MAPPINGS FROM 3-MANIFOLDS ONTO 3-MANIFOLDS(*)

BY

ALDEN WRIGHT

Abstract. Let \( f \) be a compact, boundary preserving mapping from the 3-manifold \( M^3 \) onto the 3-manifold \( N^3 \). Let \( \mathbb{Z}_p \) denote the integers mod a prime \( p \), or, if \( p = 0 \), the integers. (1) If each point inverse of \( f \) is connected and strongly 1-acyclic over \( \mathbb{Z}_p \), and if \( M^3 \) is orientable for \( p > 2 \), then all but a locally finite collection of point inverses of \( f \) are cellular. (2) If the image of the singular set of \( f \) is contained in a compact set each component of which is strongly acyclic over \( \mathbb{Z}_p \), and if \( M^3 \) is orientable for \( p \neq 2 \), then \( N^3 \) can be obtained from \( M^3 \) by cutting out of \( \text{Int} \ M^3 \) a compact 3-manifold with 2-sphere boundary, and replacing it by a \( \mathbb{Z}_p \)-homology 3-cell. (3) If the singular set of \( f \) is contained in a 0-dimensional set, then all but a locally finite collection of point inverses of \( f \) are cellular.

I. Introduction. We suppose throughout the introduction that \( f: M^3 \rightarrow N^3 \) is a compact, boundary preserving mapping from the 3-manifold \( M^3 \) onto the 3-manifold \( N^3 \) (where \( M^3 \) and \( N^3 \) may or may not have boundary). Let \( \mathbb{Z}_p \) denote the integers modulo a prime \( p \), or, if \( p = 0 \), the integers.

If \( f^{-1}(x) \) is connected and strongly 1-acyclic over \( \mathbb{Z}_p \) for all \( x \in N^3 \), and if \( M^3 \) is orientable for \( p > 2 \), then in Corollary 1 it is shown that all but a locally finite collection of point inverses are cellular. This implies that \( N^3 \) can be obtained from \( M^3 \) by cutting out of \( \text{Int} \ M^3 \) a locally finite collection of compact 2-manifolds, each bounded by a 2-sphere, and replacing them by a 3-cell (see Corollary 3). Thus, if \( M^3 \) is compact, \( N^3 \) is a factor in a connected sum decomposition of \( M^3 \).

Now suppose that the image of the singular set of \( f \) is contained in a compact set \( X \) each component of which is strongly acyclic over \( \mathbb{Z}_p \). If \( M^3 \) is orientable for \( p \neq 2 \), then \( N^3 \) can be obtained from \( M^3 \) by cutting out of \( M^3 \) a finite number of compact 3-manifolds, each bounded by a 2-sphere, and replacing each by a \( \mathbb{Z}_p \)-homology 3-cell. In particular, if \( X \) has a neighborhood which is an irreducible 3-manifold with boundary (or if \( N^3 \) is irreducible), then \( N^3 \) is a factor in a connected sum decomposition of \( M^3 \). This extends Theorem 1 of Lambert in [9]. In the special case where the image of the singular set is contained in a Cantor set,
we can say in addition that all but a finite number of point inverses are cellular. This was previously proved by the author using other techniques.

Lemma 5 restates one of Armentrout’s results on approximating cellular maps with homeomorphisms. Using this lemma, we combine the results of Theorems 1 and 3 in Theorem 5. Thus if $M^3$ is compact and orientable for $p 
eq 2$, and if the image of the point inverses of $f$ which are not connected and strongly 1-acyclic over $\mathbb{Z}_p$ is contained in a compact set $X$ each component of which is strongly acyclic over $\mathbb{Z}_p$, then $N^3$ can be obtained from $M^3$ by cutting out of $\text{Int } M^3$ a finite number of 3-manifolds each bounded by a 2-sphere, and replacing each by a $\mathbb{Z}_p$-homology 3-cell. Theorem 6 combines Theorems 1 and 4 in a similar fashion.

In Theorem 7, we extend a result of McMillan [13] to show that if the image of the singular set of $f$ is contained in a (nonclosed) 0-dimensional set, then all but a locally finite collection of point inverses are cellular.

