TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 166, April 1972

ZEROS OF PARTIAL SUMS AND REMAINDERS OF POWER SERIES

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Abstract. For a power series $f(z)=\sum_{k=0}^{\infty} a_k z^k$ let $s_n(f)$ denote the maximum modulus of the zeros of the nth partial sum of f and let $r_n(f)$ denote the smallest modulus of a zero of the nth normalized remainder $\sum_{k=n}^{\infty} a_k z^{k-n}$. The present paper investigates the relationships between the growth of the analytic function f and the behavior of the sequences $\{s_n(f)\}$ and $\{r_n(f)\}$. The principal growth measure used is that of R-type: if $R=\{R_n\}$ is a nondecreasing sequence of positive numbers such that $\lim (R_{n+1}/R_n)=1$, then the R-type of f is $\tau_R(f)=\limsup |a_n R_1 R_2 \cdots R_n|^{1/n}$. We prove that there is a constant P such that

$$\tau_R(f) \liminf (s_n(f)/R_n) \leqq P \quad \text{and} \quad \tau_R(f) \limsup (r_n(f)/R_n) \geqq (1/P)$$

for functions f of positive finite R-type. The constant P cannot be replaced by a smaller number in either inequality; P is called the power series constant.

1. **Introduction.** The following theorem is a consequence of results of the first author [3] and J. L. Frank [4].

Theorem A. *Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$ have radius of convergence $c(f)$, $0<c(f)<\infty$. There exists an absolute constant P such that, if $\varepsilon>0$, then*

(i) *infinitely many of the partial sums*

$$S_n(f;z) = \sum_{k=0}^{n} a_k z^k \qquad (n = 1, 2, 3, \ldots)$$

have all their zeros in the disc $|z| \leqq c(f)(P+\varepsilon)$;

(ii) *infinitely many of the normalized remainders*

$$\mathscr{S}^n f(z) = \sum_{k=n}^{\infty} a_k z^{k-n} \qquad (n = 0, 1, 2, \ldots)$$

have no zero in the disc $|z| \leqq c(f)(P+\varepsilon)^{-1}$;

(iii) *the constant P cannot be replaced by a smaller number in either* (i) *or* (ii).

In view of (iii), the constant P is uniquely determined by Theorem A. We call P the *power series constant*; its numerical value is known to lie between 1.7818 and

Presented to the Society, January 23, 1970 under the title *Partial sums and remainders of power series*; received by the editors May 4, 1971.

AMS 1970 *subject classifications*. Primary 30A08, 30A10; Secondary 30A06.

Key words and phrases. The power series constant, zeros of partial sums, zeros of remainders, R-type, entire functions, extremal functions.

1.82. Our object in the present paper is to give a simpler proof of Theorem A, to investigate the extremal functions associated with it, and to obtain corresponding results for various classes of entire functions.

For $f(z)=\sum_{k=0}^{\infty} a_k z^k$, let $s_n=s_n(f)$ denote the largest of the moduli of the zeros of $S_n(f; z)=\sum_{k=0}^{n} a_k z^k$ $(n=1, 2, 3, \ldots)$ with the convention that $s_n=\infty$ if $a_n=0$. Let $r_n=r_n(f)$ denote the supremum of numbers r such that $\mathscr{S}^n f(z)=\sum_{k=n}^{\infty} a_k z^{k-n}$ is analytic and has no zero in the disc $|z|<r$. Theorem A is equivalent to the estimates

$$\liminf_{n\to\infty} s_n(f) \leqq c(f)P, \tag{1.1}$$

$$\limsup_{n\to\infty} r_n(f) \geqq \frac{c(f)}{P}, \tag{1.2}$$

for $0<c(f)<\infty$, together with the assertion that the constant P is best possible in both cases.

Okada [6] has shown that $\limsup_{n\to\infty} s_n(f)=\infty$ if and only if f is entire. For entire f, M. Tsuji [6] proved the surprising result that

$$\limsup_{n\to\infty} \frac{\log n}{\log s_n(f)}$$

is always equal to the order of f. For functions of positive finite order and type, we are able to sharpen Tsuji's theorem considerably.

THEOREM B. *Suppose the entire function f is of order ρ and type τ, $0<\rho, \tau<\infty$. Then*

$$\limsup_{n\to\infty} \left(\frac{\rho\tau}{n}\right)^{1/\rho} r_n(f) \geqq \frac{1}{P} \tag{1.3}$$

and

$$e^{-1/\rho} \leqq \liminf_{n\to\infty} \left(\frac{\rho\tau}{n}\right)^{1/\rho} s_n(f) \leqq P. \tag{1.4}$$

Furthermore, for each of the three inequalities, there exists an f of order ρ and type τ for which equality is assumed.

Both Theorem A and Theorem B are special cases of a result involving a more general measure of growth for analytic functions. Let $R=\{R_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that $\lim_{n\to\infty} R_{n+1}/R_n=1$. The R-type, $\tau_R(f)$, of an analytic function $f(z)=\sum_{k=0}^{\infty} a_k z^k$ is defined to be

$$\tau_R(f) = \limsup_{n\to\infty} |a_n R_1 R_2 \cdots R_n|^{1/n}.$$

If $R_n \to \infty$ as $n \to \infty$, R-type can be related to the growth of the maximum modulus of f [1, p. 6]. It follows easily from the expression for the type of an entire function in terms of its coefficients that f is of order ρ and type τ, $0<\rho$, $\tau<\infty$, if and only if

$\tau_R(f)=1$ for the sequence $R_n=(n/\rho\tau)^{1/\rho}$, $n=1, 2, 3, \ldots$. If $\lim_{n\to\infty} R_n=l<\infty$, then one sees that $\tau_R(f)=l/c(f)$.

Our principal result is the following.

THEOREM C. *If* $0<\tau_R(f)<\infty$, *then*

$$\liminf_{n\to\infty} \frac{(R_1R_2\cdots R_n)^{1/n}}{R_n} \leqq \tau_R(f) \liminf_{n\to\infty} \frac{s_n(f)}{R_n} \leqq P \tag{1.5}$$

and

$$\tau_R(f) \limsup_{n\to\infty} \frac{r_n(f)}{R_n} \geqq \frac{1}{P}. \tag{1.6}$$

Furthermore, for each of the three inequalities, there exists a function of R-type 1 *for which equality is assumed.*

If one takes $R_n\equiv 1$, Theorem C reduces to Theorem A. If one takes $R_n\equiv(n/\rho\tau)^{1/\rho}$, then Theorem C reduces to Theorem B.

Suppose $0<c(f)<\infty$ and $\varepsilon>0$. In 1906, M. B. Porter [5] proved that an infinite sequence of the partial sums of f tends uniformly to ∞ outside the disc $|z| \leqq c(f)(2+\varepsilon)$. In view of Theorem A, the constant 2 in Porter's theorem cannot be replaced by a number less than P. We prove in §2 that the best possible constant for Porter's theorem is P. This follows fairly easily from a theorem on the partial sums of polynomials which is of some interest in itself.

THEOREM D. *Let* $Q(z)=a_0+a_1z+\cdots+a_nz^n$ *be a polynomial of degree n. Then for at least one integer k,* $0\leqq k\leqq n$, *we have*

$$|a_0+a_1z+\cdots+a_kz^k| \geqq |a_n|\,|z|^k/(n+1) \tag{1.7}$$

for all $|z|\geqq P$.

