HYPERBOLIC LIMIT SETS(*)

BY
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Abstract. Many known results for diffeomorphisms satisfying Axiom A are shown to be true with weaker assumptions. It is proved that if the negative limit set $L^-(f)$ of a diffeomorphism $f$ is hyperbolic, then the periodic points of $f$ are dense in $L^-(f)$. A spectral decomposition theorem and a filtration theorem for such diffeomorphisms are obtained and used to prove that if $L^-(f)$ is hyperbolic and has no cycles, then $f$ satisfies Axiom A, and hence is $\Omega$-stable. Examples are given where $L^-(f)$ is hyperbolic, there are cycles, and $f$ fails to satisfy Axiom A.

1. In [10] and [11], Smale obtained results for a diffeomorphism $f$ of a compact manifold $M$ satisfying Axiom A. Axiom A requires (a) the nonwandering set $\Omega = \Omega(f)$ has a hyperbolic structure, and (b) the periodic points of $f$ are dense in $\Omega(f)$. The purpose of this paper is to point out that many of Smale's results may be obtained under weaker hypotheses. One consequence of our observations is that most of the known results for diffeomorphisms satisfying Axiom A are true for those satisfying Axiom A(a) alone.

Our main result is the following. Let $L^-(f)$ be the closure of the set of $\alpha$-limit points of $f$. Then,

**Theorem 4.5.** If $L^-(f)$ is hyperbolic and $f$ has no cycles, then $f$ satisfies Axiom A (and has no cycles).

This gives strengthening of Smale's $\Omega$-stability theorem in two directions. On the one hand, it is not necessary to assume Axiom A(b), and on the other hand, it is not necessary to assume the whole nonwandering set is hyperbolic.

For the theorem to hold, a natural change in the usual definition of cycle is needed (see the definitions preceding (3.9) and Examples 1 and 4 at the end of §3). Our definition reduces to the usual one when $f$ satisfies Axiom A(a).

The basic idea of the proof of Theorem (4.5) is as follows. First we prove that $L^-(f)$ hyperbolic implies that the periodic points of $f$ are dense in $L^-(f)$ and that there is a spectral decomposition theorem. Then, using methods similar to the proof of Smale's $\Omega$-stability theorem, we obtain a filtration theorem for $f$. In the case of no cycles, this filtration separates the pieces in the spectral decomposition of $L^-(f)$ from which it follows that $L^-(f)$ is the whole nonwandering set.

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Another sufficient condition for $\Omega$-stability is contained in Theorem (4.7).

Although we deal here only with diffeomorphisms, the corresponding results for flows may be obtained by combining slight modifications of the methods used here with the techniques of [7].

At the beginning of §2 we collect several notations and definitions which will be used throughout the paper. Then we prove a spectral decomposition theorem for diffeomorphisms such that the closure of the periodic points is hyperbolic.

In §3 we obtain results when $L^-(f)$ is hyperbolic. In particular, we prove that $L^-(f)$ hyperbolic implies the periodic points of $f$ are dense in $L^-(f)$. We also obtain a filtration theorem and present some examples which show that our results are true generalizations of Smale’s results.

The main results of §4 are Theorems (4.5) and (4.7).

I wish to thank J. Palis and R. C. Robinson for some helpful comments and suggestions.

2. Here we obtain some results about periodic points. But we first establish some notation which will be used throughout the paper.

Throughout it is assumed that $f$ is a $C^r$ diffeomorphism, $0 < r < \infty$, of a compact $C^\infty$ manifold without boundary. Let $P = P(f)$ be the set of hyperbolic periodic points of $f$ and assume $P \neq \emptyset$. Let $\Omega = \Omega(f)$ denote the nonwandering set of $f$. For a subset $D \subseteq M$, $\overline{D}$ or $\text{Cl}(D)$ will denote its closure in $M$, and $\text{int}D$ will denote its interior in $M$.

For $x \in M$, define $\alpha(x) = \alpha(x, f) = \{y \in M : \text{there is a sequence of integers } n_i \to \infty \text{ such that } f^{-n_i}(x) \to y \text{ as } i \to \infty\}$. Let $\omega(x) = \omega(x, f) = \alpha(x, f^{-1})$. $\omega(x) = \{f^n(x) : -\infty < n < \infty\}$, $L^-(f) = \{x \in M : \exists y \in M \text{ such that } x \in \alpha(y)\}$, and $L^+ = L^+(f) = \{x \in M : \exists y \in M \text{ such that } y \in \alpha(x)\}$. Also, set $L^- = L^-(f^p)$, $L^+ = L^+(f^p)$, and $L = L^- \cup L^+$. $L^p_\omega$, $L^\omega$, $L_\omega$, $L^\omega$, $L^\omega$, and $L$ are called, respectively, the $\omega$-limit set of $f$, negative limit set of $f$, positive limit set of $f$, and limit set of $f$. $\alpha(x)$ is called the orbit of $x$.

A compact $f$-invariant set $\Lambda$ is hyperbolic if there are a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$ preserved by the derivative $Tf$ of $f$, a riemannian metric $| \cdot |$ on $M$, and a constant $0 < \lambda < 1$ such that $|Tf(v)| \leq \lambda |v|$ for $v \in E^s$ and $|Tf(v)| \geq \lambda^{-1} |v|$ for $v \in E^u$. A metric $| \cdot |$ such as that referred to in the preceding sentence is said to be adapted to $\Lambda$.

Let $\Lambda$ be a hyperbolic set, and $| \cdot |$ be an adapted metric. Let $d$ be the topological metric on $M$ induced by $| \cdot |$. For $x \in \Lambda$, $\varepsilon > 0$, let

$$W^s_\varepsilon(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon \text{ for } n \geq 0\},$$

$$W^u_\varepsilon(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\},$$

$$W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \leq \varepsilon \text{ for } n \geq 0\},$$

$$W^u(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}.$$
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(see [2]). For a subset $D \subset \Lambda$, let $W^u_x(D) = \bigcup_{x \in D} W^u_x(x)$, and make similar definitions for $W^s_x(D)$, $W^u_x(D)$, and $W^s(D)$.

We now proceed to establish some facts about the set $P$ of hyperbolic periodic points of the diffeomorphism $f$. Recall we have assumed $P \neq \emptyset$.

Let $p_1, p_2 \in P$. We will say that $p_1$ is homoclinically related to $p_2$ or $h$-related to $p_2$, denoted $p_1 \sim p_2$, if $W^u(o(p_1))$ has a point of transversal intersection with $W^s(o(p_2))$ and $W^s(o(p_2))$ has a point of transversal intersection with $W^u(o(p_1))$.

(2.1) **Proposition.** The relation $\sim$ on $P$ is an equivalence relation.

**Proof.** Reflexivity and symmetry are obvious. For transitivity, suppose $p_1 \sim p_2$ and $p_2 \sim p_3$. Let $x_2$ be a point of transversal intersection of $W^u(o(p_2))$ and $W^s(o(p_3))$, $p_i \in P$, $i=1, 2, 3$. Since $W^u(o(p_2))$ is a finite union of injectively immersed cells, there are a point $p'_2 \in o(p_2)$ and a closed disk $D^3 \subset W^u(o(p_2))$ such that $p'_2$ and $x_2$ are in $D^3$. Since $W^u(o(p_1))$ has a point of transversal intersection with $W^s(o(p_2))$, by the Palis $\lambda$-lemma [4], $W^u(o(p_1))$ contains disks $D^i$ arbitrarily $C^1$ close to $D^3$. Thus we can choose such a $D^i$ which will have a point of transversal intersection with $W^u(o(p_3))$ near $x_2$. So $W^u(o(p_1))$ has a point of transversal intersection with $W^s(o(p_3))$. Similarly, $W^s(o(p_3))$ has a point of transversal intersection with $W^u(o(p_2))$ and so $p_1 \sim p_3$.

We will call the equivalence classes of $P$, $h$-classes. For $p \in P$, the $h$-class of $p$ will be denoted by $P_p$.

The following lemma is essentially due to Birkhoff [1, p. 205].

(2.2) **Lemma.** (1) Let $X$ be a second countable, complete metric space. Let $h: X \to X$ be a continuous map. If for every nonempty open set $V$ in $X$, $\bigcup_{n \geq 0} h^n(V)$ is dense in $X$, then there is an $x_0 \in X$ such that $o(x_0) = o(x_0, h) = X$.

(2) If $h$ is a homeomorphism, and for every nonempty open set $V$, $\bigcup_{n \geq 0} h^n(V)$ and $\bigcup_{n \leq 0} h^n(V)$ are dense, then there is an $x_0 \in X$ such that $o(x_0) = o(x_0, h) = X$.

**Proof.** Since for every open $V$, $\bigcup_{n \geq 0} h^n(V)$ is dense, we have that for every open $V$, $\bigcup_{n \leq 0} h^n(V)$ is a dense open set in $X$. If $\{V_i\}_{i \in I}$ is a countable basis for the topology of $X$, then $\bigcap_{i \in I} (\bigcup_{n \leq 0} h^n(V_i))$ is dense in $X$ by the Baire Category Theorem.

If $x \in \bigcap_{i \in I} (\bigcup_{n \leq 0} h^n(V_i))$, then $\{h^n(x) : n \geq 0\}$ is dense in $X$, so $o(x) = X$. (2.2.2) is proved similarly by choosing

$$x \in \bigcap_{i \in I} \left( \bigcup_{n \geq 0} h^n(V_i) \right) \cap \bigcap_{i \in I} \left( \bigcup_{n \leq 0} h^n(V_i) \right).$$

If $p \in P$, then a point $q$ of transversal intersection of $W^u(o(p))$ and $W^s(o(p))$ such that $q \notin o(p)$ is called a transversal homoclinic point of $p$. Let $H_p$ be the set of all such points and suppose $H_p \neq \emptyset$.

(2.3) **Lemma.** $H_p$ is a closed, $f$-invariant set such that there is an $x \in H_p$ such that $o(x) = o(x) = H_p$.
Proof. That $H_\rho$ is closed and $f$-invariant is obvious. For the last statement we wish to apply Lemma (2.2.2).

To this end, let $V_1, V_2$ be any nonempty open sets in $H_\rho$. Thus there are $q_i \in H_\rho$ and open sets $U_i$ in $M$ such that $q_i \in V_i = U_i \cap H_\rho$, $i=1, 2$. Let $D_1^x$ be a disk in $W^u(o(p))$ of the same dimension as $W^u(o(p))$ which contains a point $p_1 \in o(p)$ and $q_1$ in its interior in $W^u(o(p))$.

Let $D_2^x$ be a small disk in $W^u(q_2) \cap U_2$ of the same dimension as $W^u(q_2)$ which contains $q_2$ in its interior in $W^u(q_2)$.

Since $D_1^x$ meets $W^u(q_2) \subset W^u(o(p))$ transversely, the $\lambda$-lemma says that $\bigcup_{n \geq 0} f^n(D_2^x)$ contains disks arbitrarily $C^1$ close to $D_1^x$. Thus $\bigcup_{n \geq 0} f^n(D_2^x)$ contains points in $H_\rho$ arbitrarily close to $q_1$. Thus, $q_1 \in \bigcup_{n \geq 0} \text{Cl } (f^n(D_2^x) \cap H_\rho) \subset \bigcup_{n \geq 0} \text{Cl } (f^n(U_2) \cap H_\rho) \subset \bigcup_{n \geq 0} \text{Cl } (f^n(U_2 \cap H_\rho)) = \bigcup_{n \geq 0} \text{Cl } (f^n(V_2))$. So

$$V_1 \cap \left( \bigcup_{n \geq 0} f^n(V_2) \right) \neq \emptyset.$$  

Similarly, $V_1 \cap (\bigcup_{n \leq 0} f^n(V_2)) \neq \emptyset$. Since $V_1$ and $V_2$ were arbitrary we may apply (2.2.2) to give (2.3).