Let $G$ be a nontrivial abelian group. A compact set $X \subset M$ is strongly $k$-acyclic over $G$ if for each open set $U \subset M$ containing $X$, there is an open set $V$ such that $X \subset V \subset U$ and such that the inclusion induced homomorphism $i_*: H_k(V; G) \to H_k(U; G)$ is zero. (If $X$ is connected and strongly $k$-acyclic over $G$ for $1 \leq k \leq n$, then $X \subset M$ has property $uv_k(G)$ in the sense of [8].) The compact set $X \subset M$ is strongly acyclic over $G$ if it is connected and strongly $k$-acyclic over $G$ for all $k \geq 1$.

We refer the reader to [13 (especially Lemma 1)] for further facts about strong acyclicity. In particular, for any positive integer $k$, a compact set $X$ in the interior of a 3-manifold $M^3$ is strongly $k$-acyclic over $G$ if and only if each component of $X$ is strongly $k$-acyclic over $G$. Also $X$ is strongly acyclic over $Z$ if and only if $X$ is connected and $H^*(X; Z) = 0$ (see [7]).

The compact set $X \subset M$ has property $uv^\infty$ if for each open set $U \subset M$ containing $X$, there is an open set $V$ such that $X \subset V \subset U$ and such that $V$ is contractable in $U$. A set $X$ in a 3-manifold $M^3$ is cellular in $M^3$ if $X = \bigcap_{i=1}^n F_i$ where each $F_i$ is a 3-cell, and $F_{i+1} \subset \text{Int } F_i$ for all $i$.

If $a$ is a loop in a space $M$, we will denote its homology class in $H_1(M; G)$ by $[a]$. The symbol $\mathbb{Z}_p$ for $p > 0$ will denote the finite cyclic group of order $p$. The symbol $\mathbb{Z}_0$ will denote the integers.

A manifold will be assumed to be connected and to have no boundary unless otherwise specified. We assume that all manifolds have a piecewise-linear structure. A 3-manifold is irreducible if every polyhedral 2-sphere in it bounds a polyhedral 3-cell. If $M^3$ and $N^3$ are 3-manifolds, possibly with boundary, the connected sum $M^3 \# N^3$ of $M^3$ and $N^3$ is obtained by removing the interior of a 3-cell from the interior of each, and then sewing the two manifolds together along the resulting boundary components, using an orientation reversing homeomorphism if $M^3$ and $N^3$ are oriented.

A map or mapping is a continuous function. A monotone map is a map all of whose point inverses are connected. A map $f: M \to N$ is compact (proper) if, for any compact set $K$ in $N$, $f^{-1}(K)$ is compact. If $f: M \to N$ is a compact monotone
map, then the point inverses of $M$ form a monotone upper semicontinuous decomposition of $M$ whose associated decomposition space is homeomorphic to $N$. Conversely, if $G$ is a monotone upper semicontinuous decomposition of $M$, the projection map $p: M \to M/G$ is a compact monotone map.

Let $\{X_a\}_{a \in A}$ be a collection of compact subsets of a space $M$. Then $\{X_a\}_{a \in A}$ is a locally finite collection if for $y \in M$, $y$ has a neighborhood $U$ which intersects only a finite number of elements of the collection.

11. Maps all of whose point inverses are strongly acyclic.

**Lemma 1.** If $X$ is a compact connected subset of a space $M$ and if $X$ is strongly $k$-acyclic over $\mathbb{Z}$ in $M$ for $1 \leq k \leq n$, then $X$ is strongly $k$-acyclic over $\mathbb{Z}_p$ in $M$ for $1 \leq k \leq n$ and for any prime $p > 1$.