Theorem D guarantees that the partial sum $a_0+a_1z+\cdots+a_kz^k$ has all its zeros in the disc $|z|\leqq P$. Since (1.7) holds for large $|z|$, we must have

$$|a_k| \geqq |a_n|/(n+1). \tag{1.8}$$

In applications, this yields information about the value of k for which (1.7) holds.

2. **The remainder polynomials.** The treatment of the power series constant in [3] and [4] involves the *remainder polynomials* $B_n(z; z_0, z_1, \ldots, z_{n-1})$, defined recursively by

$$\begin{aligned} B_0(z) &= 1,\\ B_n(z; z_0, z_1, \ldots, z_{n-1}) &= z^n - \sum_{k=0}^{n-1} z_k^{n-k} B_k(z; z_0, z_1, \ldots, z_{k-1}). \end{aligned} \tag{2.1}$$

Let

$$H_n = \max |B_n(0; z_0, \ldots, z_{n-1})|,$$

where the maximum is taken over all sequences $\{z_k\}_{k=0}^{n-1}$ whose terms lie on $|z|=1$. Buckholtz [3] proved that

$$P = \lim_{n\to\infty} H_n^{1/n} = \sup_{1\leqq n<\infty} H_n^{1/n}.$$

For a power series $f(z)=\sum_{k=0}^{\infty} a_k z^k$, we write (2.1) in the form

$$z^n = \sum_{k=0}^{n} z_k^{n-k} B_k(z; z_0, \ldots, z_{k-1})$$

and substitute this expression into the power series for f. We obtain the formal expansion

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \left\{ \sum_{k=0}^{n} z_k^{n-k} B_k(z; z_0, \ldots, z_{k-1}) \right\} \\ &= \sum_{k=0}^{\infty} B_k(z; z_0, \ldots, z_{k-1}) \sum_{n=k}^{\infty} a_n z_k^{n-k} = \sum_{k=0}^{\infty} \mathscr{S}^k f(z_k) B_k(z; z_0, \ldots, z_{k-1}), \end{aligned} \tag{2.2}$$

which holds whenever the interchange in the order of summation can be justified. In particular, (2.2) holds if f is a polynomial and yields considerable information when f is taken to be a remainder polynomial. In the latter case, an easy induction argument establishes the identity

$$\mathscr{S}^k B_n(z; z_0, \ldots, z_{n-1}) = B_{n-k}(z; z_k, \ldots, z_{n-1}), \tag{2.3}$$

for $0 \leqq k \leqq n$.

The remainder polynomials also satisfy the following properties:

(2.4) $B_n(\lambda z; \lambda z_0, \ldots, \lambda z_{n-1}) = \lambda^n B_n(z; z_0, \ldots, z_{n-1})$,

(2.5) $B_n(z_0; z_0, \ldots, z_{n-1}) = 0$,

(2.6) $z^n B_n(1/z; z_n, \ldots, z_1) = \sum_{k=0}^{n} B_k(0; z_k, \ldots, z_1) z^k$,

(2.7) $B_n(z; z_n, \ldots, z_1) = \sum_{k=0}^{n_1} B_k(0; z_k, \ldots, z_1) B_{n-k}(z; z_n, \ldots, z_{n_1+1}, 0, \ldots, 0)$ for $0 \leqq n_1 \leqq n$,

(2.8) $H_{m+n} \geqq H_m H_n$ for nonnegative integers m and n.

The proofs of these identities may be found in [3].

We are now ready to prove Theorem D. Thus let $Q(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n. Define $f(z) = z^n Q(1/z) = b_0 + b_1 z + \cdots + b_n z^n$; note that $b_{n-k} = a_k$, $0 \leqq k \leqq n$. Let $\{z_j\}_{j=0}^{n}$ be a sequence of complex numbers satisfying

$$|\mathscr{S}^j f(z_j)| = \min_{|z|\leqq 1/P} |\mathscr{S}^j f(z)|, \qquad 0 \leqq j \leqq n.$$

From (2.2),

$$|f(0)| \leqq \sum_{k=0}^{n} |\mathscr{S}^k f(z_k)|\, |B_k(0; z_0, \ldots, z_{k-1})|.$$

Setting $w_k = P z_k$, $0 \leqq k \leqq n$, we have $|w_k| \leqq 1$ and, by (2.4),

$$\begin{aligned} |B_k(0; z_0, \ldots, z_{k-1})| &= |B_k(0; w_0/P, \ldots, w_{k-1}/P)| \\ &= (1/P^k)|B_k(0; w_0, \ldots, w_{k-1})| \leqq (1/P^k) H_k \leqq 1, \end{aligned}$$

for $0 \leqq k \leqq n$. Hence $|f(0)| \leqq \sum_{k=0}^{n} |\mathscr{S}^k f(z_k)|$ and so $|f(0)| \leqq (n+1)|\mathscr{S}^m f(z_m)|$ for some m, $0 \leqq m \leqq n$. Since $f(0)=b_0$, we have $|\mathscr{S}^m f(z)| \geqq |b_0|/(n+1)$ for all $|z| \leqq 1/P$. Now

$$\mathscr{S}^m f(z) = b_m + b_{m+1}z + \cdots + b_n z^{n-m}$$

and therefore, replacing z by $1/z$, we obtain

$$|b_m z^{n-m} + b_{m+1} z^{n-m-1} + \cdots + b_n| \geqq |z|^{n-m}|b_0|/(n+1)$$

for all $|z| \geqq P$. Letting $p = n-m$, this inequality is equivalent to

$$|a_0 + a_1 z + \cdots + a_p z^p| \geqq |z|^p |a_n|/(n+1)$$

for all $|z| \geqq P$, and this completes the proof.

Corollary 1. *Suppose that the power series $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence less than* 1. *Then there are infinitely many integers k such that*

$$\left|\sum_{j=0}^{k} a_j z^j\right| \geqq |z|^k \tag{2.9}$$

for all $|z| \geqq P$.

Proof. For each positive integer n such that $a_n \neq 0$, let $k(n)$ denote the least positive integer k for which (1.7) holds. The condition $\limsup |a_n|^{1/n} > 1$ implies that there is an infinite set I of positive integers such that $|a_n|/(n+1) > n$ for all $n \in I$. For each $n \in I$ we therefore have $|\sum_{j=0}^{k(n)} a_j z^j| \geqq |z|^{k(n)}$ and, by (1.8), $|a_{k(n)}| \geqq |a_n|/(n+1) > n$. The latter condition guarantees that $k(n)$ assumes infinitely many values as n ranges over I, and this completes the proof.

Suppose f has radius of convergence t, $0 < t < \infty$. Let $\varepsilon > 0$ and define $g(z) = f(tz/(1-\varepsilon))$. Then $c(g) < 1$ and (2.9) implies that $s_n(g) \leqq P$ for infinitely many integers n. Thus $\liminf_{n\to\infty} s_n(g) \leqq P$. But $s_n(g) = ((1-\varepsilon)/t)s_n(f)$ and therefore $\liminf_{n\to\infty} s_n(f) \leqq tP/(1-\varepsilon)$. It follows that $\liminf_{n\to\infty} s_n(f) \leqq c(f)P$ and this proves (1.1).