(2.4) Theorem. Let $p$ be a hyperbolic periodic point whose $h$-class $P_p$ contains more than one orbit, i.e. there is a point $p_1 \in P_p$, such that $p_1 \notin o(p)$. Then $H_\rho = \overline{P}_p$.

In particular, $H_\rho \neq \emptyset$.

(2.5) Corollary. If $P_p$ is the $h$-class of $p$, then $\overline{P}_p$ is a closed, $f$-invariant set such that there is an $x \in \overline{P}_p$ such that $\alpha(x) = \omega(x) = P_p$.

Proof of (2.4). The proof that $P_p \subset H_\rho$ is very similar to the proofs of (2.1) and (2.3). Let $p_1 \in P_p, p_1 \notin o(p)$. Let $x$ be a point of transversal intersection of $W^u(o(p))$ and $W^s(o(p_1))$ and let $x_1$ be a point of transversal intersection of $W^u(o(p_1))$ and $W^s(o(p))$. Let $D_{x_1}$ be a disk in $W^u(o(p_1))$ containing $x_1$ in its interior in $W^u(o(p_1))$. By the $\lambda$-lemma, there are disks in $W^u(o(p))$ which are arbitrarily $C^1$ close to $D_{x_1}$.

Thus $x_1 \in H_\rho$. Hence $p_1 \in H_\rho$, so $P_p \subset H_\rho$.

The fact that $H_\rho \subset \overline{P}_p$ is a consequence of the following version of Smale's theorem on transversal homoclinic points.

(2.6) Theorem (Smale [9]). Let $p$ be a hyperbolic periodic point of the diffeomorphism $f$, and let $q$ be a transversal homoclinic point of $p$. Then in every neighborhood of $q$ there are infinitely many periodic points which are $h$-related to $p$.

In [9], Smale made use of Sternberg's linearization theorem, and this required eigenvalue assumptions other than hyperbolicity and additional smoothness assumptions. However, he expressed the feeling that Sternberg's theorem was probably not needed for his result. We wish to point out that a proof very close to Smale's original one and avoiding Sternberg's theorem can be given using the
tubular family theorems of [6](2). Taking tubular families for \( W^u(o(p)) \) and \( W^s(o(p)) \), one can get continuous coordinates on a neighborhood of \( o(p) \) on which there is a continuous splitting of the tangent bundle \( T_0 M = E^s \oplus E^u \) such that, for \( x \in U \cap f^{-1}(U) \),

\[
T_x f = \begin{pmatrix} A_x & 0 \\ 0 & D_x \end{pmatrix}
\]

with respect to the splitting \( E^s \oplus E^u \). Also this can be done so that \( \|A_x\| < 1 \) and \( \|D_x^{-1}\| < 1 \) on \( f^{-1}(U) \cap U \). Now one can proceed as Smale did in [9]. However, for the reader's convenience we will give a different and more elementary proof of (2.6) in the appendix at the end of the paper.

We will need the following corollary to the proof of (2.4).

(2.7) Corollary. Let \( P_\ell \) be an \( h \)-class. Suppose \( p_1, p_2 \in P_\ell \) and \( y \) is a point of transversal intersection of \( W^u(p_1) \) and \( W^s(p_2) \). Then \( y \in P_\ell \).

Proof. By the first part of the proof of Theorem (2.4), \( y \in \overline{P_\ell} \). By the second part, \( \overline{P_\ell} = \overline{P_\ell} \), so (2.7) is proved.

(2.8) Proposition. Suppose \( P \) is a hyperbolic set. Then there are only a finite number of \( h \)-classes of \( P \) and their closures are pairwise disjoint.

Proof. If there were infinitely many \( h \)-classes, \( \{P_\ell\} \), let \( p_1, p_2, \ldots \) be a sequence of points such that \( p_i \in P_\ell \) and \( P_\ell \neq P_j \) for \( i \neq j, i, j \geq 1 \).

We may assume, by taking a subsequence if necessary that \( \dim W^u(p_i) = \dim W^s(p_j) \) for all \( i, j \). Let \( x \) be a limit point of \( \{p_i\} \). Then by continuous dependence of the stable and unstable manifolds on \( P \) (see [2]), if \( p_i \) and \( p_j \) are close to \( x \), then \( p_i \sim p_j \). Similarly, if \( p_i \sim p_j \), then \( \overline{P_{p_i}} \cap \overline{P_{p_j}} = \emptyset \) where \( P_{p_i} \) is the \( h \)-class of \( p_i \) and \( P_{p_j} \) is the \( h \)-class of \( p_j \).

The next theorem is the analog of Smale's spectral decomposition theorem [10].

(2.9) Theorem. If \( P \) is hyperbolic, then \( \overline{P} = \Lambda_1 \cup \cdots \cup \Lambda_n \) where the \( \Lambda_\ell \) are the closures of the distinct \( h \)-classes. Thus each \( \Lambda_\ell \) is a closed, invariant, topologically transitive set with periodic points dense. Further, each \( \Lambda_\ell \) has a local product structure, i.e. for \( \varepsilon > 0 \) small, \( W^u_\varepsilon(\Lambda_i) \cap W^s_\varepsilon(\Lambda_i) = \emptyset \) (see [3]).

Proof. All we need prove is the local product structure statement. Thus we need to show if \( x, y \in \Lambda_\ell \) and \( \epsilon \) is small then

\[
W^u_\varepsilon(x) \cap W^s_\varepsilon(y) \subset \Lambda_\ell.
\]

Choose \( \varepsilon \) such that \( W^u_\varepsilon(x) \) is transverse to \( W^s_\varepsilon(y) \) and \( W^u_\varepsilon(x) \cap W^s_\varepsilon(y) \) is at most one point. Then if \( z \in W^u_\varepsilon(x) \cap W^s_\varepsilon(y) \) and \( V \) is a neighborhood of \( z \), there are periodic points \( x_1, y_1 \in \Lambda_\ell \) such that \( W^u_\varepsilon(x_1) \) has a point \( z_1 \) of transversal intersection with \( W^s_\varepsilon(y_1) \) in \( V \). Then \( z_1 \in \overline{P_{x_1}} = \Lambda_\ell \) by Corollary (2.7).

(*) It seems that another proof of (2.6) without Sternberg's theorem appears in [14].
We now proceed to state and prove a technical lemma which will be needed for the proof of Theorem (3.1).

We need some notation. By a disk $D'$ we mean a closed ball in some Euclidean space with the usual metric. If $D'$ is a disk, we let $r(D')$ denote its radius and, for a real number $c > 0$, we let $cD'$ denote the disk whose center is the same as that of $D'$ and whose radius is $cr(D')$. Let $0 < s, u$ be integers and let $D = D^s \times D^u \subset R^{s+u}$ where $D^s$ is a disk in the Euclidean space $R^s$ for $s = s, u$. Assume $r(D^s) = r(D^u)$. Let $D_1 \subset D$ and let $g : D_1 \to R^{s+u}$ be a smooth injection. For $z \in D_1$, suppose $T_2g : R^s \times R^u \to R^s \times R^u$ is given by

$$T_2g = \begin{pmatrix} A_z & B_z \\ C_z & D_z \end{pmatrix}$$

where $A_z : R^s \to R^s$, $B_z : R^u \to R^s$, $C_z : R^s \to R^u$, and $D_z : R^u \to R^u$. Let

$$a = \sup_{z \in D_1} \|A_z\|,$$

$$c = \sup_{z \in D_1} \|C_z\|,$$

$$e = \sup_{z \in D_1} \{|A_z d|/|Sc| : z \in D_1, v \text{ is a unit vector in } R^s\},$$

$$d = \inf_{z \in D_1} \{|D_z v| : z \in D_1, v \text{ is a unit vector in } R^u\}.$$ 

Here the norms $\| \cdot \|$ are the usual matrix norms, and $e$ is assumed to be finite.

(2.10) Lemma. Using the above notation, suppose there is a subdisk $D_1 \subset D^u$ centered at $y_0 \in \frac{1}{2}D^u$ such that if $D_2 = D_1 \times D^u$ then $g : D_1 \to R^{s+u}$ is a smooth injection such that

1. $g(D_1) \subset D,$
2. $g(D^s \times \{y_0\}) \subset D^s \times \frac{1}{2}D^u,$
3. $a < 1$ and $d(1 - ce(1 - a)^{-1})r(D^s) > \frac{1}{2}r(D^u) + r(D^u).$

Then $g$ has a unique fixed point in $D_1$.

Proof. For $z \in D$, let $z = (x, y)$ with $x \in D^s$, $y \in D^u$, and let $\pi^s : (x, y) \mapsto x$, $\pi^u : (x, y) \mapsto y$ denote the natural projections on $D$.

For each $y \in D^u$, the map $\varphi_1 : x \mapsto \pi^s g(x, y)$ takes $D^s$ into $D^s$ by (1). Further, $\|T_x \varphi_1\| \leq a < 1$ for all $x \in D^s$. Thus $\varphi_1$ is a contraction and, hence for each $y \in D^u$, there is a unique $x(y)$ such that $\varphi_1(x(y)) = \pi^s g(x(y), y) = x(y)$.

If $\psi$ is the mapping $(x, y) \mapsto \pi^u g(x, y) - x$, then since $a < 1$, the partial derivative $\partial \psi/\partial x$ has rank $s$ on $D_1$, so the implicit function theorem gives that the mapping $y \mapsto x(y)$ is smooth.

Consider the mapping $\varphi_2 : y \mapsto \pi^u g(x(y), y)$ on $D^u$. We claim $\varphi_2(D^u) \supset D^u$ and $\varphi_2$ is a uniform expansion on $D^u$. Once this is shown, it follows that $\varphi_2$ has a unique fixed point $y_1$ and hence $(x(y_1), y_1)$ is the unique fixed point of $g$ in $D_1$.

So we first show $\varphi_2$ is an expansion on $D^u$. Let $v$ be a unit vector in $R^u$. Then

$$|T_2 \varphi_2(v)| = |Cx(y)v + Dv|$$
where \( T_{x(y),y}g = (\frac{d}{d} \xi \ 0) \) and \( x'(y) \) is the derivative of \( y \mapsto x(y) \) at \( y \). Further, \( x'(y) = -(A-I)^{-1}B \) where \( I \) is the \( s \times s \) identity matrix. Thus,
\[
|T_y \phi_2(v)| \geq |Dv| - (c(1-a)^{-1})|Bv| \\
\geq |Dv|(1-ce(1-a)^{-1}) \geq d(1-ce(1-a)^{-1}) > 1
\]
where the last inequality follows from (3). So,
\[
|T_y \phi_2(v)| \geq d(1-ce(1-a)^{-1}) > 1
\]
which shows \( \phi_2 \) is an expansion.

It remains to show \( \phi_2(D^i) \supset D^j \). From (3) and (4) it follows that \( \phi_2(D^i) \) contains a disk of radius \( \frac{1}{2}r(D^u) \) centered at \( \phi_2(y_0) \). But \( y_0 \in \frac{1}{2}D^u \) and (2) implies that \( \phi_2(y_0) \in \frac{1}{2}D^u \). Thus \( |\phi_2(y_0) - y_0| < \frac{1}{2}r(D^u) \), so \( \phi_2(D^i) \supset D^j \).

Several conversations with R. C. Robinson were helpful in working out the proof of Lemma (2.10).

(2.11) Remark. If \( D^i \) exists for \( z \in D^i \), then \( e = \sup_{z \in D^i} \{||B_z D^i||\} \). In the application of Lemma (2.10) to the proof of Theorem (3.1), one cannot use the version of the lemma in which \( e \) is replaced by \( e' = \sup_{z \in D^i} \{||B_z \cdot || D^i||\} \). For, in the proof of (3.1), it is essential to keep the appropriate counterpart of \( e \) or \( e' \) bounded as one takes larger integers \( N \). In general, the counterpart of \( e' \) will not remain bounded, whereas that of \( e \) will.