**Proof.** Let $W$ and $V$ be chosen so that $X \subseteq W \subseteq V \subseteq U$ and so that the inclusion induced homomorphisms $i_*: H_k(V; \mathbb{Z}) \to H_k(U; \mathbb{Z})$ and $j_*: H_k(W; \mathbb{Z}) \to H_k(V; \mathbb{Z})$ are zero for $1 \leq k \leq n$. Consider the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & H_k(W; \mathbb{Z}) \otimes \mathbb{Z}_p & \to & H_k(W; \mathbb{Z}_p) & \to & \text{Tor}_1(H_{k-1}(W; \mathbb{Z}), \mathbb{Z}_p) & \to & 0 \\
& & \downarrow i_* \otimes \text{id} & \downarrow i_* & & & & \downarrow \text{id} & \\
0 & \to & H_k(V; \mathbb{Z}) \otimes \mathbb{Z}_p & \to & H_k(V; \mathbb{Z}_p) & \to & \text{Tor}_1(H_{k-1}(V; \mathbb{Z}), \mathbb{Z}_p) & \to & 0 \\
& & \downarrow j_* \otimes \text{id} & \downarrow j_* & & & & \downarrow \text{id} & \\
0 & \to & H_k(U; \mathbb{Z}) \otimes \mathbb{Z}_p & \to & H_k(U; \mathbb{Z}_p) & \to & \text{Tor}_1(H_{k-1}(U; \mathbb{Z}), \mathbb{Z}_p) & \to & 0 \\
\end{array}
\]

The horizontal rows, which are exact, are from the universal coefficient theorem. By our choice of $W$ and $V$, the outer vertical maps are zero. Using a diagram chasing argument, we see that $j_*i_*$ is the zero homomorphism.

**Lemma 2.** Let $M^3$ and $N^3$ be 3-manifolds, and let $f: M^3 \to N^3$ be a compact, monotone, onto map. Let $p$ be 0 or a prime, and suppose $M^3$ is orientable if $p \neq 2$. If $f^{-1}(y)$ is strongly 1-acyclic over $\mathbb{Z}_p$ for every $y \in N^3$, then each $f^{-1}(y)$ is strongly acyclic over $\mathbb{Z}_p$ in $M^3$.

**Proof.** By Alexander duality and Theorem 3 of [8] we see that $H^k(f^{-1}(y); \mathbb{Z}_p) = 0$ for $k \geq 2$. Then the continuity of $H^*$ and the universal coefficient theorem for cohomology show that $f^{-1}(y)$ is strongly acyclic over $\mathbb{Z}_p$ for all $y \in N^3$. (For more details, see Theorems 4.4 and 3.2 of [7].)

**Lemma 3.** Let $M^3$ and $N^3$ be 3-manifolds, and let $f: M^3 \to N^3$ be a compact, monotone, onto map such that $f^{-1}(y)$ is strongly 1-acyclic over $G$ for each $y \in N^3$. If $H_1(N^3; G) = 0$, then $H_1(M^3; G) = 0$.

The proof of Lemma 3 is similar to the proof of Theorem 2.1 of [15].
If $M^n$ and $N^n$ are $n$-manifolds with boundary, a map $f: M^n \to N^n$ is said to be boundary preserving if $f|\partial M^n$ is a homeomorphism of $\partial M^n$ onto $\partial N^n$, and if $f^{-1}(\partial N^n) = \partial M^n$. A 2-manifold with boundary $S$ is properly embedded in a 3-manifold with boundary $M^3$ if $S \cap \partial M^3 = \partial S$.

A $Z_p$-homology (homotopy) 3-cell is a compact $Z_p$-acyclic (contractible) 3-manifold with boundary. A cube-with-handles is obtained by adding orientable 1-handles to a 3-cell. We define a $Z_p$-homology (homotopy) cube-with-handles similarly. We will say that a set $X$ is the intersection of a decreasing sequence of ($Z_p$-homology, homotopy) cubes-with-handles if $X = \bigcap_{i=1}^{\infty} K^3_i$ where each $K^3_i$ is a ($Z_p$-homology, homotopy) cube-with-handles and $K^3_{i+1} \subset \text{Int} K^3_i$.

**Theorem 1.** Let $p$ denote 0 or a prime, and let $M^3$ and $N^3$ be compact 3-manifolds, possibly with boundary, where $M^3$ is orientable if $p > 2$. Let $f: M^3 \to N^3$ be a monotone, onto, boundary preserving map. Let $U$ be an open subset of $N^3$. If $f^{-1}(x)$ is strongly 1-acyclic over $Z_p$ for all $x \in U$, then $\{x \in U : f^{-1}(x) \text{ is not cellular} \}$ is a finite set.