Lemma 1. *If n is a nonnegative integer, then*

$$1 \leqq P^n/H_n \leqq 17. \tag{2.10}$$

This will be proved in §3.

Let m be a positive integer and suppose $z_0, z_1, \ldots, z_{m-1}$ lie on $|z|=1$. If $k \geqq m$, then (2.1) implies

$$B_k(0; z_0, \ldots, z_{m-1}, 0, \ldots, 0) = -\sum_{j=0}^{m-1} z_j^{k-j} B_j(0; z_0, \ldots, z_{j-1}).$$

It follows that

$$|B_k(0; z_0, \ldots, z_{m-1}, 0, \ldots, 0)| \leqq \sum_{j=0}^{m-1} H_j \leqq \sum_{j=0}^{m-1} P^j < \frac{P^m}{P-1}. \tag{2.11}$$

The assertion that the constant P is best possible in (1.1) depends on the existence of a function f such that $\liminf s_n(f)=c(f)P$. It suffices to construct such an f satisfying $c(f)=1$.

LEMMA 2. *There exists a power series* $\sum_{k=0}^{\infty} A_k z^k$, *with radius of convergence* 1, *such that each partial sum* $\sum_{k=0}^{m} A_k z^k$ *has a zero of modulus* P.

Proof. For each nonnegative integer n, let $\{z_j^{(n)}\}_{j=1}^{n}$ be a sequence of complex numbers of modulus $1/P$ such that

$$|B_n(0; z_n^{(n)}, z_{n-1}^{(n)}, \ldots, z_1^{(n)})| = H_n/P^n. \tag{2.12}$$

Here, we have used (2.4). If n, n_1 and j are positive integers such that $j \leqq n_1 \leqq n$, then (2.7) implies

$$\begin{aligned} &|B_n(0; z_n^{(n)}, \ldots, z_1^{(n)})| \\ &\leqq \sum_{k=0}^{n_1-j} |B_k(0; z_k^{(n)}, \ldots, z_1^{(n)})|\, |B_{n-k}(0; z_n^{(n)}, \ldots, z_{n_1+1}^{(n)}, 0, \ldots, 0)| \\ &\quad + \sum_{k=n_1-j+1}^{n_1} |B_k(0; z_k^{(n)}, \ldots, z_1^{(n)})|\, |B_{n-k}(0; z_n^{(n)}, \ldots, z_{n_1+1}^{(n)}, 0, \ldots, 0)|. \end{aligned} \tag{2.13}$$

If $0 \leqq k \leqq n_1-j$, we have

$$|B_k(0; z_k^{(n)}, \ldots, z_1^{(n)})| \leqq H_k/P^k \leqq 1$$

and (2.11) implies

$$|B_{n-k}(0; z_n^{(n)}, \ldots, z_{n_1+1}^{(n)}, 0, \ldots, 0)| \leqq P^{n-n_1}/(P^{n-k}(P-1)).$$

The first sum on the right of (2.13) therefore does not exceed

$$\sum_{k=0}^{n_1-j} \frac{P^{k-n_1}}{(P-1)} = \frac{P^{-j+1}-P^{-n_1}}{(P-1)^2} < \frac{1}{P^{j-1}(P-1)^2}.$$

If $n_1-j+1 \leqq k \leqq n_1$, then

$$|B_{n-k}(0; z_n^{(n)}, \ldots, z_{n_1+1}^{(n)}, 0, \ldots, 0)| \leqq H_{n-k}/P^{n-k} \leqq 1.$$

In view of (2.10), (2.13) now yields

$$\sum_{k=n_1-j+1}^{n_1} |B_k(0; z_k^{(n)}, \ldots, z_1^{(n)})| \geqq \frac{1}{17} - \frac{1}{P^{j-1}(P-1)^2}.$$

Taking $j=7$ and using the bound $P>1.78$, we have $\sum_{k=n_1-6}^{n_1} |B_k(0; z_k^{(n)}, \ldots, z_1^{(n)})| > 1/1000$. Therefore $|B_k(0; z_k^{(n)}, \ldots, z_1^{(n)})| > 1/7000$ for at least one integer k, $n_1-6 \leqq k \leqq n_1$. Moreover, $|B_k(0; z_k^{(n)}, \ldots, z_1^{(n)})| \leqq 1$ for all n and k. Now define

$$P_n(z) = z^n B_n(1/z; z_n^{(n)}, \ldots, z_1^{(n)}), \qquad n = 0, 1, 2, \ldots.$$

Since (2.6) implies $P_n(z)=\sum_{k=0}^{n} B_k(0; z_k^{(n)}, \ldots, z_1^{(n)})z^k$, it follows that the coefficients of P_n are bounded by 1 and that in a set of 7 consecutive coefficients, at least one coefficient has modulus greater than $1/7000$. The sequence $\{P_n\}$ is uniformly bounded on compact subsets of the unit disc. Extract a uniformly convergent subsequence of $\{P_n\}$ and let F denote the limit function. Writing $F(z)=\sum_{k=0}^{\infty} A_k z^k$, it follows that $|A_k|\leqq 1$, $0\leqq k<\infty$, and that in a set of 7 consecutive coefficients A_k, at least one coefficient has modulus greater than $1/7000$. Hence $c(F)=1$. If $m<n$, then (2.6) implies that the mth partial sum of P_n is given by

$$S_m(P_n; z) = z^m B_m(1/z; z_m^{(n)}, \ldots, z_1^{(n)}).$$

By (2.5), $S_m(P_n; 1/z_m^{(n)})=0$. Since $S_m(F; z)$ is the uniform limit of a subsequence of $\{S_m(P_n; z)\}$, it follows that $S_m(F; z)$ has a zero of modulus P. This completes the proof of the lemma.

The function F of the preceding lemma satisfies $c(F)=1$ and $\liminf_{n\to\infty} s_n(F)\geqq P$. It follows that the constant P is best possible in (1.1).

We now show that P is the sharp constant in Porter's theorem. If $f(z)=\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence t, then Corollary 1 implies that there are infinitely many integers k such that $|\sum_{j=0}^{k} a_j z^j|\geqq(|z|/t(1+\varepsilon))^k$ for all $|z|\geqq tP(1+\varepsilon)$. The corresponding subsequence of partial sums $\{S_k(f; z)\}$ therefore tends uniformly to ∞ outside the disc $|z|\leqq c(f)P(1+\varepsilon)$. On the other hand, we can, by Lemma 3, construct a function F such that $c(F)=t$ and such that each partial sum of F has a zero in $|z|\leqq c(F)P$.

The inequality (1.2) is a special case of (1.6); the latter will be proved in §4. To show that P is the sharp constant in (1.2), it suffices to construct a function G satisfying $c(G)=1$ and $\limsup r_n(G)\leqq 1/P$.

LEMMA 3. *There exists a power series* $G(z)=\sum_{k=0}^{\infty} A_k z^k$, *with* $c(G)=1$, *such that each normalized remainder of G has a zero of modulus* $1/P$. *In particular*, $\limsup r_n(G)\leqq 1/P$.

