mu = \int f \circ R_{n_k} d\mu \geq \int_{\mathcal{O}_\zeta} f \circ R_{n_k} d\mu \rightarrow f(z_0)\mu(\mathcal{O}_\zeta) = \mu(\mathcal{O}_\zeta) > 0.$$

Thus μ must have an atom at z_0 , which is impossible, so the support of $\mu \subseteq F$.

To show the reverse inclusion, suppose $\zeta \in F$ and there is a neighborhood \mathcal{O} containing ζ with $\mu(\mathcal{O}) = 0$. By Lemma 3, $\mu(R_n(\mathcal{O})) = 0$, $n = 1, 2, \dots$, so that $\mu(\hat{C} - \{\text{exceptional points}\}) = (\bigcup_{n=0}^\infty R_n(\mathcal{O})) = 0$ which would imply that μ was atomic.

III. The uniqueness and ergodicity of the weak limit measure. Let \mathcal{F} be a countable set of continuous functions dense in the set of continuous functions on \hat{C} in the uniform norm sense. Let

$$C_f = \left\{ z : \frac{1}{n+1} \sum_{k=0}^n A^k f(z) \text{ converges as } n \rightarrow \infty \right\}$$

for each f in \mathcal{F} . By the ergodic theorem of Hopf, $\mu(C_f) = 1$ for any A -invariant probability measure μ . Let $\mathcal{C} = \bigcap_{f \in \mathcal{F}} C_f - \{\text{exceptional points}\}$, then $\mu(\mathcal{C}) = 1$ for any nonatomic A -invariant probability measure.

LEMMA 6. *For any z in \mathcal{C} and any continuous function f on \hat{C} we have that*

$$\frac{1}{n+1} \sum_{k=0}^n A^k f(z) = \int f d\nu_{n,z} \rightarrow \int f d\mu_z \text{ as } n \rightarrow \infty$$

where μ_z is the measure constructed above using z as a starting point.

Proof. Trivial.

Set

$$\tilde{f}(z) = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n A^k f(z).$$

This limit exists for all z in \mathcal{C} and all continuous f by Lemma 6. If μ is any A -invariant measure this limit exists μ -a.e. for any f belonging to $L_1(F, \mathcal{B}, \mu)$.

LEMMA 7. *For ω any nonatomic A -invariant probability measure and any continuous f we have*

$$\int f d\omega = \int \tilde{f} d\omega = \int \left(\int f d\mu_z \right) d\omega.$$

Proof. By the dominated convergence theorem

$$\int f d\omega = \int \left(\frac{1}{n+1} \sum_{k=0}^n A^k f \right) d\omega \rightarrow \int \tilde{f} d\omega.$$

We wish now to show that there is only one A -invariant nonatomic probability measure μ and to compute its logarithmic potential function

$$u(z) = \int \log |z-w|^{-1} d\mu(w)$$

and its energy integral

$$I(\mu) = \iint \log |z-w|^{-1} d\mu(z) d\mu(w).$$

LEMMA 8. Suppose $Q(z) = c \prod_{j=1}^M (z - q_j)$, then

$$A \log |Q|(z) = \log |c| + \frac{1}{N} \sum_{j=1}^M \log |P(q_j)|.$$

Proof.

$$\begin{aligned} A \log |Q|(z) &= \frac{1}{N} \sum_{R(w)=z} \log |Q(w)| = \frac{1}{N} \log \left| c^N \prod_{j=1}^M \prod_{R(w)=z} (w - q_j) \right| \\ &= \log |c| + \frac{1}{N} \sum_{j=1}^M \log |P(q_j) - zQ(q_j)| \\ &= \log |c| + \frac{1}{N} \sum_{j=1}^M \log |P(q_j)|. \end{aligned}$$

We have used here the fact that $\prod_{R(w)=z} (q - w) = P(q) - zQ(q)$.

THEOREM 4. For $R(z) = P(z)/Q(z)$, P and Q polynomials with the leading coefficient of P equal to one and the degree of $P = N \geq$ the degree of $Q + 2$, the weak limit measure μ_z is unique; i.e., μ_z is independent of z where z is any nonexceptional point. In fact, all nonatomic A -invariant probability measures are identical. Furthermore, the logarithmic potential function and the energy integral of this μ are given by

$$(i) \quad u(w) = -\frac{1}{N} \sum_{j=0}^{\infty} N^{-j} \log |Q \circ R_j(w)| \quad \text{for any } w \text{ in } F,$$

$$(ii) \quad \begin{aligned} I(\mu) &= \frac{1}{1-N} \int \log |Q(w)| d\mu(w) \\ &= \frac{1}{1-N} \log |c| + \frac{1}{N(1-N)} \sum_{j=1}^M \log |P(q_j)| \end{aligned}$$

where $Q(z) = c \prod_{j=1}^M (z - q_j)$.