3. In this section we establish some properties of diffeomorphisms with hyperbolic negative limit sets.

A slight change in the proof of our first result will also yield a proof of the so-called Anosov closing lemma which says that if \( f \) satisfies Axiom A(a), then \( \overline{P} = \Omega(f/\Omega) \).

(3.1) Theorem. If \( L^- \) is hyperbolic, then \( \overline{P} = L^- \).

Proof. First note that \( L^- = L_1 \cup \cdots \cup L_{n_0} \) where \( L_i \) is closed invariant, the hyperbolic splitting on \( L_i \) has constant dimension, and \( L_i \neq L_j \) for \( 1 \leq i < j \leq n_0 \). Secondly, an argument used by Smale [11, p. 782] applied to \( f^{-1} \) shows that, for each \( y \in M \), there is an \( i \) such that \( \omega(y) \subset L_i \).

We show if \( x_0 \in L_i \) and \( V \) is any neighborhood of \( x_0 \), then there is a periodic point in \( V \). Let \( x \in V \cap L_{n_0} \). We show there is a periodic point in \( V \) near \( x \).

Choose a compact neighborhood \( U \) of \( L_i \) such that there are semi-invariant disk families \( \widetilde{W}^u_z, \widetilde{W}^s_z \), through \( U \) (see [3]). If \( E^s_z (E^s_z) \) is the tangent space to \( \widetilde{W}^s(z) (\widetilde{W}^s(z)) \) at \( z \), then \( E^s \oplus E^u \) is a continuous splitting of \( T_x M \) which is preserved by \( T_x f \) for \( x \in f^{-1}(U) \cap U \).

Assume \( U \) and \( V \) are small enough so that
(1) \( E^s \) and \( E^u \) are defined on \( f^{-1}(U) \cap U \cap f(U) \).
(2) \( ||Tf||_{E^s} < 1 \) on \( U \cap f^{-1}(U) \) and \( ||Tf^{-1}||_{E^u} < 1 \) on \( U \cap f(U) \).
(3) \( V \subset U \) and for \( u_1, u_2 \in V \), \( \widetilde{W}^s(u_1) \cap \widetilde{W}^s(u_2) \) is a single point.

(12) Another proof of the Anosov closing lemma is in [12].
Let

\[ V_1 = \bigcup_{z \in W^u_z(x)} \tilde{W}^u_\delta(z), \quad V_2 = \bigcup_{z \in W^s_z(x)} \tilde{W}^s_\delta(z). \]

Then it is proved in [3] that \( V_1 \) and \( V_2 \) are neighborhoods of \( x \) in \( M \). Let \( y_0 \in M \) be such that \( x \in a(y_0) \subseteq L_4 \). Then there is an integer \( n_1 > 0 \) such that if \( n \geq n_1 \), \( f^{-n}(y_0) \subseteq U \). If \( \delta > 0 \) is small enough, \( f^{-n}(\tilde{W}^u_\delta(y_0)) \subseteq U \) for \( n \geq n_1 \) since \( f^{-1} \) is a contraction on each \( \tilde{W}^u_\delta(z) \).

Let \( \exp_x \) denote the exponential map associated to the Riemannian metric on \( M \). We claim

\[ \text{(4) there are a disk } D = D^i \times D^s \subseteq E^s_x \oplus E^s_u, \text{ a subdisk } D_1 \subseteq D, \text{ integers } N_1 > N_2 > n_1, \text{ and a diffeomorphism } g_1 : \exp_x(D) \rightarrow D \text{ such that the map} \]

\[ g = g_1 f^{N_1-N_2} g_1^{-1} | D_1 \]

satisfies the hypotheses of (2.10) and \( \exp(D) \subseteq V \).

Once (4) is shown, we can apply (2.10) to get a fixed point \( z_1 \) of \( g \) in \( D_1 \). Then \( g_1^{-1}(z_1) \) is a fixed point of \( f^{N_1-N_2} \) in \( V \).

We now prove (4).

For a linear map \( H \) from one Euclidean space to another \( \| H \| \) denotes its norm, and \( m(H) \) denotes its minimum norm which is defined by \( m(H) = \inf_{|v|=1} |Hv| \).

Let \( \epsilon > 0 \) be small enough such that if \( E^s \) and \( E^u \) are subspaces of \( T_xM = E^s_x \oplus E^u_x \) which are \( \epsilon \)-close to \( E^s_x \) and \( E^u_x \), respectively, in the induced metric on the Grassmann bundles of \( M \), then the following is true. There is a linear automorphism \( H : E^s_x \oplus E^u_x \rightarrow E^s_x \oplus E^u_x \) such that \( H(E^s_x) = E^s_x \), \( H(E^u_x) = E^u_x \), and if

\[ H = \begin{pmatrix} I + \sigma_1 & \sigma_2 \\ \sigma_3 & I + \sigma_4 \end{pmatrix} \quad \text{and} \quad H^{-1} = \begin{pmatrix} I + \sigma_5 & \sigma_6 \\ \sigma_7 & I + \sigma_8 \end{pmatrix} \]

with the \( I \)'s denoting identity operators, then \( \| \sigma_i \| < \frac{1}{4} \) for \( i = 1, \ldots, 8 \) and \( m(I + \sigma_i) > \frac{1}{4} \) for \( i = 1, 4, 5, 8 \).

Choose \( D = D^i \times D^s \subseteq E^s_x \oplus E^u_x \) and \( \delta > 0 \) small enough such that

\[ \text{(5) } \exp_x(D) \subseteq U \cap V_1 \cap V_2 \cap V \cap V_3, \]

\[ \text{(6) the manifolds } \{ \exp_x(z \times D^u) : z \in D^i \} \text{ are } \epsilon-C^1 \text{ close to each other, and the manifolds } \{ \exp_x(D^i \times z) : z \in D^u \} \text{ are } \epsilon-C^1 \text{ close to each other.} \]

\[ \text{(7) } T_{y_1} \tilde{W}^u_\delta(z_1) \text{ is } \epsilon \text{-close to } T_{y_2} \tilde{W}^u_\delta(z_2) \text{ for } y_i, z_i \in \exp_x(D), \ y_i \in \tilde{W}^u_\delta(z_i), \ i = 1, 2. \]

\[ \text{(8) For } z \in \exp_x(D), \text{ there is a tubular neighborhood retraction } r_z : F_z \rightarrow \tilde{W}^u_\delta(z) \text{ such that} \]

\[ \text{(a) } F_z \subseteq \bigcup_{y \in \tilde{W}^u_\delta(z)} \tilde{W}^u_\delta(z) \cap V_3; \]

\[ \text{(b) the tangent spaces } T_{y_1}r_z^{-1}(y_1) \text{ and } T_{y_2}r_z^{-1}(y_2) \text{ are } \epsilon \text{-close to those of the } \tilde{W}^u_\delta(z) \text{ for } z_i \in \exp_x(D), \ y_i \in \tilde{W}^u_\delta(z_i) \cap \exp_x(D), \ i = 1, 2; \]

\[ \text{(c) there is a neighborhood } V_4 \text{ of } x \text{ such that } V_4 \subseteq \text{int } \exp_x(D) \cap V_3 \text{ and such that if } z \in V_4 \text{ then } \pi_\delta \exp_z^{-1}(F_z) \supset D^u \text{ and } \pi_\delta \exp_z^{-1}(F_z) \subseteq \text{int } D^i; \]

\[ \text{(d) there is a } \delta_2 > 0 \text{ such that for } y \in \tilde{W}^u_\delta(z) \text{ and } z \in \exp_x(D), \ \tilde{W}^u_\delta(y) \subseteq F_z. \]

The \( \pi_\delta \) and \( \pi_\delta \) in (c) are the natural projections on \( E^s_x \oplus E^u_x \).
Let $N > 0$. If $z, f^{-N}(z) \in V_4$, let $\Sigma_N$ be the connected component of $f^{-N}(F_z) \cap \exp_x(D)$ containing $f^{-N}(z)$. If $N > 0$ is large enough, $z, f^{-N}(z) \in V_4$, and $y \in \tilde{W}_{\delta_2}(z)$, then $f^{-N}(y) \in \exp_x(D)$. In this case, let $\Sigma_{N, y}$ be the connected component of $f^{-N}(r_z^{-1}(y)) \cap \exp_x(D)$ containing $f^{-N}(y)$.

If $\tilde{g}$ is a diffeomorphism of a subset of $\exp_x(D)$ into $D$, we define its $s$-submanifolds to be $\{\tilde{g}^{-1}(D^s \times z) : z \in D^s\}$ and its $u$-submanifolds to be $\{\tilde{g}^{-1}(z \times D^u) : z \in D^u\}$.

To define a diffeomorphism from $\exp_x(D)$ to $D$ it suffices to say what its $s$-submanifolds and $u$-submanifolds are. This is what we will do to prove (4).

Since $f^{-1}$ stretches each $\tilde{W}_s^*$ and contracts each $\tilde{W}_s^*$, there is an integer $N_0 > n_0 > 0$ such that if $N \geq N_0$, then

(9) if $f^{-N}(z) \in U$ for $0 \leq n \leq N$, $z, f^{-N}(z) \in V_4$, and $y \in \tilde{W}_{\delta_2}(z)$, then $\pi^s \circ \exp_x^{-1} \circ f^{-N}|_{\Sigma_{N, y}}$ is a diffeomorphism of $\Sigma_{N, y}$ onto $D^s$.

Assume $N \geq N_0$ so that (9) holds. Define a diffeomorphism $g_2^N : \Sigma_N \to D$ so that its $s$-submanifolds are the $\Sigma_{N, y}$ and these submanifolds are $\epsilon$-$C^1$ close to each other and to $\exp_x(D^s \times 0)$. Extend $g_2^N$ to $\exp_x(D)$ such that its $s$-submanifolds are $\epsilon$-$C^1$ close to each other and to $\exp_x(D^s \times 0)$.

Now define a diffeomorphism $g_1^N : \exp_x(D) \to D$ such that its $s$-submanifolds are those of $g_2^N$ and its $u$-submanifolds are $\{\exp_x(z \times D^u) : z \in D^u\}$. By (6) and the construction of $g_1^N$, the $s$-submanifolds of $g_1^N$ are $\epsilon$-$C^1$ close to each other and the $u$-submanifolds of $g_1^N$ are $\epsilon$-$C^1$ close to each other.

Now we assert that it is possible to choose $N_1 \geq N_0$ so that $f^{-N_1}(y_0), f^{-N_1}(y_0) \in V_4$ and if $z = f^{-N_1}(y_0)$, $D_1 = g_1^{N_1-N_2}(\Sigma_{N_1-N_2})$, and $g_1 = g_1^{N_1-N_2}$, then $g_1$ is the diffeomorphism required in (4). That is, $g_1 f^{N_1-N_2} g_1^{-1}$ satisfies the hypotheses of (2.10).

For $N \geq N_0$, set

\[ T(g_1^N) T f^{N} T(g_1^N)^{-1} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \]

on $g_1^N(\Sigma_N)$ with respect to the splitting $E^s_\alpha \oplus E^u_\alpha$ on $D$. To prove the last assertion it suffices to show that, as $N \to \infty$,

(10) $\|A_0\| \to 0$

and

(11) $m(D_0) \left( 1 - \frac{\|C_0\| \sup_{|v| = 1} |B_0 v|/|D_0 v|}{1 - \|A_0\|} \right) \to \infty.$

This will complete the proof of Theorem (3.1).