Proof. Consider the sequence of complex numbers $\{B_n(0; z_n^{(n)}, \ldots, z_1^{(n)})\}_{n=1}^{\infty}$ constructed in Lemma 2. For each n we have $|z_j^{(n)}|=1/P$, for $1\leqq j\leqq n$, $|B_j(0; z_j^{(n)}, \ldots, z_1^{(n)})|\leqq 1$, for $0\leqq j\leqq n$, and $|B_n(0; z_n^{(n)}, \ldots, z_1^{(n)})|=H_n/P^n$. Furthermore, if $n_1\leqq n$, then $|B_k(0; z_k^{(n)}, \ldots, z_1^{(n)})|\geqq 1/7000$ for at least one integer k such that $n_1-6\leqq k\leqq n_1$. By (2.6),

$$B_n(z; z_n^{(n)}, \ldots, z_1^{(n)}) = \sum_{k=0}^{n} B_k(0; z_k^{(n)}, \ldots, z_1^{(n)}) z^{n-k}.$$

The sequence $\{B_n(z; z_n^{(n)}, \ldots, z_1^{(n)})\}_{n=1}^{\infty}$ is therefore uniformly bounded on compact subsets of the unit disc. Extract a uniformly convergent subsequence from $\{B_n\}$ and let G denote the limit function. If $G(z)=\sum_{k=0}^{\infty} A_k z^k$, then $|A_k|\leqq 1$ for all k and $|A_k|\geqq 1/7000$ for infinitely many k; thus $c(G)=1$. The identities

$$\mathscr{S}^k B_n(z; z_n^{(n)}, \ldots, z_1^{(n)}) = B_{n-k}(z; z_{n-k}^{(n)}, \ldots, z_1^{(n)}),$$

$$B_{n-k}(z_{n-k}^{(n)}; z_{n-k}^{(n)}, \ldots, z_1^{(n)}) = 0,$$

for $0 \leqq k < n$, show that B_n and each of its first $(n-1)$ normalized remainders have zeros of modulus $1/P$. Furthermore, if m is a nonnegative integer, then $\mathscr{S}^m G(z)$ is the uniform limit of a subsequence of $\{\mathscr{S}^m B_n(z; z_n^{(n)}, \ldots, z_1^{(n)})\}$ on the compact set $|z| \leqq (1/P) + \varepsilon < 1$. It follows that $\mathscr{S}^m G(z)$ has a zero of modulus $1/P$.

3. **The functions $T_m(\mathscr{U})$.** For $m = 1, 2, 3, \ldots$, and $0 \leqq \mathscr{U} < 1$, define

$$T_m(\mathscr{U}) = \max \sum_{k=m}^{\infty} \mathscr{U}^k |B_k(0; w_0, w_1, \ldots, w_{m-1}, 0, \ldots, 0)|$$

where the maximum is taken over all sequences $\{w_k\}_{k=0}^{m-1}$ whose terms lie on $|z| = 1$. The functions $T_n(\mathscr{U})$ were characterized by Buckholtz [3]. For each m, T_m is increasing; the unique solution to the equation $T_m(\mathscr{U}) = 1$ is denoted by $\mathscr{U}_m$. The most important property of the sequence $\{\mathscr{U}_m\}$ is the determination

$$\text{(3.1)} \qquad \mathrm{P} = \lim_{m \to \infty} \mathscr{U}_m^{-1} = \inf_{1 \leqq m \leqq \infty} \mathscr{U}_m^{-1}.$$

Since T_m is increasing, (3.1) implies

$$\text{(3.2)} \qquad T_m(1/P) > 1, \qquad m = 1, 2, 3, \ldots.$$

Proof of Lemma 1. By (2.11) and (3.2), we have

$$1 \leqq T_m(1/P) \leqq \frac{H_m}{P^m} + \frac{H_{m+1}}{P^{m+1}} + \frac{H_{m+2}}{P^{m+2}} + \sum_{k=m+3}^{\infty} (1/P)^k \frac{P^m}{P-1},$$

for each positive integer m.

In view of (2.8), the previous inequality implies

$$1 \leqq \left(\frac{H_{m+2}}{P^{m+2}}\right)\left[1 + \frac{P}{H_1} + \frac{P^2}{H_2}\right] + \frac{P^m}{P-1} \frac{P^{-m-3}}{(1-(1/P))};$$

therefore,

$$1 \leqq (H_{m+2}/P^{m+2})[1 + P + P^2/2] + P^{-2}(P-1)^{-2}.$$

Using the bounds $1.78 < P < 1.82$, we obtain $H_{m+2}/P^{m+2} \geqq 1/17$. It is easily verified that $H_j/P^j > 1/17$ for $j = 1, 2$. Since $P = \sup_{1 \leqq n < \infty} H_n^{1/n}$, we have $1/17 \leqq H_n/P^n \leqq 1$ for all n.

4. **Main results.** In this section, we prove (1.5) and (1.6).

LEMMA 4. *Let m be a positive integer and $\{A_k\}_{k=1}^{\infty}$ a sequence of complex numbers $(A_0 = 1)$ such that $|A_k| \leqq 1$ for $k \geqq m$. Then for at least one integer p, $0 \leqq p \leqq m-1$, the function $A_p + A_{p+1}z + A_{p+2}z^2 + \cdots$ has no zero in the disc $|z| < \mathscr{U}_m$.*

Proof. Let $f(z) = 1 + \sum_{k=1}^{\infty} A_k z^k$. We have to show that for some p, $0 \leqq p \leqq m-1$, $\mathscr{S}^p f(z)$ has no zero in $|z| < \mathscr{U}_m$. Let $\{z_k\}_{k=0}^{\infty}$ be a sequence of points in $|z| < 1$ such that $z_k = 0$ for $k \geqq m$. Then, by (2.1),

$$\sum_{k=0}^{m-1} \mathscr{S}^k f(z_k) B_k(z; z_0, \ldots, z_{k-1})$$

$$= \sum_{j=0}^{m-1} A_j \sum_{k=0}^{j} z_k^{j-k} B_k(z; z_0, \ldots, z_{k-1}) + \sum_{j=m}^{\infty} A_j \sum_{k=0}^{m-1} z_k^{j-k} B_k(z; z_0, \ldots, z_{k-1})$$

$$= \sum_{j=0}^{m-1} A_j z^j + \sum_{j=m}^{\infty} A_j [z^j - B_j(z; z_0, \ldots, z_{m-1}, 0, \ldots, 0)]$$

$$= \sum_{j=0}^{\infty} A_j z^j - \sum_{j=m}^{\infty} A_j B_j(z; z_0, \ldots, z_{m-1}, 0, \ldots, 0).$$

By transposing, we obtain the important identity

$$f(z) = \sum_{k=0}^{m-1} \mathscr{S}^k f(z_k) B_k(z; z_0, \ldots, z_{k-1}) + \sum_{k=m}^{\infty} A_k B_k(z; z_0, \ldots, z_{m-1}, 0, \ldots, 0). \tag{4.1}$$

Without loss of generality, we may assume that each of $\mathscr{S}^k f(z)$, $0 \leqq k \leqq m-1$, has a zero in $|z| < 1$. For $0 \leqq k \leqq m-1$, let w_k denote the smallest modulus of a zero of $\mathscr{S}^k f(z)$. It follows from (4.1) that

$$1 = f(0) \leqq \sum_{k=m}^{\infty} |B_k(0; w_0, \ldots, w_{m-1}, 0, \ldots, 0)|.$$

If $\mathscr{U} = \max_{0 \leqq k \leqq m} |w_k|$, then

$$1 \leqq \sum_{k=m}^{\infty} \mathscr{U}^k |B_k(0; w_0/\mathscr{U}, \ldots, w_{m-1}/\mathscr{U}, 0, \ldots, 0)| \leqq T_m(\mathscr{U})$$

and therefore $\mathscr{U} \geqq \mathscr{U}_m$. Thus there is an integer p, $0 \leqq p \leqq m-1$, such that $|w_p| \geqq \mathscr{U}_m$ and it follows that $\mathscr{S}^p f(z)$ has no zero in $|z| < \mathscr{U}_m$.