**Remark.** This theorem was first proved for $p = 0, 2$ in [16]. It has since been generalized by D. R. McMillan in [13].

**Proof.** The case where $p = 0$ reduces to the case where $p = 2$ by Lemma 1. By the proofs of Theorems 1 and 2 of [11] and by Kneser's Theorem [6] it is sufficient to prove that $\{x \in U : f^{-1}(x) \text{ is not } U\circ\alpha \}$ is finite.

We can apply Lemma 2 to see that $f^{-1}(x)$ is strongly acyclic over $Z_p$ for each $x \in U$. By Theorem 2 of [12], $f^{-1}(x)$ is the intersection of a decreasing sequence of $Z_p$-homology cubes-with-handles.

Let $q$ be the rank (i.e. the minimum number of generators) of $\pi_1(M^3)$. By a corollary to the Grushko-Neumann Theorem (p. 192 of [10]), there are at most $q$ disjoint $Z_p$-homology 3-cells in $M^3$ which are not homotopy 3-cells. Thus there are at most $q$ points in $U$ whose inverse images are not the intersection of a decreasing sequence of homotopy cubes-with-handles.

Let $x \in U$, where $f^{-1}(x)$ is the intersection of a decreasing sequence of homotopy cubes-with-handles. We will complete the proof by showing that $f^{-1}(x)$ is $U\circ\alpha$. Let $U'$ be an open set in $M^3$ containing $f^{-1}(x)$. There is a homotopy cube-with-handles $H^3$ such that

$$f^{-1}(x) \subset \text{Int } H^3 \subset H^3 \subset U' \cap f^{-1}(U).$$

Let $W$ be an open 3-cell in $U$ such that $x \in W$ and $f^{-1}(W) \subset \text{Int } H^3$. Define inductively $G_0, G_1, G_2, \ldots$ by letting $G_0 = \pi_1(f^{-1}(W))$, and by letting

$$G_i = G_{i-1}(X_1X_2X_1^{-1}X_2^{-1}X_3^p).$$

(See p. 74 of [10] for notation.) In other words, $G_i$ is the subgroup of $G_{i-1}$ generated by all elements of the form $uwv^{-1}t^{-1}$ where $u, v, t \in G_{i-1}$. Let $F_0, F_1, F_2, \ldots$ be the corresponding subgroups of $\pi_1(H^3)$. 

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The subgroup $G_1$ certainly contains the commutator subgroup of $G_0$. The image of $G_1$ in $H_1(f^{-1}(W); Z)$ is $p \cdot H_1(f^{-1}(W); Z)$. Thus

$$\pi_1(f^{-1}(W))/G_1 \cong H_1(f^{-1}(W); Z)/p \cdot H_1(f^{-1}(W); Z) \cong H_1(f^{-1}(W); Z_p).$$

Let $\delta \in \pi_1(f^{-1}(W))$. Since $H_1(f^{-1}(W); Z_p) = 0$ (by Lemma 3), $\delta \in G_1$. Thus $\delta$ is a product of elements of the form $uw^{-1}r^{-1}r^p$ where $u, r, \tau \in G_0$. By applying the same argument to $u, r,$ and $\tau$, we see that $u, r, \tau \in G_1$. Thus $\delta \in G_2$. By repeating this argument, $\delta \in \bigcap_{i=0}^{\infty} G_i$. By Corollary 2.12 on p. 109 of [10], $\bigcap_{i=1}^{\infty} F_i = 1$. Thus $\delta = 1$ in $\pi_1(H^3)$, and $f^{-1}(x)$ is $UV^\infty$.

**Corollary 1.** Let $M^3$ and $N^3$ be 3-manifolds, possibly with boundary, and let $f: M^3 \to N^3$ be a compact, monotone, boundary preserving, onto map. Let $p$ denote 0 or a prime, and suppose that $M^3$ is orientable if $p > 2$. If $f^{-1}(x)$ is strongly 1-acyclic over $Z_p$ in $M^3$ for all $x \in U$, then $\{x \in U : f^{-1}(x)$ is not cellular $\}$ is a locally finite set in $N^3$.