Proof. We first need the formula for the logarithmic potential function. For ζ in C we have

$$\begin{aligned} u(w) &= \int \log \frac{1}{|w-z|} d\mu_{\zeta}(z) = \int A^k \log \frac{1}{|w-z|} d\mu_{\zeta}(z) \\ &= \int N^{-k} \left(\sum_{R_k(\eta)=z} \log \frac{1}{|\eta-w|} \right) d\mu_{\zeta}(z) \\ &= \int N^{-k} \left(\log \frac{1}{|R_k(w)-z|} + \log \frac{1}{|Q_k(w)|} \right) d\mu_{\zeta}(z). \end{aligned}$$

If $w \in F$, $\log |R_k(w) - z|^{-1} \geq \log (\text{diameter of } F)^{-1}$ for $z \in F$, so

$$\begin{aligned} u(w) &\geq \lim_{k \rightarrow \infty} N^{-k} \int \log \frac{1}{|Q_k(w)|} d\mu_z(z) \\ &= \lim_{k \rightarrow \infty} -N^{-k} \log |Q_k(w)| \\ &= \lim_{k \rightarrow \infty} -(N^{-k} \log |Q|^{N^{k-1}} + N^{-k} \log |Q_{k-1} \circ R|)(w) \\ &= \dots = -\frac{1}{N} \sum_{j=0}^{\infty} N^{-j} \log |Q \circ R_j(w)|. \end{aligned}$$

Now for arbitrary w , if $R_k(w) \rightarrow \infty$ then $\log |R_k(w) - z|^{-1} \rightarrow -\infty$ for all z in F , so by a similar computation

$$u(w) \leq -\frac{1}{N} \sum_{j=0}^{\infty} N^{-j} \log |Q \circ R_j(w)|.$$

If $w_k \rightarrow w \in F$, and $\lim_{j \rightarrow \infty} |R_j(w_k)| = \infty$, then

$$\begin{aligned} u(w) &= \lim_{T \rightarrow \infty} \int \min \left(\log \frac{1}{|w-z|}, T \right) d\mu_z \\ &= \lim_{T \rightarrow \infty} \lim_{k \rightarrow \infty} \int \min \left(\log \frac{1}{|w_k-z|}, T \right) d\mu_z \\ &\leq \liminf_{k \rightarrow \infty} -\frac{1}{N} \sum_{j=0}^{\infty} N^{-j} \log |Q \circ R_j(w_k)|. \end{aligned}$$

It remains to find a sequence $\{w_k\}$ for which the last limit can be taken term by term. If $M=0$ so that Q is constant, this can easily be done. Otherwise set $L=N-M$ and proceed as follows: for large enough a , $|z| \geq a$ implies

- (i) $|R(z)| > 2|z|$,
- (ii) $|R(z)| \leq b|z|^L$,
- (iii) $|Q(z)| \leq b|z|^M$, and
- (iv) $|Q(z)| \geq 1$

where b is some constant. For w in F we can find a sequence $\{w_k\}$ converging to w such that for each w_k there is an $l=l(k)$ with $|R_j(w_k)| < a$ for $j < l$ and $a \leq |R_l(w_k)| < 2a$, since otherwise there would be a neighborhood V of w for which $R_j(V)$ missed the entire annulus $a \leq |z| < 2a$. For $j < l(k)$ we have

$$N^{-j} \log \delta \leq N^{-j} \log |Q \circ R_j(w_k)| \leq N^{-j} \log \gamma$$

where

$$\delta = \inf \{|Q(z)| : |z| < a\} > 0 \quad \text{and} \quad \gamma = \sup \{|Q(z)| : |z| < a\} < \infty.$$

For $j \geq l(k)$,

$$\begin{aligned} 0 &\leq N^{-j} \log |Q \circ R_j(w_k)| \leq N^{-j} (\log b + M \log |R_j(w_k)|) \\ &\leq N^{-j} (\log b + M \log |R_{j-l}(R_l(w_k))|) \\ &\leq N^{-j} (\log b + M((L + \dots + L^{j-l}) \log b + L^{j-l} \log |R_l(w_k)|)) \\ &\leq N^{-j} (\log b + M((L + \dots + L^{j-l}) \log b + L^{j-l} \log (2a))) \\ &\leq K(L/N)^j. \end{aligned}$$

Thus the limit above can be taken term by term, and the formula for the logarithmic potential is justified.