LEMMA 5. *Let m be a positive integer and $a_0 + a_1 z + \cdots + a_n z^n$ a polynomial of degree n, $n \geqq m-1$, such that $|a_k| \leqq |a_n|$, $0 \leqq k \leqq n$. Then for at least one integer p, $n-m+1 \leqq p \leqq n$, the polynomial $a_0 + a_1 z + \cdots + a_p z^p$ has all its zeros in the disc $|z| \leqq \mathscr{U}_m^{-1}$.*

Proof. Let $A_k = a_{n-k}/a_n$, $0 \leqq k \leqq n$. Lemma 4 implies that there exists an integer q, $0 \leqq q \leqq m-1$, such that $A_q + A_{q+1} z + \cdots + A_n z^{n-q}$ does not vanish in $|z| < \mathscr{U}_m$. Therefore, the function $(a_{n-q}/a_n) + (a_{n-q-1}/a_n) z + \cdots + (a_0/a_n) z^{n-q}$ has no zero in $|z| < \mathscr{U}_m$, so the same is true of $(z^{n-q}/a_n)(a_0 + a_1/z + \cdots + a_{n-q}/z^{n-q})$. It follows that $(1/a_n z^{n-q})(a_0 + a_1 z + \cdots + a_{n-q} z^{n-q})$ has no zero in the region $|z| > \mathscr{U}_m^{-1}$, hence $a_0 + a_1 z + \cdots + a_{n-q} z^{n-q}$ has all its zeros in $|z| \leqq \mathscr{U}_m^{-1}$. Taking $p = n-q$, we obtain the desired result.

LEMMA 6. *Suppose $f(z) = \sum_{k=0}^{\infty} A_k z^k$ has R-type greater than 1. Then*

$$\liminf_{n \to \infty} \frac{s_n(f)}{R_n} \leqq P.$$

Proof. If $f(z)$ is written

$$f(z) = \sum_{k=0}^{\infty} (a_k/R_1R_2\cdots R_k)z^k,$$

then $\tau_R(f) = \limsup_{n\to\infty} |a_n|^{1/n}$. The condition $\tau_R(f) > 1$ implies that there exists an infinite set N of positive integers such that $n \in N$ implies $|a_n| > |a_k|$, $0 \leqq k < n$. Let m be a positive integer and suppose $n \in N$ is such that $n \geqq m-1$. The nth partial sum of $f(R_nz)$ is given by

$$\begin{aligned} S_n(f; R_nz) &= a_0 + \frac{a_1R_n}{R_1}z + \frac{a_2R_n^2}{R_1R_2}z^2 + \cdots + \frac{a_nR_n^n}{R_1R_2\cdots R_n}z^n \\ &= \frac{a_nR_n^n}{R_1R_2\cdots R_n}\left(z^n + \frac{a_{n-1}R_n}{a_nR_n}z^{n-1} + \frac{a_{n-2}R_{n-1}R_n}{a_nR_n^2}z^{n-2} + \cdots + \frac{a_0R_1R_2\cdots R_n}{a_nR_n^n}\right). \end{aligned}$$

For $n \in N$ and $n \geqq m-1$, Lemma 5, applied to the polynomial

$$z^n + \frac{a_{n-1}R_n}{a_nR_n}z^{n-1} + \frac{a_{n-2}R_{n-1}R_n}{a_nR_n^2}z^{n-2} + \cdots + \frac{a_0R_1R_2\cdots R_n}{a_nR_n^n},$$

implies that at least one of the partial sums $S_n(f; R_nz)$, $S_{n-1}(f; R_nz), \ldots,$ $S_{n-m+1}(f; R_nz)$ has all its zeros in the disc $|z| \leqq \mathscr{U}_m^{-1}$. In view of $s_k(f(R_nz)) = R_n^{-1}s_k(f)$, for $n-m+1 \leqq k \leqq n$, it follows that

$$\min\{s_n(f)/R_n, s_{n-1}(f)/R_n, \ldots, s_{n-m+1}(f)/R_n\} \leqq \mathscr{U}_m^{-1} \tag{4.2}$$

for all $n \in N$, $n \geqq m-1$. If $n-k(n)$ denotes the subscript for which the minimum in (4.2) is assumed, then

$$(s_{n-k(n)}(f)/R_{n-k(n)})(R_{n-m+1}/R_n) \leqq \mathscr{U}_m^{-1} \tag{4.3}$$

for $n \in N$, $n \geqq m-1$. Since $\lim_{n\to\infty}(R_{n-m+1}/R_n) = 1$, then (4.3) implies

$$\liminf_{j\to\infty} \frac{s_j(f)}{R_j} \leqq \mathscr{U}_m^{-1}.$$

Since m is arbitrary, (3.1) implies $\liminf_{j\to\infty} s_j(f)/R_j \leqq P$, which is the desired result.

For a power series $f(z) = 1 + \sum_{k=1}^{\infty} a_kz^k$, the estimate

$$s_n(f) \geqq |a_n|^{-1/n} \qquad (a_n \neq 0) \tag{4.4}$$

follows from the fact that the geometric mean of the moduli of the zeros of $S_n(f; z)$ does not exceed the maximum modulus of its zeros. The following lemma, whose proof we omit, is an extension of (4.4).

LEMMA 7. *Suppose the power series $f(z) = \sum_{k=0}^{\infty} a_kz^k$ has positive radius of convergence and is not a polynomial. If $N = \{n : a_n \neq 0\}$, then*

$$\liminf_{n\to\infty;\, n\in N} |a_n|^{1/n}s_n(f) \geqq 1. \tag{4.5}$$

We are now ready to prove (1.5) of Theorem C.