Let

\[ T f^{N} = \begin{pmatrix} A_N & 0 \\ 0 & D_N \end{pmatrix} \]
with respect to the splitting \( E^s \oplus E^u \) on \( \Sigma_N \). Then,

\[
\begin{pmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{pmatrix} = \begin{pmatrix}
I + \sigma_1 & \sigma_2 \\
\sigma_3 & I + \sigma_4
\end{pmatrix} \begin{pmatrix}
A_N & 0 \\
0 & D_N
\end{pmatrix} \begin{pmatrix}
I + \sigma_5 & \sigma_6 \\
\sigma_7 & I + \sigma_8
\end{pmatrix},
\]

where \( \|\sigma_i\| < \frac{1}{2}, \) \( i = 1, \ldots, 8 \), and \( m(I + \sigma_i) > \frac{1}{2}, \) \( i = 1, 4, 5, 8. \) Then if \( (0, v) \in \{0\} \times E_x^u \) and \( |v| = 1, \)

\[
\begin{pmatrix}
B_0 \\
D_0
\end{pmatrix} = \begin{pmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{pmatrix} \begin{pmatrix}
0 \\
v
\end{pmatrix},
\]

so

\[
\frac{|B_0v|}{|D_0v|} \leq \frac{\frac{5}{16} \|A_N\|}{\frac{1}{16} \|A_N\|} + \frac{\|\sigma_2\| |v_1|}{m(I + \sigma_4) |v_1| - \frac{1}{16} \|A_N\|}
\]

where \( v_1 = D_N(I + \sigma_0)v. \) As \( N \to \infty, \) \( \|A_N\| \to 0 \) and \( m(D_N) \to \infty, \) so \( |v_1| \to \infty. \) Since \( \|\sigma_2\| < \frac{1}{2} \) and \( m(I + \sigma_4) > \frac{1}{2}, \) for \( N \) large, \( |B_0v|/|D_0v| < \frac{1}{2}. \) Thus for \( N \) large, \( \sup|v_1|=1 \) \( |B_0v|/|D_0v| \leq \frac{1}{2}. \) Further, using the expression in (12), it is easy to see that \( m(D_0) \to \infty \) as \( N \to \infty. \) A similar but easier calculation using the construction of the \( s \)-submanifolds on \( \Sigma_N \) shows that \( \|A_0\| \leq e_1 \|A_N\| \) where \( e_1 \) is a constant independent of \( N. \) It also follows from the construction of the \( s \)-submanifolds on \( \Sigma_N \) that \( \|C_0\| \leq e_2 \|A_0\| \) where \( e_2 \) is a constant independent of \( N. \) Thus, as \( N \to \infty, \) \( \|C_0\| \to 0 \) and \( \|A_0\| \to 0. \) From these facts (10) and (11) follow.

Combining Theorems (3.1) and (2.9), we obtain

(3.2) **Theorem.** If \( L^- \) is hyperbolic, then \( L^- = \Lambda_1 \cup \cdots \cup \Lambda_n \) where the \( \Lambda_i \) are pairwise disjoint closed invariant topologically transitive sets with periodic points dense. Further, each \( \Lambda_i \) has a local product structure.

(3.3) **Proposition.** For \( L^- = \Lambda_1 \cup \cdots \cup \Lambda_n \) as in (3.2), and \( x \in M, \) \( \alpha(x) \) meets at most one \( \Lambda_i. \)

**Proof.** Use the argument at the bottom of p. 782 of [10] for \( f^{-1}. \)

(3.4) **Corollary.** \( M = W^s(\Lambda_1) \cup \cdots \cup W^s(\Lambda_n). \)

**Proof.** If \( x \in M, \) there is an \( i \) such that \( \alpha(x) \subseteq \Lambda_i. \) Since \( \Lambda_i \) has a local product structure, there is a \( y \in \Lambda_i \) such that \( x \in W^s(y) \) by Theorem (1.1) of [3] applied to \( f^{-1}. \)

Following Smale's convention, if \( \overline{P} \) is hyperbolic we will call the sets \( \Lambda_i \) of Theorem (2.9) **basic sets.**

A sequence \( M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset \) of compact submanifolds with boundary such that \( f(M_i) \subseteq \text{int} M_i \) is called a **filtration** for \( f. \) In [11] Smale constructs a filtration which "separates" basic sets if \( f \) satisfies Axiom A and has no cycles.

In this case, if \( \Lambda_i \) is a basic set, then \( W^s(\Lambda_i) \cap W^u(\Lambda_i) = \emptyset. \) If \( f \) satisfies Axiom A and has no cycles, one can obtain a filtration for
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which "separates" certain closed invariant sets. In case the limit set \( L(f) \) is hyperbolic, these sets turn out to be the intersections of the stable and unstable manifolds of c-loop classes of basic sets (see definition below). Our filtration will be constructed by modifying the methods in [11].

We will also examine some consequences of the filtration theorem.

For the remainder of this section we assume \( L^- = L^-(f) \) is hyperbolic. Thus \( L^- = F = \Lambda_1 \cup \cdots \cup \Lambda_n \) as in Theorem (3.2) and \( M = W^s(\Lambda_1) \cup \cdots \cup W^s(\Lambda_n) \).

We define two relations on \( \{A_i\} \).
1. \( A_i \cong_1 A_j \) if there is a sequence \( A_i = A_{i_1}, \ldots, A_{i_r} = A_j \) such that \( \text{Cl} (W^u(A_{i_k})) \cap W^s(A_{i_{k+1}}) \neq \emptyset \) for \( 1 \leq k < r \).
2. \( A_i \cong_2 A_j \) if there is a sequence \( A_i = A_{i_1}, \ldots, A_{i_r} = A_j \) such that \( \text{Cl} (W^u(A_{i_k})) \cap \text{Cl} (W^s(A_{i_{k+1}})) \neq \emptyset \) for \( 1 \leq k < r \).

We will show that these relations are the same, i.e. \( A_i \cong_1 A_j \) if and only if \( A_i \cong_2 A_j \). To each of these relations there is a corresponding equivalence relation \( \sim_k \) defined by \( A_i \sim_k A_j \) if \( A_i \cong_k A_j \) and \( A_i \cong_k A_j \), \( k = 1, 2 \). It will follow that the equivalence classes of these two relations are the same.

Let \( \gamma'_1, \ldots, \gamma'_m \) be the distinct equivalence classes of \( \{A_i\} \) under \( \sim_1 \). Note that \( \{\gamma'_i\} \) is partially ordered by \( \gamma'_i \preceq_1 \gamma'_j \) if there are \( A_i \in \gamma'_i, A_j \in \gamma'_j \) such that \( A_i \cong_1 A_j \).

Let \( \preceq \) be a simple ordering on \( \{\gamma'_i\} \) such that if \( \gamma'_i \preceq \gamma'_j \), then \( \gamma'_i \not\succ \gamma'_j \) (i.e., \( \gamma'_i \) does not strictly precede \( \gamma'_j \) in the \( \preceq_1 \) ordering). We will call such a simple ordering a filtration ordering for \( \{\gamma'_i\} \).

For \( i = 1, \ldots, m, \sigma = u, s \), define \( W^\sigma(\gamma'_i) = \bigcup \{W^\sigma(A) : A \in \gamma'_i\} \). Write \( \bigcup \gamma'_i = \bigcup \{A : A \in \gamma'_i\} \).

We will need a lemma due to Smale.

(3.5) Lemma (See [11]). Suppose \( F \) is a compact \( f \)-invariant subset of \( M \) and \( Q \) is a compact neighborhood of \( F \) such that \( \bigcap_{n \geq 0} f^n(Q) = F \). Then there is a compact neighborhood \( V \) of \( F \) such that \( V \subset Q \) and \( f(V) \subset \text{int} V \).

Proof (due to Smale). Let \( A_1 = Q \cap f(Q) \cap \cdots \cap f^r(Q), r \geq 0 \).

Then \( A_0 \supset A_1 \supset \cdots \) and \( \bigcap_{n \geq 0} A_n = F \). Since \( f(F) \subset F \), there is an integer \( r > 0 \) such that \( A_r \subset \text{int} Q \) and \( f(A_r) \subset \text{int} Q \). But then \( f(A_r) = A_{r+1} \subset A_r \) and \( f'(A_r) = A_{r+j}, j \geq 0 \). Thus there is an integer \( r_1 > 0 \) such that \( f^{r_1}(A_r) \subset \text{int} A_r \). If \( r_1 = 1 \), we are done, so suppose \( r_1 > 1 \). Let \( W_0 \subset \text{int} Q \) be a compact neighborhood of \( A_r \) such that \( f^{r_1}(W_0) \subset \text{int} A_r \). Let \( W_1 = (W_0 \cap f^{r_1}(W_0)) \cup A_r \subset \text{int} Q \). Since \( r_1 \geq 1, f^{r_1-1}(A_r) = A_{r-1+r} \subset \text{int} f^{r_1-1}(W_0) \subset \text{int} W_0 \cap \text{int} f^{r_1-1}(W_0) \subset \text{int} (W_0 \cap f^{r_1-1}(W_0)) \subset f^{r_1-1}(W_0) \subset \text{int} A_r \). Thus \( f^{r_1-1}(W_0) \subset \text{int} W_1 \). Continue by downward induction to prove the lemma.

(3.6) Theorem. Let \( \preceq \) be any filtration ordering for \( \{\gamma'_i\} \) and label \( \{\gamma'_i\} \) such that \( \gamma'_i \preceq \gamma'_{i-1} \preceq \cdots \preceq \gamma'_i \) where \( \gamma'_i \preceq \gamma'_j \) is taken to mean \( \gamma'_i \preceq_1 \gamma'_j \) but \( \gamma'_i \not\succ \gamma'_j \). Then there is a filtration for \( f, M = M_m \supset M_{m-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset \), such that for \( 1 \leq i \leq m \)

1. \( \bigcup \gamma'_i \subset \text{int} (M_i - M_{i-1}) \),
Proof. Say that \( f \) satisfies (*)\( k \) if there is a sequence \( M_k \supset M_{k-1} \supset \cdots \supset \emptyset \) of compact manifolds with boundary such that \( f(M_i) \subset \text{int } M_i \) and (1)–(5) hold for \( 1 \leq i \leq k \).

To begin, take \( V \) to be a compact neighborhood of \( W^+(\gamma_i) \) such that \( V \cap \bigcup_{j \geq 1} (\bigcup \gamma_j) = \emptyset \). Then if \( x \in \cap_{n \geq 0} f^n(V) \), then \( \alpha(x) \subset V \), so \( x \in W^+(\gamma_i) \). Thus \( \cap_{n \geq 0} f^n(V) = W^+(\gamma_i) = \text{Cl} (W^+(\gamma_i)) \). By Lemma (3.5), there is a compact neighborhood \( \tilde{V} \) of \( \text{Cl} (W^+(\gamma_i)) \) such that \( \tilde{V} \subset V \) and \( f(\tilde{V}) \subset \text{int } V \). Further, we may suppose \( \tilde{V} \) is a compact manifold with boundary. Taking \( M_1 = \tilde{V} \), we see that \( f \) satisfies (*)\( 1 \).

Now suppose \( f \) satisfies (*)\( k \). Then \( \text{Cl} (W^+(\gamma_{k+1})) \cup M_k \) is a closed set which does not meet \( \bigcup_{j \geq k+1} (\bigcup \gamma_j) \). Let \( V \) be a compact neighborhood of \( \text{Cl} (W^+(\gamma_{k+1})) \cup M_k \) such that \( V \cap \bigcup_{j \geq k+1} (\bigcup \gamma_j) = \emptyset \). Then if \( x \in \cap_{n \geq 0} f^n(V) \), \( \alpha(x) \subset V \), so \( x \in W^+(\gamma_i) \). Thus \( \cap_{n \geq 0} f^n(V) = \bigcup_{j \geq k+1} W^+(\gamma_j) \) is closed and in the interior of \( V \). Again applying (3.5) there is a compact submanifold with boundary \( M_{k+1} \) such that \( \cap_{n \geq 0} f^n(V) \subset f(M_{k+1}) \subset \text{int } M_{k+1} \subset V \). Now clearly properties (1), (3), and (4) hold for \( 1 \leq i \leq k+1 \). If \( x \in \cap_{n \geq 0} f^n(V) \), \( \alpha(x) \subset (M_{k+1} - M_k) \cap L^- = \bigcup \gamma_{k+1} \), so (2) holds. Finally, since \( f(M_{k+1}) \subset \text{int } M_{k+1} \), if \( \text{Cl} (W^+(\gamma_i)) \cap M_{k+1} \neq \emptyset \), then \( W^+(\gamma_j) \cap M_{k+1} \neq \emptyset \), so \( \bigcup \gamma_j \cap V \neq \emptyset \). Hence, by the construction of \( V \), \( \gamma_{k+1} \supseteq \gamma_j \) which shows that (5) holds. Thus \( f \) satisfies (*)\( k+1 \) and we are done.