We can now obtain the formula for $I(\mu_\zeta)$ for any ζ in C . Using the Fubini and dominated convergence theorems

$$\begin{aligned} I(\mu_\zeta) &= \iint \log \frac{1}{|z-w|} d\mu_\zeta(z) d\mu_\zeta(w) = \int S(w) d\mu_\zeta(w) \\ &= -\frac{1}{N} \int \sum_{j=0}^{\infty} N^{-j} \log |Q \circ R_j(w)| d\mu_\zeta(w) \\ &= -\frac{1}{N} \left(\sum_{j=0}^{\infty} N^{-j} \right) \int \log |Q(w)| d\mu_\zeta(w) \\ &= \frac{1}{1-N} \int \log |Q(w)| d\mu_\zeta(w) \\ &= \frac{1}{1-N} \int A \log |Q(w)| d\mu_\zeta(w) \\ &= \frac{1}{1-N} \log |c| + \frac{1}{N(1-N)} \sum_{j=1}^M \log |P(q_j)|. \end{aligned}$$

We shall next show the uniqueness of μ_ζ for all ζ in C . Let ζ_1 and ζ_2 be in C . Let $\sigma = \mu_{\zeta_1} - \mu_{\zeta_2}$. $I(|\sigma|) \leq I(\mu_{\zeta_1} - \mu_{\zeta_2}) < \infty$. Now using Fubini's Theorem again

$$\begin{aligned} I(\sigma) &= \int \left(\int \log |z-w|^{-1} \right) d(\mu_{\zeta_1}(z) - \mu_{\zeta_2}(z)) d\sigma(w) \\ &= \int (S(w) - S(w)) d\sigma(w) = 0. \end{aligned}$$

So, by Theorem 16.42 of Hille [3, p. 241], $\mu_{\zeta_1} = \mu_{\zeta_2}$. Since ζ_1 and ζ_2 are arbitrary in C , we have shown that all ζ in C produce identical μ_ζ . Finally, by Lemma 7, if $\mu_\zeta = \mu$ for all ζ in C , and if ν is any nonatomic A -invariant probability measure then

$$\int f d\nu = \int \left(\int f d\mu_\zeta \right) d\nu = \int f d\mu$$

for all continuous f , so $\nu = \mu$.

THEOREM 5. For $R(z) = P(z)/Q(z)$ with P a monic polynomial and Q a polynomial such that the degree of $P \geq$ the degree of $Q + 2$, μ is ergodic with respect to the transformation R where μ is the unique A -invariant nonatomic probability measure.

Proof. If $R^{-1}(B) = B$ a.e. and B is a Borel set, then for any Borel measurable f in L_1 ,

$$\begin{aligned} \int \chi_B f d\mu &= \int \left(\frac{1}{n+1} \sum_{k=0}^n \chi_B \circ R_k \right) f d\mu \\ &= \int \chi_B \left(\frac{1}{n+1} \sum_{k=0}^n A^k f \right) d\mu \rightarrow \int f d\mu \int \chi_B d\mu \end{aligned}$$

i.e., χ_B is a constant a.e.

In general, for compact $E \subset C$ either $I(\nu)$ is infinite for all probability measures supported on E , or there exists a unique ν_0 , called the equilibrium measure, which minimizes $I(\nu)$. In this case the capacity C of E is defined by $C = \exp [-I(\nu_0)]$. The above facts can be found in Hille [3]. Brolin [1] proved that μ is the equilibrium measure when R is a polynomial.

We will write $\|f\|_F$ for the uniform norm of f on F ; i.e.,

$$f\|_F = \sup \{|f(z)| : z \in F\}.$$

THEOREM 6. *We have*

$$\exp \left(\frac{1}{N-1} \int \log |Q| d\mu \right) \leq C \leq \|Q\|_F^{1/(N-1)}.$$

μ is the equilibrium measure if and only if Q is constant; i.e., R is a polynomial, and in this case $C = Q^{1/(N-1)}$.

Proof. Let ν be the equilibrium measure. Then

$$\log C = -I(\nu) \geq -I(\mu) = \frac{1}{N-1} \int \log |Q| d\mu,$$

or

$$\exp \left(\frac{1}{N-1} \int \log |Q| d\mu \right) \leq C.$$

We also have $C = \lim_{k \rightarrow \infty} (\|T_k\|_F)^{1/k}$ where T_k is the k th Tchebycheff polynomial for F , i.e., the monic polynomial of degree k with minimum $\|T_k\|_F$. Since $R(F) \subset F$,

$$\|T_{nN}\|_F^{1/nN} \leq \|Q^n T_n \circ R\|_F^{1/nN} \leq \|Q\|_F^{1/N} \|T_n\|_F^{1/nN}.$$

Taking limits on n we get $C \leq \|Q\|_F^{1/N} C^{1/N}$ from which the desired inequality follows.

If Q is constant we have

$$C = |Q|^{1/(N-1)} = \exp (-I(\mu)).$$

On the other hand, if μ is the equilibrium measure then u is constant almost everywhere on F so that

$$\begin{aligned} u(w) &= u(R(w)) = -\frac{1}{N} \sum_{j=0}^{\infty} N^{-j} \log |Q \circ R_{j+1}(w)| \\ &= Nu(w) + \log |Q|(w), \end{aligned}$$

and Q must be constant.

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