THEOREM 1. *If* $0<\tau_R(f)<\infty$, *then*

$$\liminf_{n\to\infty}\frac{(R_1R_2\cdots R_n)^{1/n}}{R_n} \leqq \tau_R(f)\liminf_{n\to\infty}\frac{s_n(f)}{R_n} \leqq P. \tag{4.6}$$

Proof. If $f(z)=\sum_{k=0}^{\infty} A_kz^k=\sum_{k=0}^{\infty}(a_k/R_1R_2\cdots R_k)z^k$, then

$$\begin{aligned}\tau_R(f) &= \limsup_{n\to\infty}|a_n|^{1/n} = \limsup_{n\to\infty}|A_n|^{1/n}(R_1\cdots R_n)^{1/n}\\ &\geqq R_1\limsup_{n\to\infty}|A_n|^{1/n} = R_1/c(f)\end{aligned}$$

and therefore $c(f)>0$. Since $\tau_R(f)>0$, f is not a polynomial. By Lemma 7,

$$\liminf_{n\to\infty;\, n\in N}|A_n|^{1/n}s_n(f) \geqq 1,$$

where $N=\{n : A_n\neq 0\}$. Therefore,

$$\begin{aligned}\liminf_{n\to\infty}\frac{(R_1R_2\cdots R_n)^{1/n}}{R_n} &\leqq \liminf_{n\to\infty;\, n\in N}\frac{(R_1R_2\cdots R_n)^{1/n}}{R_n}\liminf_{n\to\infty;\, n\in N}|A_n|^{1/n}s_n(f)\\ &\leqq \liminf_{n\to\infty;\, n\in N}\frac{(R_1R_2\cdots R_n)^{1/n}}{R_n}|A_n|^{1/n}s_n(f)\\ &\leqq \limsup_{n\to\infty}(R_1R_2\cdots R_n)^{1/n}|A_n|^{1/n}\liminf_{n\to\infty;\, n\in N}\frac{s_n(f)}{R_n}\\ &= \tau_R(f)\liminf_{n\to\infty}\frac{s_n(f)}{R_n},\end{aligned}$$

which is the left side of (4.6). For the right side of (4.6), suppose $\tau_R(f)=1$, let $\alpha>1$ and define $f_1(z)=f(\alpha z)$. Then $\tau_R(f_1)=\alpha$ and Lemma 6 implies $\liminf_{n\to\infty} s_n(f_1)/R_n \leqq P$. Since $s_n(f_1)=\alpha^{-1}s_n(f)$, we have $\liminf_{n\to\infty} s_n(f)/R_n\leqq P\alpha$. Letting $\alpha\to 1$, we obtain $\liminf_{n\to\infty} s_n(f)/R_n\leqq P$. Now suppose $\tau_R(f)=t$ and define $g(z)=f(z/t)$. Since $\tau_R(g)=1$, the previous inequality implies $\liminf_{n\to\infty} s_n(g)/R_n\leqq P$. But $s_n(g)=ts_n(f)$ and therefore

$$\tau_R(f)\liminf_{n\to\infty}\frac{s_n(f)}{R_n} \leqq P.$$

This completes the proof of the theorem and establishes (1.5).

For the proof of (1.6), we require the following lemma.

LEMMA 8. *If* $0<\tau_R(f)<1$, *then*

$$\limsup_{n\to\infty}\frac{r_n(f)}{R_n} \geqq 1/P. \tag{4.7}$$

Proof. Let $f(z)=\sum_{k=0}^{\infty} A_kz^k=\sum_{k=0}^{\infty}(a_k/R_1R_2\cdots R_k)z^k$. Since $\tau_R(f)=\limsup|a_n|^{1/n}$ and $0<\tau_R(f)<1$, then there is an infinite set N of positive integers such that $n\in N$ implies $|a_n|>|a_k|$ for $k>n$. Let m be a positive integer, let $n\in N$ and suppose k is an integer such that $0\leqq k\leqq m-1$. The expression

$$\frac{R_n^k(R_1\cdots R_n)}{a_n}\mathscr{S}^{n+k}f(R_nz) = \frac{a_{n+k}R_n^k}{a_nR_{n+1}\cdots R_{n+k}}+\frac{a_{n+k+1}R_n^{k+1}}{a_nR_{n+1}\cdots R_{n+k+1}}z+\cdots$$

is the kth normalized remainder of

$$1+\frac{a_{n+1}R_n}{a_nR_{n+1}}z+\frac{a_{n+2}R_n^2}{a_nR_{n+1}R_{n+2}}z^2+\cdots.$$

By Lemma 4, there is an integer $k(n)$, $0\leqq k(n)\leqq m-1$, such that

$$\frac{a_{n+k(n)}R_n^{k(n)}}{a_nR_{n+1}\cdots R_{n+k(n)}}+\frac{a_{n+k(n)+1}R_n^{k(n)+1}}{a_nR_{n+1}\cdots R_{n+k(n)+1}}z+\cdots$$

does not vanish in $|z|\leqq\mathscr{U}_m$. Therefore $\mathscr{S}^{n+k(n)}f(R_nz)$ has no zero in $|z|\leqq\mathscr{U}_m$, so that $r_{n+k(n)}(f)/R_n\geqq\mathscr{U}_m$ for all $n\in N$. It follows that $(r_{n+k(n)}(f)/R_{n+k(n)})(R_{n+m-1}/R_n)\geqq\mathscr{U}_m$ and, therefore, $\limsup_{j\to\infty} r_j(f)/R_j\geqq\mathscr{U}_m$. By (3.1), $\limsup_{j\to\infty} r_j(f)/R_j\geqq 1/P$, and this completes the proof.

The proof of (1.6) of Theorem C is contained in the following theorem.

THEOREM 2. *If* $\tau_R(f)>0$, *then*

$$\tau_R(f)\limsup_{n\to\infty}\frac{r_n(f)}{R_n}\geqq 1/P. \tag{4.8}$$

Proof. Suppose first that $\tau_R(f)=1$, let $0<\alpha<1$, and define $f_1(z)=f(\alpha z)$. Then $r_n(f_1)=\alpha^{-1}r_n(f)$ and $\tau_R(f_1)=\alpha$. By Lemma 8, $\limsup_{n\to\infty} r_n(f_1)/R_n\geqq 1/P$. Thus $\limsup_{n\to\infty} r_n(f)/R_n\geqq\alpha/P$ and, letting $\alpha\to 1$, we have $\limsup_{n\to\infty} r_n(f)/R_n\geqq 1/P$.

Now suppose $\tau_R(f)=t$. If $t=\infty$, there is nothing to prove. For finite t, define $g(z)=f(z/t)$. Then $\tau_R(g)=1$ and $r_n(g)=tr_n(f)$. By the previous inequality, $\tau_R(f)\limsup_{n\to\infty} r_n(f)/R_n\geqq 1/P$, which is the desired result.

5. **Extremal functions.** In this section, we construct extremal functions which show that P is the sharp constant in each of the three inequalities of Theorem C.

THEOREM 3. *There is a function f of R-type* 1 *such that* $\liminf_{n\to\infty} s_n(f)/R_n=P$.

Proof. Let $F(z)=\sum_{k=0}^\infty A_kz^k$ be the function constructed in Lemma 2. Recall that $c(F)=1$, $s_n(F)\geqq P$, $|A_n|\leqq 1$ and $\max\{|A_n|, |A_{n+1}|,\ldots,|A_{n+6}|\}\geqq 1/7000$ for all n. Let

$$f(z)=\sum_{k=0}^\infty (A_k/R_1R_2\cdots R_k)z^k \qquad (R_0=1)$$

and

$$x=\liminf_{n\to\infty}\frac{s_n(f)}{R_n}.$$

Let A denote an infinite set of positive integers such that $x=\lim_{n\to\infty;\,n\in A} s_n(f)/R_n$. For $n\in A$, define

$$P_n(z)=z^nS_n(f;R_n/z)(R_1R_2\cdots R_n)/R_n^n$$

and

$$Q_n(z)=z^nS_n(F;1/z)=\sum_{k=0}^n A_{n-k}z^k.$$

The bound

$$|P_n(z)-Q_n(z)| \leqq \sum_{k=1}^{n} |z|^k(1-(R_nR_{n-1}\cdots R_{n-k+1})/R_n^k) \leqq (1-|z|)^{-1}$$