Remark. For basic sets \( \Lambda_i \) and \( \Lambda_j \), call a sequence from \( \Lambda_i \) to \( \Lambda_j \) as in the definition of \( \Lambda \) a \( c \)-path from \( \Lambda_i \) to \( \Lambda_j \). Let \( \mathcal{P}(\Lambda_i) = \{ \Lambda_j : \text{there is a } c \text{-path from } \Lambda_i \text{ to } \Lambda_j \} \). The techniques in the proof of (3.6) can be used to show that for any \( \Lambda_i \), there is a compact neighborhood \( V \) of \( \text{Cl} (W^+(\mathcal{P}(\Lambda_i))) \) such that \( f(V) \subset \text{int } V \) and

\[
\bigcap_{n \geq 0} f^n(V) = \text{Cl} (W^+(\mathcal{P}(\Lambda_i))) = W^+(\mathcal{P}(\Lambda_i)).
\]

(Of course, \( W^+(\mathcal{P}(\Lambda_i)) = \bigcup_{\Lambda_j \in \mathcal{P}(\Lambda_i)} W^+(\Lambda_j) \).)

(3.7) Theorem. If \( \Lambda_i \supseteq \Lambda_j \), then \( \Lambda_i \supseteq \Lambda_j \).

(3.8) Corollary. The equivalence relations \( \sim_1 \) and \( \sim_2 \) give the same equivalence classes.

Proof of (3.7). We prove that if \( \Lambda_i \supseteq \Lambda_j \), then \( \Lambda_i \supseteq \Lambda_j \). If \( \Lambda_i \supseteq \Lambda_j \), there is a filtration ordering \( \supseteq \) such that \( [\Lambda_i] > [\Lambda_j] \) where \( [\Lambda] \) is the equivalence class of \( \Lambda \) under \( \sim \), \( \Lambda = \Lambda_i \) or \( \Lambda_j \).

Let \( \{M_i\} \) be a filtration corresponding to \( \supseteq \) as in Theorem (3.6). Then if \( \Lambda_i \supseteq \Lambda_j \), an easy induction on the length of a sequence from \( \Lambda_i \) to \( \Lambda_k \) as in the definition of \( \supseteq \) shows that \( \Lambda_k \subset M_i \). But since \( \Lambda_j \cap M_i = \emptyset \), we get \( \Lambda_i \supseteq \Lambda_j \).
We will use the notation \( A_i \sim A_j \) to mean \( A_i \sim_1 A_j \) or \( A_i \sim_2 A_j \) which is justified by Corollary (3.8).

By analogy with the usual definition of cycles in the case of Axiom A (see [5] and [11]), we define an \( r \)-cycle to be a sequence \( A_{i_0}, \ldots, A_{i_r} \) such that \( A_{i_0} = A_{i_r} \) and \( \hat{W}^u(\Lambda_{i_k}) \cap \hat{W}^s(\Lambda_{i_{k+1}}) \neq \emptyset \) for \( 0 \leq k < r \) where \( \hat{W}^\sigma(\Lambda) = W^\sigma(\Lambda) - \Lambda \), \( \sigma = u, s \). A cycle will mean an \( r \)-cycle for some \( r \). The reason we need to use \( \hat{W}^\sigma(\Lambda) \) instead of \( W^\sigma(\Lambda) \) as in the case of Axiom A is that 1-cycles (our definition) can occur for \( L^- \) hyperbolic (see Examples 1 and 4 at the end of this section), whereas they cannot occur for diffeomorphisms satisfying even Axiom A(a).

Define a c-cycle (for closure cycle) to be a sequence \( A_{i_0}, \ldots, A_{i_r} \) such that \( A_{i_0} = A_{i_r} \) and \( (\text{Cl}(W^u(A_{i_k}))) - A_{i_k} \) \( \cap \hat{W}^s(A_{i_{k+1}}) \neq \emptyset \) for \( 0 \leq k < r \).

A sequence \( A_{i_0}, \ldots, A_{i_r} \) such that \( A_{i_0} = A_{i_r} \) and \( (\text{Cl}(W^u(A_{i_k}))) \cap \hat{W}^s(A_{i_{k+1}}) \neq \emptyset \) for \( 0 \leq k < r \) will be called a c-loop. Also we will call the equivalence classes of \( \{ A_i \} \) under \( \sim \), c-loop classes.

The proof of the following lemma was worked out with the aid of J. Palis.

\[(3.9) \text{Lemma. Suppose } \Lambda_1 \neq \Lambda_2 \text{ are basic sets such that } (\text{Cl}(W^u(\Lambda_1)) - \Lambda_1) \cap \hat{W}^s(\Lambda_2) \neq \emptyset. \text{ Then } (\text{Cl}(W^u(\Lambda_1)) - \Lambda_1) \cap \hat{W}^s(\Lambda_2) \neq \emptyset.\]

**Proof.** Since \( \Lambda_2 \) has a local product structure, there is a proper fundamental neighborhood \( V \) for \( W^u(\Lambda_2) \) (see [3]). Moreover, we may choose \( V \) to be arbitrarily close to a proper fundamental domain \( D \subset W^u(\Lambda_2) - \Lambda_2 \). By Theorem (1.1) of [3], \( V' = \bigcup_{n \geq 0} f^n(V) \cup W^u(\Lambda_2) \) is a neighborhood of \( \Lambda_2 \) in \( M \). But then for \( \varepsilon \) small enough, \( V' \) is the neighborhood of \( W^\varepsilon(\Lambda_2) \). Since \( (\text{Cl}(W^u(\Lambda_1)) - \Lambda_1) \cap \hat{W}^u(\Lambda_2) \neq \emptyset \), \( (\text{Cl}(W^u(\Lambda_1)) - \Lambda_1) \cap W^\varepsilon(\Lambda_2) - \Lambda_2 \neq \emptyset \). But then \( (W^u(\Lambda_1) - \Lambda_1) \cap \bigcup_{n \geq 0} f^n(V) \neq \emptyset \) so \( (W^u(\Lambda_1) - \Lambda_1) \cap V \neq \emptyset \). Since \( V \) was arbitrarily close to \( D \),

\[(\text{Cl}(W^u(\Lambda_1)) - \Lambda_1) \cap \hat{W}^s(\Lambda_2) \neq \emptyset.\]

\[(3.10) \text{Proposition. A c-loop class contains a cycle if and only if it contains a c-cycle.}\]

**Proof.** Suppose \( \gamma \) is a c-loop class which contains a c-cycle. We prove that \( \gamma \) contains a cycle. The converse is obvious.

For \( \Lambda_1, \Lambda_2 \in \gamma \), call a sequence \( \Lambda_{i_0}, \ldots, \Lambda_{i_s} \) a proper sequence from \( \Lambda_1 \) to \( \Lambda_2 \) if \( \Lambda_{i_0} = \Lambda_1, \Lambda_{i_s} = \Lambda_2 \) and \( \hat{W}^u(\Lambda_{i_k}) \cap \hat{W}^s(\Lambda_{i_{k+1}}) \neq \emptyset \) for \( 0 \leq k < s \). Let \( (\Lambda_{i_0}, \ldots, \Lambda_{i_s}) \) be a c-cycle in \( \gamma \).

If for each \( 0 \leq j < r \) there is a proper sequence from \( \Lambda_{i_j} \) to \( \Lambda_{i_0} \), taking \( j = 0 \), we get a cycle in \( \gamma \). If there is a \( 0 \leq j < r \) such that there is no proper sequence from \( \Lambda_{i_j} \) to \( \Lambda_{i_0} \), let \( j_0 \) be the largest such integer. Then there is no proper sequence from \( \Lambda_{i_0} \) to \( \Lambda_{i_{j_0 + 1}} \).

Let \( x \in (\text{Cl}(W^u(\Lambda_{i_{j_0}})) - \Lambda_{i_{j_0}}) \cap \hat{W}^s(\Lambda_{i_{j_0 + 1}}) \). Let \( \Lambda_{i_1} \) be the basic set such that \( x \in W^u(\Lambda_{i_1}) \). Then since \( x \in \hat{W}^s(\Lambda_{i_{j_0 + 1}}) \), \( x \in \hat{W}^u(\Lambda_{i_1}) \), so we may apply Lemma (3.9) to conclude that \( \Lambda_{i_{j_0 + 1}} >_1 \Lambda_{i_1} \). Now assume \( \Lambda_{i_k} \neq \Lambda_{i_{j_0}} \) is defined for \( k \leq r \) such that \( \Lambda_{i_{j_0 + 1}} >_1 \Lambda_{i_k} \) and \( \hat{W}^u(\Lambda_{i_k}) \cap \hat{W}^s(\Lambda_{i_{k-1}}) \neq \emptyset \).
If any two of the $\Lambda_{it}$'s are equal we have a cycle so we may assume they are all distinct. Let $x \in (\text{Cl}(W^u(\Lambda_{t_0})) - \Lambda_{t_0}) \cap W^s(\Lambda_{it})$. Since there is no proper sequence from $\Lambda_{t_0}$ to $\Lambda_{it}$, if $\Lambda_{t_{i+1}}$ is the basic set such that $x \in W^u(\Lambda_{t_{i+1}})$, then $\Lambda_{i+1} \neq \Lambda_{t_0}$, and $\Lambda_{t_0} \geq 1 \Lambda_{t_{i+1}}$. So we get a sequence of distinct basic sets ($\Lambda_{t_1}, \Lambda_{t_2}, \ldots, \Lambda_{t_n}$) such that $W^u(\Lambda_{t_e}) \cap W^s(\Lambda_{t_{e-1}}) \neq \emptyset$ for $2 \leq e \leq n+1$. Continuing as above, since there are only finitely many basic sets, we eventually get a cycle.

A basic set $\Lambda$ is called a source if $W^s(\Lambda) = \emptyset$.

If $\Lambda$ is a source, there is a compact neighborhood $V$ of $\Lambda$ such that $f^{-1}(V) \subset \text{int } V$.

(3.11) Proposition. If $\gamma$ is a c-loop class which is maximal with respect to the partial ordering $>_1$ (hence $>_2$), then $\gamma$ has only one element, and that element is a source.

Proof. Suppose $\gamma$ is maximal with respect to $>_1$. Choose a filtration ordering with $\gamma$ as its largest element. Let $M = M_m \supset M_{m-1} \supset \cdots \supset M_1 \supset \emptyset$ be the corresponding filtration for $f$ so that $\bigcup \gamma \subset \text{int}(M_{m} - M_{m-1})$. By Theorem (3.6) if $\Lambda$ is a basic set such that $W^u(\Lambda) \cap (M_{m} - M_{m-1}) \neq \emptyset$, then $[\Lambda] \geq 1$. Thus $\Lambda \in \gamma$ since $\gamma$ is maximal. Thus $M_{m} - M_{m-1} \subset W^u(\gamma)$. So $W^u(\gamma) - M_{m-1}$ is an open neighborhood of $W^u(\gamma)$. Thus there is a $\Lambda \in \gamma$ such that $W^u(\Lambda)$ contains an open subset of $M$. Let $\epsilon > 0$. Then $\bigcap_{n \geq 0} f^n(W^u(\Lambda)) = W^u(\Lambda)$ so $W^u(\Lambda)$ contains an open subset of $M$, say $V$. Since the periodic points of $\Lambda$ are dense in $\Lambda$, there is a periodic point $p \in \Lambda$ such that $W^u(\Lambda(p)) \subset V$. By an application of the $\lambda$-lemma, $W^u(\Lambda(p)) \subset \text{Cl}(\bigcap_{n \geq 0} f^{-n}(V))$. Further, as observed by Smale, if $\Lambda$ is a basic set, then $W^u(\Lambda) \subset \text{Cl}(W^u(\Lambda(p)))$. Thus, $W^u(\Lambda) \subset \text{Cl}(W^u(\Lambda(p)) \subset \text{Cl}(\bigcap_{n \geq 0} f^{-n}(V)) = W^u(\Lambda)$. Here the last inclusion follows since $W^u(\Lambda) \subset \Lambda$ and $f^{-1}$-invariant. Thus $W^u(\Lambda) \subset W^u(\Lambda)$ for all $\epsilon > 0$. But $\bigcap_{\epsilon > 0} W^u(\Lambda) = \emptyset$, so $W^u(\Lambda) \subset \Lambda$ and we are done.

(3.12) Remark. 1. The preceding results are true with $L^+$ replacing $L^-$ where the obvious changes are made; e.g., if $L^+$ is hyperbolic, then $\bar{P} = L^+ = \Lambda_1 \cup \cdots \cup \Lambda_n$, $M = W^s(\Lambda_1) \cup \cdots \cup W^s(\Lambda_n)$, there is a filtration for $f^{-1}$ as in Theorem (3.6), etc.

2. If $L = L^+ \cup L^+$ is hyperbolic and $M_m \supset M_{m-1} \supset \cdots \supset M_1 \supset \emptyset$ is the filtration for $f$ as in Theorem (3.6), then $\bigcap_{-\infty < n < \infty} f^n(\text{Cl}(M_i - M_{i-1})) = W^s(\gamma_i) \cap W^u(\gamma_i)$. For, if $x \in \bigcap_{n \geq 0} f^n(\text{Cl}(M_i - M_{i-1}))$ then $\omega(x) \in M_i - M_{i-1}$, so $x \in W^u(\gamma_i)$.

3. Clearly, if $f$ satisfies Axiom A(a), then $L$ is hyperbolic. In [5], Palis shows that if $f$ satisfies Axiom A and has a cycle, then $f$ may be perturbed to given an $\Omega$-explosion. Notice that Remark (3.12.2) can be used to give some kind of control on the size of the $\Omega$-explosion. That is, if $g$ is close to $f$, then $\Omega(g)$ is close to $\bigcup_{1 \leq i \leq n} (W^s(\gamma_i) \cap W^u(\gamma_i))$.

Before proceeding, we consider some examples. All of these examples will be diffeomorphisms on the two-sphere $S^2$.

1. This example is such that $L^-$ is hyperbolic, but $L^+$ is not hyperbolic. It also shows that minimal elements of a filtration ordering do not have to consist of
single basic sets (see Proposition (3.10)). We take $f$ to be the time-one map $\varphi_1$ of the flow $\varphi_t$ pictured below.

Figure 1

Here, for $\varphi_t$, there is an expanding spiral source at infinity. There are two hyperbolic saddle points and three expanding spiral fixed points as in the picture. The saddle points taken together form a minimal element in the filtration ordering and all orbits except the saddle connections and the fixed points spiral in to the saddle connections.

In this case $L^-(f)$ is the set of fixed points of $f$ so it is hyperbolic. $L^+(f)$ is the set of fixed points together with the saddle connections (which are the stable and unstable manifolds of the saddle points).

2. Here $\bar{P}(f)$ is hyperbolic and finite, but $L^-(f)$ and $L^+(f)$ are neither. Again $f$ is the map $\varphi_1$ for a flow $\varphi_t$. This flow is described as follows. On a two disk $D_1$, let $\psi_t$ be a flow transversal to the boundary whose $\omega$-limit set is two spiral sources and a figure eight as in Figure 2a.

Figure 2a

Figure 2b
Here $\psi_t$ has only three critical points, so $P(f) = \bar{P}(f)$ is the set of those three points. Let $\eta_t$ be the inverse of the flow $\psi_t$ on another copy $D_2$ of $D_1$, i.e. $\eta_t(x) = \psi_{-t}(x)$ (see Figure 2b). Then $\eta_t$ has three critical points and its $\alpha$-limit set is not hyperbolic. Now glue the two disks $D_1$ and $D_2$ together along their boundaries and fit $\psi_t$ and $\eta_t$ together to give a flow as required.

3. This example was shown to me by J. Palis. In it the limit set $L(f) = L^-(f) \cup L^+(f)$ is finite and hyperbolic, but $\Omega(f)$ is neither.

Start with a flow $\varphi_t$ having two sources, two sinks and two saddle points $x_1, x_2$ connected by trajectories as in Figure 3a.

![Figure 3a](image)

The circles represent the sources and sinks and all the critical elements are assumed hyperbolic. Thus $\varphi_t = f$ satisfies Axiom A and has a 2-cycle. Now by a slight change of $\varphi_t = f$ in the space of diffeomorphisms we make one component of $W^u(x_1) - \{x_1\}$ have nonempty transversal intersection with one component of $W^s(x_2) - \{x_2\}$ as in Figure 3b.

![Figure 3b](image)
This can be done so as not to change the other components $V^s (V^u)$ of $W^s(x_1) - \{x_1\} (W^u(x_2) - \{x_2\}).$ Of course, $V^s = V^u.$ For the new diffeomorphism $g$, $\Omega(g)$ will consist of $P(g) \cup V^s.$

We can also make $\Omega$ countable and keep $L$ finite by making $V^s$ and $V^u$ intersect nontransversely in an appropriate way. An example is depicted in Figure 3c.

4. Here we have $L$ hyperbolic, Axiom A(a) is not satisfied and there is a 1-cycle. Start with the familiar horseshoe example of Smale on $S^2.$ Thus $\Omega(f)$ consists of a source $p_0$, a sink $p_1$, and a Cantor set $\Lambda$ on which $f$ is topologically conjugate to a shift automorphism on two symbols. This is pictured in Figure 4a.
There is a single fixed point $x_0 \in \Lambda$ such that one component $V_1$ of $W^u(x_0) - \{x_0\}$ is contained in $W^s(p_1)$ and one component $V_2$ of $W^s(x_0) - \{x_0\}$ is contained in $W^u(p_0)$.

There are open intervals $V^s$ and $V^u$ in the other components of $W^s(x_0) - \{x_0\}$ and $W^u(x_0) - \{x_0\}$ such that $\text{Cl}(V^s)$ and $\text{Cl}(V^u)$ are closed intervals bounded on one side by $x_0$. We suppose $V^s$ and $V^u$ are as depicted in Figure 4b, so that $V^s \cap V^u$ consists of four points.

![Figure 4b](image)

Now one can modify the diffeomorphism away from $\Lambda$ so as to produce a unique tangency $y$ off $\Lambda$ of $V^s$ and $V^u$. This is depicted in Figure 4c.

The modification can be done so that one gets a diffeomorphism $g$ such that

(a) $\Omega(g) = \Omega(f) \cup o(y)$.

(b) For each $x \in M$ and each small neighborhood $U$ of $y$, $o(x) \cap U$ has at most two points.

It follows from (a), (b) and the construction of such a $g$, that $L(g) = \Omega(f)$ and $g$ has a 1-cycle ($\Lambda_0, \Lambda_1$) with $\Lambda_0 = \Lambda_1 = \Lambda$.

4. In this section we complete the proof that if $L^-(f)$ is hyperbolic and there are no cycles, then $f$ satisfies Axiom A. We also give another sufficient condition for $\Omega$-stability. The latter result is that if $\bar{P}$ is hyperbolic, $L^u_\Lambda \subset \bar{P}$ (see definition before (4.7)), and there are no $c$-cycles, then $f$ satisfies Axiom A (and has no cycles).

(4.1) Theorem. Suppose $L^-(f)$ is hyperbolic, and $f$ has no $c$-cycles. Then $L^- = \bar{P} = \Omega$, so $f$ satisfies Axiom A. Further, $f$ has no cycles, so $f$ is $\Omega$-stable.
Proof. Let $L^{-}(f) = \Lambda_{1} \cup \cdots \cup \Lambda_{n}$ as in Theorem (3.2). Since $f$ has no $c$-cycles, each $c$-loop class has only one element. Let $\{ \Lambda_{n} \} \supseteq \{ \Lambda_{n-1} \} \supseteq \cdots \supseteq \{ \Lambda_{1} \}$ be a filtration ordering for $f$ with filtration $M_{n} \supset M_{n-1} \supset \cdots \supset M_{1} \supset \emptyset$ as in Theorem (3.6). We claim
\begin{equation}
\text{for each } 1 \leq i \leq n, \quad \bigcap_{-\infty < n < \infty} f^{n}(\text{Cl}(M_{i} - M_{i-1})) = \Lambda_{i}.
\end{equation}

This will prove that $L^{-} = \Omega$, so $L^{-} = \bar{P} = \Omega$. Thus we will have that $f$ satisfies Axiom A. Clearly, the hypotheses imply that $f$ has no cycles. Further, once (4.2) is proved we will, in fact, have a filtration for $f$ exactly the same as the filtration Smale obtains in [11]. Then $\Omega$-stability will follow as in his proof by proving $\Omega$-stability on each $M_{i} - M_{i-1}$. This last step follows from Theorem (7.3) of [2].

So we need only prove (4.2). Fix $i$. By (3.6.2), $\bigcap_{n \geq 0} f^{n}(\text{Cl}(M_{i} - M_{i-1})) \subseteq W^{s}(\Lambda_{i})$. We claim
\begin{equation}
\bigcap_{n \leq 0} f^{n}(\text{Cl}(M_{i} - M_{i-1})) \subseteq W^{s}(\Lambda_{i}).
\end{equation}

Once this is shown (4.2) follows since we will have
\[ \Lambda_{i} \subseteq \bigcap_{-\infty < n < \infty} f^{n}(\text{Cl}(M_{i} - M_{i-1})) \subseteq W^{u}(\Lambda_{i}) \cap W^{s}(\Lambda_{i}) = \Lambda_{i} \]
where the last equality holds since there are no 1-cycles.

For the proof of (4.3) suppose $x \in \bigcap_{n \leq 0} f^{n}(\text{Cl}(M_{i} - M_{i-1}))$. Then $\omega(x) \subseteq \text{Cl}(M_{i} - M_{i-1})$, so $\omega(x) \subseteq \bigcap_{n \geq 0} f^{n}(\text{Cl}(M_{i} - M_{i-1})) \subseteq W^{u}(\Lambda_{i})$ where the first inclusion follows from the $f$-invariance of $\omega(x)$ and the second one follows from (3.6.2).
Since there are no 1-cycles, $W^u_\varepsilon(\Lambda_i)$ is a neighborhood of $\Lambda_i$ in $W^u(\Lambda_i)$ for $\varepsilon > 0$. But $\omega(x)$ is a compact subset of $W^u(\Lambda_i)$ and $\bigcup_{n \geq 0} f^n(W^u_\varepsilon(\Lambda_i)) = W^u(\Lambda_i)$. So there is an integer $n_1 > 0$ such that $\omega(x) \subset F^{n_1 / n}(W^u_\varepsilon(\Lambda_i))$. Since $\omega(x)$ is $f^{-1}$-invariant, $\omega(x) \subset \bigcap_{n \geq 0} f^{-n}(W^u_\varepsilon(\Lambda_i)) = \Lambda_i$. But $\Lambda_i$ has a local product structure, so $x \in W^u(\Lambda_i)$ and (4.3) is proved.