holds for all $n \in A$ and $|z|<1$. Thus there is an infinite set of integers $B \subset A$ such that the sequence $\{P_n-Q_n\}_{n\in B}$ converges uniformly on compact subsets of $|z|<1$ to a function $g(z)=\sum_{k=0}^{\infty} \alpha_k z^k$ analytic in the unit disc. Since

$$\alpha_m = \lim_{n\to\infty;\, n\in B} A_{n-m}(1-(R_nR_{n-1}\cdots R_{n-m+1})/R_n^m) = 0,$$

for $m=1, 2, 3, \ldots$, and $\alpha_0=0$, then $g\equiv 0$. For $n \in B$, we also have the bound $|Q_n(z)|<(1-|z|)^{-1}$, $|z|<1$. Thus there is an infinite subset $C\subset B$ such that $\{Q_n\}_{n\in C}$ converges uniformly on compact subsets of $|z|<1$ to a function $Q(z)=\sum_{k=0}^{\infty} \beta_k z^k$ analytic in the unit disc. The bound $\max\{|\beta_k|, |\beta_{k+1}|, \ldots, |\beta_{k+6}|\}>1/7000$ holds for the coefficients of Q; in particular, Q is not identically zero. The sequence $\{P_n(1/z)\}_{n\in C}$ converges uniformly to $Q(1/z)$ in $|z|\geqq 1/\rho$ for all $\rho<1$. Moreover, $Q_n(1/z)=(1/z^n)S_n(F; z)$ has a zero of modulus P for all $n \in C$. Thus $Q(1/z)$ has a zero of modulus P. If $\varepsilon>0$, it follows from Hurwitz's Theorem that $P_n(1/z)$ has a zero of modulus at least $P-\varepsilon$ for $n \in C$ sufficiently large, i.e., if Γ_n denotes the maximum modulus of the zeros of $P_n(1/z)$, then $\Gamma_n \geqq P-\varepsilon$ for $n \in C$ sufficiently large. Since $\Gamma_n=R_n^{-1}s_n(f)$, then $s_n(f)/R_n \geqq P-\varepsilon$ for large $n \in C$. Therefore

$$x = \lim_{n\to\infty;\, n\in C} \frac{s_n(f)}{R_n} \geqq P-\varepsilon;$$

letting $\varepsilon \to 0$, we obtain the desired result.

THEOREM 4. *There is a function g of R-type* 1 *such that* $\limsup r_n(g)/R_n=1/P$.

Proof. Let $G(z)=\sum_{k=0}^{\infty} A_k z^k$ denote the function constructed in Lemma 3. We have $c(G)=1$, $|A_n|\leqq 1$ and $\max\{|A_n|, |A_{n+1}|, \ldots, |A_{n+6}|\}\geqq 1/7000$ for all n. Let

$$g(z) = \sum_{k=0}^{\infty} (A_k/R_1R_2\cdots R_k)z^k \qquad (R_0 = 1),$$

and

$$x = \limsup_{n\to\infty} \frac{r_n(g)}{R_n}.$$

Let A denote an infinite set of positive integers for which $x=\lim_{n\to\infty;\, n\in A} r_n(g)/R_n$. For $m \in A$, define

$$E_m(z) = \mathscr{S}^m G(z)-(R_1R_2\cdots R_m/R_m^m)\mathscr{S}^m g(R_m z)$$

and let $0<\alpha<1$. If $m \in A$, $|z|\leqq \alpha$ and N is a positive integer, then

$$\begin{aligned}|E_m(z)| &\leqq \sum_{k=1}^{\infty} |A_{m+k}|(1-R_m^k/(R_{m+1}\cdots R_{m+k}))|z|^k \\ &\leqq \sum_{k=1}^{N} |z|^k(1-R_m^k/(R_{m+1}\cdots R_{m+k}))+\sum_{k=N+1}^{\infty} \alpha^k \\ &\leqq (1-R_m^N/(R_{m+1}\cdots R_{m+N}))(1-\alpha)^{-1}+\alpha^{N+1}(1-\alpha)^{-1}.\end{aligned}$$

Let $\varepsilon>0$ and choose N so that $\alpha^{N+1}(1-\alpha)^{-1}<\varepsilon/2$. Let $m_0 \in A$ be a positive integer such that $m \geqq m_0$ implies $(1-R_m^N/(R_{m+1}\cdots R_{m+N}))(1-\alpha)^{-1}<\varepsilon/2$. Then the conditions $m \geqq m_0$ and $|z| \leqq \alpha$ imply $|E_m(z)|<\varepsilon$. Thus $\{E_m\}_{m\in A}$ converges uniformly to zero on compact subsets of $|z|<1$. For $m \in A$, we also have $|\mathscr{S}^m G(Z)| \leqq (1-|z|)^{-1}$. Thus there is an infinite subset $B \subset A$ of integers such that $\{\mathscr{S}^m G(Z)\}_{m\in B}$ converges uniformly on compact subsets of $|z|<1$ to a function $S(z)=\sum_{k=0}^{\infty} b_k z^k$. The relation

$$|b_k|+|b_{k+1}|+\cdots+|b_{k+6}| \geqq 1/1000$$

holds for all k; in particular $S \not\equiv 0$. Since $\mathscr{S}^m G(z)$ has a zero of modulus $1/P$ for all $m \in B$, then $S(z)$ has a zero of modulus $1/P$. Moreover, $S(z)$ is the uniform limit of the sequence $\{(R_1R_2\cdots R_m/R_m^m)\mathscr{S}^m g(R_m z)\}_{m\in B}$ and it follows from Hurwitz's Theorem that, if $\varepsilon>0$, then $\mathscr{S}^m g(R_m z)$ has a zero of modulus at most $(1/P)+\varepsilon$ for $m \in B$ sufficiently large. Therefore $r_m(g)/R_m \leqq (1/P)+\varepsilon$ for large $m \in B$, and it follows that

$$x = \lim_{m\to\infty;\, m\in B} \frac{r_m(g)}{R_m} \leqq (1/P)+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we obtain $\limsup_{n\to\infty} r_n(g)/R_n \leqq 1/P$ and this completes the proof.

For the left-hand side of (1.5), we begin by considering the infinite matrix (a_{mn}), where

$$\begin{aligned} a_{mn} &= 2(m-n+1)/m^2, && 1 \leqq n \leqq m,\\ &= 0, && m < n. \end{aligned}$$

It is easily verified that

(1) $\lim_{m\to\infty} a_{mn}=0$, $n=1, 2, 3, \ldots$,

(2) $\sup_m \sum_{n=1}^{\infty} |a_{mn}|=2$,

(3) $\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}=1$.

Thus (a_{mn}) provides a regular method of summability. If $\{R_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of positive numbers $(R_0=1)$ such that $R_{n+1}/R_n \to 1$, then (a_{mn}) transforms the sequence $\{\log(R_n/R_{n-1})\}_{n=1}^{\infty}$ into $\{2\log(R_1R_2\cdots R_n)^{1/n^2}\}_{n=1}^{\infty}$. Therefore

$$\lim_{n\to\infty} [2\log(R_1R_2\cdots R_n)^{1/n^2}] = \lim_{n\to\infty} [\log(R_n/R_{n-1})] = 0,$$

or

$$\lim_{n\to\infty} (R_1R_2\cdots R_n)^{1/n^2} = 1. \tag{5.1}$$

We use this result to prove the following lemma.

Lemma 9. *Let $\{R_n\}_{n=1}^{\infty}$ $(R_0=1)$ be a nondecreasing sequence of positive numbers such that $(R_1R_2\cdots R_n)^{1/n}\to\infty$ and $R_{n+1}/R_n \to 1$, as $n\to\infty$. For each pair of positive integers m and p, let x_{mp} be the largest root of the equation*

$$\frac{x^{m+p}}{R_1\cdots R_{m+p}} = \frac{x^m}{R_1\cdots R_m}+\frac{x^{m-1}}{R_1\cdots R_{m-1}}+\cdots+\frac{x}{R_1}+1. \tag{5.2}$$

Then

$$\lim_{p\to\infty} \frac{x_{mp}}{(R_1\cdots R_{m+p})^{1/(m+p)}} = 1$$

for $m=1, 2, 3, \ldots$.

Proof. For all m and p we have $x_{mp} \geqq (R_1\cdots R_{m+p})^{1/(m+p)}$, and therefore $x_{mp}\to\infty$ as $p\to\infty$, $m=1,2,3,\ldots$. Let m be a positive integer and choose p so large that $x_{mp}^m/(R_1\cdots R_m) \geqq x_{mp}^{m-k}/(R_1\cdots R_{m-k})$, for $0\leqq k\leqq m$. For such integers p we have $x_{mp}^{m+p}/(R_1\cdots R_{m+p}) \leqq (m+1)x_{mp}^m/(R_1\cdots R_m)$ and hence

$$x_{mp} \leqq (m+1)^{1/p}(R_{m+1}\cdots R_{m+p})^{1/p}.$$

Thus

$$1 \leqq \frac{x_{mp}}{(R_1\cdots R_{m+p})^{1/(m+p)}} \leqq \frac{(m+1)^{1/p}}{(R_1\cdots R_m)^{1/(m+p)}} (R_{m+1}\cdots R_{m+p})^{(1/p)-(1/(m+p))}.$$

Since each of $(m+1)^{1/p}$ and $(R_1\cdots R_m)^{1/(m+p)}$ tends to 1 as $p\to\infty$, it is sufficient to show that

$$(R_{m+1}\cdots R_{m+p})^{(1/p)-(1/(m+p))} = \frac{(R_1\cdots R_{m+p})^{m/p(m+p)}}{(R_1\cdots R_m)^{m/p(m+p)}} \to 1, \qquad p\to\infty.$$

Since $(R_1\cdots R_m)^{m/p(m+p)} \to 1$, it is sufficient to show that $(R_1\cdots R_{m+p})^{1/p(m+p)} \to 1$. Now

$$(R_1\cdots R_{m+p})^{1/p(m+p)} = (R_1\cdots R_{m+p})^{1/(m+p)^2}[(R_1\cdots R_{m+p})^{1/(m+p)^2}]^{m/p},$$

and we know that $(R_1\cdots R_{m+p})^{1/(m+p)^2} \to 1, p\to\infty$. Thus $(R_1\cdots R_{m+p})^{1/p(m+p)} \to 1$, and this completes the proof.

THEOREM 5. *There is a function φ of R-type* 1 *such that*

$$\liminf_{n\to\infty} \frac{(R_1R_2\cdots R_n)^{1/n}}{R_n} = \liminf_{n\to\infty} \frac{s_n(\varphi)}{R_n}.$$

Proof. Let $\{R_n\}_{n=1}^\infty$ $(R_0=1)$ and $\{x_{mp}\}_{m,p=1}^\infty$ be defined as in Lemma 9. Let $\{n_k\}_{k=1}^\infty$ denote a sequence of positive integers such that

$$\liminf_{n\to\infty} \frac{(R_1R_2\cdots R_n)^{1/n}}{R_n} = \lim_{k\to\infty} \frac{(R_1\cdots R_{n_k})^{1/n_k}}{R_{n_k}}.$$

Let $m_1=n_1$, choose an integer p_1 such that $m_1+p_1 \in \{n_j\}$ and

$$\frac{x_{m_1p_1}}{(R_1\cdots R_{m_1+p_1})^{1/(m_1+p_1)}} < 1+\frac{1}{2},$$

and let $m_2=m_1+p_1$. If $m_k=m_{k-1}+p_{k-1}\in\{n_j\}$ has been chosen, choose the integer p_k such that $m_k+p_k\in\{n_j\}$ and

(5.3) $$x_{m_kp_k}/(R_1\cdots R_{m_k+p_k})^{1/(m_k+p_k)} < 1+1/(k+1)$$

and let $m_{k+1}=m_k+p_k$. Thus we inductively obtain the sequence $\{m_j\}\subset\{n_j\}$ such that (5.3) holds for $k=1, 2, 3, \ldots$. Now let

$$\varphi(z) = 1+z^{m_1}/(R_1\cdots R_{m_1})+z^{m_2}/(R_1\cdots R_{m_2})+\cdots.$$

Note that

$$\begin{aligned}|S_{m_j}(\varphi; z)| &\geqq \frac{|z|^{m_j}}{R_1\cdots R_{m_j}}-\frac{|z|^{m_{j-1}}}{R_1\cdots R_{m_{j-1}}}-\cdots-\frac{|z|^{m_1}}{R_1\cdots R_{m_1}}-1\\ &> \frac{|z|^{m_1}}{R_1\cdots R_{m_1}}-\sum_{k=0}^{m_j-1}\frac{|z|^k}{R_1\cdots R_k}\end{aligned}$$

for $j=1, 2, 3, \ldots$. Moreover, if $x>x_{mp}$, then

$$\frac{x^{m+p}}{R_1\cdots R_{m+p}} > \frac{x^m}{R_1\cdots R_m}+\cdots+\frac{x}{R_1}+1,$$

since x_{mp} is the largest positive root of (5.2). Thus if $|z|=x>x_{m_{j-1},p_{j-1}}$, then $|S_{m_j}(\varphi; z)|>0$. Therefore $s_{m_j}(\varphi)\leqq x_{m_{j-1}p_{j-1}}$. From (5.3) we have

$$s_{m_j}(\varphi)/(R_1\cdots R_{m_j})^{1/m_j} \leqq 1+1/j \quad \text{for } j = 1, 2, 3, \ldots.$$

Since $s_n(\varphi)=\infty$ for integers $n\notin\{m_j\}$, then

$$\begin{aligned}\liminf_{n\to\infty}\frac{s_n(\varphi)}{R_n} &= \liminf_{j\to\infty}\frac{s_{m_j}(\varphi)}{R_{m_j}}\\ &\leqq \liminf_{j\to\infty}\left[\frac{(R_1\cdots R_{m_j})^{1/m_j}}{R_{m_j}}\left(1+\frac{1}{j}\right)\right]\\ &= \lim_{k\to\infty}\frac{(R_1\cdots R_{n_k})^{1/n_k}}{R_{n_k}} = \liminf_{n\to\infty}\frac{(R_1\cdots R_n)^{1/n}}{R_n}.\end{aligned}$$

and this completes the proof.

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