(4.4) REMARK. Example 4 in §3 gives a diffeomorphism such that $L = L^- \cup L^+$ is hyperbolic, each c-loop class has only one element, and there is a single 1-cycle. Also, $L \neq \emptyset$ and Axiom A(a) does not hold.

We now come to our main result.

(4.5) THEOREM. Suppose $L^-(f)$ is hyperbolic and $f$ has no cycles. Then $f$ satisfies Axiom A.

Proof. Since $f$ has no cycles, it has no c-cycles by Proposition (3.10). Now (4.5) follows from (4.1).

(4.6) REMARK. 1. Using Theorem (4.5) one can obtain, of course, that Axiom A(a) and no cycles imply Axiom A(b). Also using Theorem (3.6) and a result similar to the theorem in §1 of [13], one can prove that if $f$ satisfies Axiom A(a) and every c-loop $(\Lambda_{i_0}, \ldots, \Lambda_{i_r})$ is 2-related in the sense that for $0 \leq j, l \leq r$, $\text{Cl}(W^u(\Lambda_{i_l})) \cap W^s(\Lambda_{i_j}) \neq \emptyset$, then $f$ satisfies Axiom A(b). However, it is still unknown if Axiom A(a) implies Axiom A(b) in general.

2. J. Robbin has recently proved that if $f$ is $C^2$ and satisfies Axiom A and the strong transversality condition, then $f$ is structurally stable [8], thus confirming part of a conjecture of Smale. Using Theorem (4.5) and some other well-known results, Robbin’s theorem may be restated in the following way. If $f$ is $C^2$, $L^-(f)$ is hyperbolic, and for $x, y \in L^-(f)$, $W^u(x)$ is transverse to $W^s(y)$, then $f$ is structurally stable.

For $V$ a closed subset of $M$, let $L_\alpha^0(V) = V$ and $L_\alpha^1(V) = \{y \in V : \exists x \in L_\alpha^{i-1}(V) \text{ such that } y \in \alpha(x) \}$, for $N > 0$. Note that, for $N > 0$, $L_\alpha^N(V) = \{y \in V : \text{there is a sequence } x_0, x_1, \ldots, x_N \text{ in } V \text{ such that } x_N = y \text{ and } x_i \in \alpha(x_{i-1}) \subset V \text{ for } 1 \leq i \leq N\}$. Let $L_\alpha^N(M) = L_\alpha^N$, so that $L_\alpha^N = L_\alpha$ as defined earlier. Notice also that $P \subset L_\alpha^2 \subset \cdots \subset L_\alpha = \Omega$ for all $N > 0$.

Our final result is the following.

(4.7) THEOREM. If $\bar{P}$ is hyperbolic, $L_\alpha^N \subset \bar{P}$ for some $N > 0$, and there are no c-cycles for the basic sets in the spectral decomposition of $\bar{P}$, then $f$ satisfies Axiom A.

Proof. Let $N > 0$ be such that $L_\alpha^N \subset \bar{P}$. We prove that $L_\alpha \subset \bar{P}$ and then (4.7) follows from (4.1).

Let $\bar{P} = \Lambda_1 \cup \cdots \cup \Lambda_n$ as in Theorem (2.9). Since there are no c-cycles, each c-loop class has only one element, so there is a simple ordering $\Lambda_n \geq \Lambda_{n-1} \geq \cdots \geq \Lambda_1$ such that if $\Lambda_i \geq \Lambda_j$, then $\text{Cl}(W^u(\Lambda_j)) \cap W^s(\Lambda_i) = \emptyset$, i.e. $\Lambda_j \not\prec \Lambda_i$, as defined before.
Say that $f$ satisfies $(*)_k$ if there is a sequence of compact submanifolds with boundary $M_k \supset M_{k-1} \supset \cdots \supset M_1$ such that for $1 \leq i \leq k$

(a) $\Lambda_i \subseteq \text{int} (M_i - M_{i-1}),$
(b) $f(M_i) \subseteq \text{int} M_i,$
(c) $\bigcap_{n \geq 0} f^n(M_i) = \bigcup_{j \leq i} \text{Cl} (W^u(\Lambda_j)),$
(d) $f^{-1}(\Lambda_i) \subseteq \bigcup_{j \leq i} \text{Cl} (W^s(\Lambda_j)).$

Then one proves by induction on $k$ that $f$ satisfies $(*)_k$ for $1 \leq k \leq n.$ It follows from (d) that $L_n \subseteq \overline{P}.$

We will prove $(*)_1.$ The induction step for $(*)_k + 1$ from $(*)_k$ is similar, so we omit its proof.

Let $V$ be a compact neighborhood of $\text{Cl} (W^u(\Lambda_1))$ such that $V \cap \Lambda_i = \emptyset$ for $i > 1.$ Now $L_n^u \subseteq \overline{P},$ so $L_n^u(V) \subseteq \Lambda_1.$ We claim that $L_n^u(V) \subseteq \Lambda_1$ implies that $L_n^{u-1}(V) \subseteq \Lambda_1.$

For, if $x \in L_n^{u-1}(V),$ then $L^n_x(V) \subseteq \Lambda_1,$ so $x \in W^u(\Lambda_1).$ But there is a $y \in L_n^{u-2}(V)$ such that $x \in \alpha(y)$ and $\alpha(y) \subset V.$ Also $\alpha^2(y) \subset L_n^{u-4}(V) \subset W^u(\Lambda_1).$ Now, as in the proof of (4.3), $\alpha(y) \subseteq \Lambda_1$ since there are no 1-cycles. But $x \in \alpha(y),\;$ so $x \in \Lambda_1.$ Thus $L_n^{u-1}(V) \subseteq \Lambda_1.$

Proceeding by downward induction we get that $L_n(V) \subseteq \Lambda_1.$ But then $\bigcap_{n \geq 0} f^n(V) = W^u(\Lambda_1) = \text{Cl} (W^u(\Lambda_1)).$ Now, as in the proof of Theorem (3.6), by Smale's lemma, there is a compact submanifold with boundary $M_1$ such that $f(M_1) \subseteq \text{int} M_1$ and $\bigcap_{n \geq 0} f^n(M_1) = W^u(\Lambda_1).$ Again as in the proof of (4.3), $M_1 = \bigcap_{n \geq 0} f^n(M_1) \subset W^u(\Lambda_1).$ So $W^u(\Lambda_1) \subset W^u(\Lambda_1)$ and hence $W^u(\Lambda_1) = \Lambda_1.$ This proves $(*)_1.$

Questions and Remarks. 1. Does (4.7) remain true if one replaces the no c-cycles assumption by the assumption that there are no cycles?

2. Let $F_1 = \overline{P}, F_2 = L^-,$ $F_3 = L,$ and $F_4 = \Omega.$ Say that $f$ is $F_i$-stable, $i = 1, \ldots, 4,$ if there is a neighborhood $N$ of $f$ in Diff $(M)$ such that if $g \in N,$ there is a homeomorphism $h: F_i(f) \to F_i(g)$ such that $hf = gh.$ Is $F_i$-stability equivalent to $F_i$-stability for $i = 1, \ldots, 4$? Is it true that $F_i$ is $L^-$ stable if and only if $L^-$ is hyperbolic and there are no cycles? Palis has shown (unpublished) that if $f$ is $F_i$-stable and $F_i$ is hyperbolic, then there are no cycles. Therefore, the main part of the last question is: does $L^-$-stability imply $L^-$ is hyperbolic? By the closing lemma this can be reduced to: does $\overline{P}$-stability imply that $\overline{P}$ is hyperbolic?

3. There are easily constructed examples where $\overline{P}$ is hyperbolic, there are no cycles, and $\overline{P} \not\subseteq L^-.$ Hence, by the closing lemma, it follows that $\overline{P}$ hyperbolic and no cycles is not sufficient for $\overline{P}$-stability.

Appendix: The homoclinic point theorem. Here we give a fairly elementary proof of Theorem (2.6) based on the stable manifold theory for a hyperbolic fixed point and Lemma (2.10).

We first need a lemma due to Hirsch and Pugh [2].

Let $E, F$ be Banach spaces and give $E \times F$ the norm $|(x, y)| = \max \{|x|, |y|\}.$

Lemma 1. Let $T = (\begin{pmatrix} a & b \\ c & d \end{pmatrix})$ be a linear map from $E \times F$ to itself such that $D^{-1}$ exists.

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Let $\eta < 1$. If

1. $\|A\| \cdot \|D^{-1}\| + \|BD^{-1}\| + C \cdot \|D^{-1}\| < \eta$ and
2. $\|A\| \cdot \|D^{-1}\| + 2\|C\| \cdot \|D^{-1}\| < 1,$

then there is a unique linear map $P : F \to E$ such that $\|P\| < \eta$ and

3. $T(\text{graph } P) = \text{graph } P$.

Further, if $m(D) - \|C\| > 1$, then $T|_{\text{graph } P}$ is expanding (recall $m(D) = \inf_{|v| = 1} |Dv|$).

Proof. Condition (3) can be written $T(Py, y) = (Pu, u)$ where $u = CPy + Dy$ or $APy + By = PCPy + PDy$ for $y \in F$. Thus as linear maps, $AP + B = PCP + PD$, or $P$ is a fixed point of the map $H : P \mapsto APD^{-1} + BD^{-1} - PCPD^{-1}$.

Let $\mathcal{H}_\nu$ be the complete metric space of bounded linear maps from $F$ to $E$ with norm less than or equal to $\nu$ where $\nu = \|A\| \cdot \|D^{-1}\| + \|BD^{-1}\| + \|C\| \cdot \|D^{-1}\|$.

For $P \in \mathcal{H}_\nu$, $\|H(P)\| \leq \|A\| \cdot \|D^{-1}\| + \|BD^{-1}\| + \|C\| \cdot \|D^{-1}\| = \nu < \eta$ by (1). Thus, $H$ maps $\mathcal{H}_\nu$ into itself.

Similarly, by (2), $H$ is a contraction and so has a unique fixed point $P$.

Also, if $(Py, y) \in \text{graph } P$, then, since $\|P\| < 1$, $|T(Py, y)| = |CPy + Dy| \geq (m(D) - \|C\||y| > |y| = |(Py, y)|$.

This proves $T|_{\text{graph } P}$ is expanding and completes the proof of Lemma 1.

Now returning to the notation of Lemma (2.10), suppose $T_z g = \begin{pmatrix} A_z & B_z \\ C_z & D_z \end{pmatrix}$ for $z \in D_1$, and

$T_w g^{-1} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ for $w \in g(D_1)$

where $T_w g^{-1}$ is assumed to exist.

By analogy with the definitions of $a$, $c$, $e$, $d$ before Lemma (2.10), define

$a_1 = \inf \{|A_1 w| : w \in g(D_1), v$ is a unit vector in $R^n\}$,

$b_1 = \sup_{w \in g(D_1)} \|B_1 w\|, \quad d = \sup_{w \in g(D_1)} \|D_1 w\|,$

$e_1 = \sup \{|C_1 w|/|A_1 w| : w \in g(D_1), v$ is a unit vector in $R^n\}$.

Notice if $A_1 w$ is invertible for all $w$, then $e_1 = \sup_{w \in g(D_1)} \|C_1 w A_1 w^{-1}\|$, and if $D_2$ is invertible for all $z$, then $e = \sup_{z \in D_2} \|B_z D_z^{-1}\|.$

Lemma 2. Suppose the hypotheses of Lemma (2.10) are satisfied and $g(D_1) \subset (\frac{1}{4} D^6) \times D^6$. Let $\eta < 1$, and let $z_1 = (x_1, y_1)$ be the fixed point of $g$ in $D_1$. Suppose,

1. $a/d + e + c/d < \eta,$
2. $a/d + 2c/d < 1,$
3. $d - c < 1,$

and

1. $d_1/a_1 + e_1 + b_1/a_1 < \eta,$
2. $d_1/a_1 + 2b_1/a_1 > 1,$
3. $a_1 - b_1 > 1.$
Then,

(4) \( z_1 \) is a hyperbolic fixed point of \( g \),
(5) \( W^u(z_1, g, D_1) = \{ z \in D_1 : g^{-n}(z) \in D_1 \text{ for all } n \geq 0 \text{ and } g^{-n}(z) \to z_1 \text{ as } n \to \infty \} \),
and \( W^s(z_1, g, D_1) = \{ z \in D_1 : g^n(z) \in D_1 \text{ for all } n \geq 0 \text{ and } g^n(z) \to z_1 \text{ as } n \to \infty \} \) are smooth manifolds.
(6) if \( d - c \) dist \((z_1, \text{ boundary } D_1) > \text{dist} \((z_1, D^s \times 0) \), then \( gW^u(z_1, g, D_1) \) has a unique point of transversal intersection with \( D^s \times 0 \),
(7) \( W^s(z_1, g, D_1) \) has a point of transversal intersection with \( 0 \times D^u \).

Proof. Applying Lemma 1 to \( T_{z_1}g \) and \( T_{z_1}g^{-1} \), we see that there are unique \( T_{z_1}g \) invariant subspaces \( E^u, E^s \) in \( R^{n+u} \) such that

(a) \( E^u \) is the graph of a linear function \( P^u \colon R^u \to R^s \) such that \( \|P^u\| < \eta \) and \( E^s \) is the graph of a linear function \( P^s \colon R^s \to R^u \) such that \( \|P^s\| < \eta \), and
(b) \( \|T_{z_1}g^{-1}E^u\| < 1, \|T_{z_1}gE^s\| < 1 \).

Now (4) follows easily from (a) and (b). Also (5) follows from the stable and unstable manifold theorem for the hyperbolic fixed point \( z_1 \).

Let \( \pi^u \colon D \to D^u, \sigma = s, u, \) denote the natural projections on \( D \).

To prove (6), we will show that

(c) \( gW^u(z_1, g, D_1) \) is the graph of a smooth function \( \varphi^u \colon \pi^u(gW^u(z_1, g, D_1)) \to D^u \) such that if \( \mathcal{L}(\varphi^u) \) is the Lipschitz constant of \( \varphi^u \), then \( \mathcal{L}(\varphi^u) < \eta < 1 \), and
(d) the center 0 of \( D^u \) is in \( \varphi^u \).

We first prove (c). Let \( z_1 = (x_1, y_1) \). From the unstable manifold theorem for the point \( z_1 \), if \( D_2 \) and \( D_2^u \) are small disks in \( R^u \) and \( R^u \) centered at \( x_1 \) and \( y_1 \), then
\[
W^u(z_1, g, D_2 \times D_2^u) = \bigcap_{n \geq 0} g^n(D_2 \times D_2^u) = \{ z \in D_2 \times D_2^u : g^{-n}(z) \to z_1 \} \text{ is a smooth manifold tangent to } E^u \text{ at } z_1.
\]
Thus, if \( D_2 \) and \( D_2^u \) are small enough, \( gW^u(z_1, g, D_2 \times D_2^u) \) is the graph of a smooth function \( \varphi_0 \colon \pi^u(gW^u(z_1, g, D_2 \times D_2^u)) \to D^u \) such that \( \mathcal{L}(\varphi_0) < \eta \).

Let \( W_0^u = W^u(z_1, g, D_2 \times D_2^u) \) and let \( W^u = W^u(z_1, g, D_1) \).

For \( i > 0 \), define \( W_i^u = g(W_{i-1}^u) \cap D_1 \). We claim, for each \( i > 0 \), \( gW_i^u \) is the graph of a smooth function \( \varphi_i \colon \pi^u(gW_i^u) \to D_i \) such that \( \mathcal{L}(\varphi_i) < \eta \), and there is an integer \( n_0 > 0 \) such that \( W^u_j = W^u_{n_0} = W^u \) for \( j \geq n_0 \). Once the claim is proved, (c) follows by taking the function \( \varphi_{n_0} \).

Suppose \( \varphi_i \colon \pi^u(gW_i^u) \to D_i \) has been defined such that graph \( (\varphi_i) = gW_i^u \) and \( \mathcal{L}(\varphi_i) < \eta < 1 \).

By (3), the mapping \( \psi_i \colon y \mapsto \pi^u(g(\psi_i(y)), y) \) is a uniform expansion and hence a diffeomorphism on \( \pi^u(gW_i^u \cap D_i^u) \) (recall \( D_i^u = \pi^uD_i \)). Further, \( \psi_i(\pi^u(gW_i^u \cap D_i^u)) = \pi^u(gW_{i+1}^u) \).

On \( \pi^u(gW_i^u \cap D_i^u) \), define \( \varphi_{i+1}(y) = g(\psi_i^{-1}(y)), \psi_i^{-1}(y) \) and observe that
\[
\text{graph } (\varphi_{i+1}) = gW_{i+1}^u.
\]

To prove \( \mathcal{L}(\varphi_{i+1}) < \eta \), it suffices to prove the following fact.
Let $v = (v^u, v^s)$ be a tangent vector to $g \mathcal{W}^u \cap D$ at $z = (\varphi_t(y), y)$ (so $v^s = \mathcal{T}_y \varphi_t(v^u)$) and $|v^s|/|v^u| < \eta < 1$. Then, letting $T_z g(v) = (v^1, v^2)$, we have $|v^1|/|v^2| < \eta$.

Indeed, $T_z g(v) = (v^1, v^2) = (A_z \mathcal{T}_y \varphi_t(v^u) + B_z v^u, C_z \mathcal{T}_y \varphi_t(v^u) + D_z v^u)$, so if $v^u = D_z^{-1} w$, then

$$
\frac{|v^1|}{|v^2|} \leq \frac{\|A_z\| \cdot (\|D_z^{-1}\| + \|B_z D_z^{-1}\|)}{1 - \|C_z\| \cdot (\|D_z^{-1}\|)} \leq \frac{a/d + e}{1 - c/d}
$$

since $\|T_y \varphi_t\| < 1$.

But, since $\eta < 1$, (1) implies that $(a/d + e)/(1 - c/d) < \eta$. Thus $\mathcal{L}(\varphi_{t+1}) < \eta$.

Now each map $\varphi_t$ is a uniform expansion (in fact

$$\inf \{|T_y \varphi_t(v^u)| : y \in \mathcal{W}^u \cap D, |v^u| = 1\} > d - c > 1$$,

$\pi^* D_1 \subset 1/4 D^u$ and $\pi^* g_{-1} D_1 \subset 1/4 D^s$, and $\mathcal{L}(\varphi_1) < 1$. Thus there is an integer $n_0 > 0$ such that $\mathcal{W}^u_{n_0} = D_1$. Further, if $n_0$ is the least such integer, then $W_{n_j} = W_{n_0}$ for $j \geq n_0$. Notice also that $W_{n_j} \subset W_{n_j+1} \subset \cdots$. Since $W^u \subset \bigcup_{n \geq 0} W^u_1$, we have that $W^u = W^u_{n_0}$. This completes the proof of (c).

For the proof of (d), since $\psi_{n_0}$ expands everywhere more than $d - c$ and $(d - c) \text{dist} (z_1, \text{boundary } D^s) > \text{dist} (z_1, D^s \times 0)$, it follows that the center $0$ of $D^u$ is in the image of $\psi_{n_0}$ which equals $\pi^* g W^u$. This completes the proof of (6). The proof of (7) is similar.

We now apply Lemma 2 to prove Theorem (2.6).

Let $p$ be the hyperbolic periodic point of the diffeomorphism $f$. Let $q$ be a transversal homoclinic point of $p$. Let $D = D^s \times D^u$ be a disk in $T_q M$ such that there is a diffeomorphism $g_1 : D \rightarrow M$ such that

1. $g_1(0) = q$.
2. $g_1(D^s \times 0) \subset W^s(q)$, $g_1(0 \times D^u) \subset W^u(q)$,
3. the manifolds $g_1(D^s \times y)$, $y \in D^u$, are $C^1$ close to each other, and the manifolds $g_1(x \times D^u)$, $x \in D^s$, are $C^1$ close to each other.

We claim if $D$ is small enough, there is a subdisk $D^s_1 \subset D^u$, an integer $N > 0$, and a diffeomorphism $g_2 : D \rightarrow M$ such that

4. $g_2(D^s \times 0) \subset W^s(q)$, $g_2(0 \times D^u) \subset W^u(q)$,
5. if $D_1 = D^s \times D^u_1$ and $g = g_2^{-1} f g_2 \mid D_1$, then $g$ satisfies the hypotheses of Lemma 2.

After this has been shown, Theorem (2.6) will be proved, for $g_2(z_1)$ will be a hyperbolic point of $f$ which is $h$-related to $p$.

Once $D$ is chosen small enough, the disk $D^s_1$, diffeomorphism $g_2$, and integer $N$ are constructed in a way similar to the analogous constructions in Theorem (3.1). One defines the $u$-submanifolds of $g_2$ ($\{g_2(x \times D^u) : x \in D^s\}$) and then the $s$-submanifolds $\{g_2(D^s \times y) : y \in D^u\}$ of $g_2$. We will indicate how to define the $u$-submanifolds of $g_2$.

Using the $\lambda$-lemma one can show that, given $\varepsilon > 0$, there is a small real number $\mu$ and an integer $N > 0$ such that there is a connected component $\Sigma_N$ of $f^N(g_1(D^s \times \mu D^u)) \cap g_1(D)$ satisfying
(6) the manifolds $\Sigma_{N,x} = f^N(g_1(x \times \mu D^u)) \cap \Sigma_N$ for $x \in D^s$ are $\epsilon$-$C^1$ close to $g_1(0 \times D^u)$, 
(7) the manifolds $f^{-N}(g_1(D^s \times y) \cap \Sigma_N)$ for $y \in D^u$ are $\epsilon$-$C^1$ close to $g_1(D^s \times 0)$, 
(8) $f^N$ expands a large amount on the manifolds $g_1(x \times \mu D^u) \cap f^{-N}(\Sigma_N) = f^{-N}(\Sigma_{N,x})$, $x \in D^s$, 
(9) $f^{-N}$ expands a large amount on the manifolds $g_1(D^s \times y) \cap \Sigma_N$, $y \in D^u$.

Now define a diffeomorphism $g_2^2 : D \to g_1(D)$ whose $u$-submanifolds are $\{g_1(x \times \mu D^u)\}$ and such that there is a subdisk $D^s_1 \subset D^u$ such that $g_2^2(D^s \times D^s_1) = f^{-N}(\Sigma_N)$.

Then define $g_2 : D \to g_1(D)$ as follows. Take the $u$-submanifolds of $g_2$ to be those of $g_2^2$. In $f^{-N}(\Sigma_N)$, take the $s$-submanifolds of $g_2$ to be 
$$\{f^{-N}(g_1(D^s \times y) \cap \Sigma_N) : y \in D^u\},$$ 
and off $f^{-N}(\Sigma_N)$ take them so that all of the $s$-submanifolds of $g_2$ are $\epsilon$-$C^1$ close to $g_1(D^s \times 0)$.

The computations needed to prove that $\epsilon$, $D_1$, $N$, and $g_2$ can be taken so that the hypotheses of Lemma 2 are satisfied for $g = g_2^{-1}f^Ng_2|_{D_1}$ are similar to those in the proof of Theorem (3.1) and will be left to the reader.

Remark. Here again, as in the proof of (3.1), one cannot use the weaker version of Lemma (2.10) in which $e$ is replaced by $\sup_{x \in D_1} \{\|B_x\| \cdot D^{-1}_x\}$ (see Remark (2.11)).

References


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