**III. Maps where the image of the singular set lies in a strongly acyclic set.** We state below a slightly strengthened version of Theorem 2 of [13]: here we assume that $M^3$ is orientable only if $p > 2$, and thus the 1-handles which are attached to Bd $Q_i$ to obtain $H_i$ may be attached in a nonorientable fashion. (See the statement of Theorem 2 for the definition of $Q_i$ and $H_i$.) The only additional difficulty in the proof is when we have $S_i \subset$ Bd $Z^*_p$ and $S_k \subset$ Bd $Z^*_p$ topologically parallel. (See p. 133 of [12].) As before, each loop in $S_i$, $Z_p$-bounds in $Z^*_p$, and the same argument shows that $S_i$ is a 2-sphere if $S_i$ is not homeomorphic to a projective plane. But if $S_i$ is a projective plane, it must contain an orientation-reversing simple closed curve since $S_i$ is two-sided. This contradicts the fact that every simple closed curve in $S_i$, $Z_p$-bounds in $Z^*_p$, since $p = 0, 2$.

**Theorem 2.** Let $p$ denote 0 or a prime. Let $X$ be a compact, proper subset of Int $M^3$, where $M^3$ is a 3-manifold, possibly with boundary. Suppose $M^3$ is orientable if $p > 2$, and suppose that $X$ has the following property relative to $M^3$ and $p$. For each open set $U \subset M^3$ with $X \subset U$, there is an open set $V$, $X \subset U \subset V \subset U$, such that, under inclusion, $H_1(V - X; Z_p) \to H_1(U; Z_p)$ is zero. Then $X = \bigcap_{i=1}^{\infty} H_i$, where $H_i$ is a compact polyhedron in $M^3$, each component of $H_i$ is a 3-manifold with nonempty boundary, $H_{i+1} \subset$ Int $H_i$ and each $H_i$ has the following structure: it is obtained from a compact polyhedron $Q_i$, each component of which is a 3-manifold whose boundary consists entirely of 2-spheres, by adding to Bd $Q_i$ a finite number of (solid, possibly nonorientable) 1-handles.

Let $f: M \to N$ be a map. Then let $S_f = \{x \in M : f^{-1}(x) \text{ is nondegenerate}$.}

**Theorem 3.** Let $p$ denote 0 or a prime. Let $M^3$ and $N^3$ be piecewise-linear 3-manifolds, possibly with boundary, where $M^3$ is orientable if $p \neq 2$. Let $X$ be a compact subset of Int $N^3$ such that each component of $X$ is strongly acyclic over $Z_p$. Let $f: M^3 \to N^3$ be a compact, boundary preserving map with $f(S_f) \subset X$. Then $N^3$ can
be obtained from $M^3$ by cutting out of $\text{Int } M^3$ a finite number of polyhedral 3-manifolds which are each bounded by a 2-sphere, and replacing each by a polyhedral $\mathbb{Z}_p$-homology 3-cell.

**Proof.** By Theorem 2 of [12], $X$ is the intersection of a decreasing sequence of $\mathbb{Z}_p$-homology cubes-with-handles. Thus we can assume that $N^3$ is a $\mathbb{Z}_p$-homology cube-with-handles, and that each two-sided surface in $\text{Int } N^3$ separates $N^3$.

The first half of the proof will be to show that $f^{-1}(X)$ has the following property in $\text{Int } M^3$: for each open set $U \subset \text{Int } M^3$ with $f^{-1}(X) \subset U$, there is an open set $V$, with $f^{-1}(X) \subset V \subset U$, such that, under inclusion, $H_1(V - f^{-1}(X); \mathbb{Z}_p) \to H_1(U; \mathbb{Z}_p)$ is zero.

Let $U$ be an open set in $\text{Int } M^3$ with $f^{-1}(X) \subset U$. Since $\text{Cl}(S_i) \subset U$, $f(U)$ is open. Let $Z^3$ be a compact polyhedron in $f(U)$ such that each component of $Z^3$ is a 3-manifold with boundary, and such that $X \subset \text{Int } Z^3$. Since $X$ is strongly 1-acyclic over $\mathbb{Z}_p$, there is an open set $W$ containing $X$ such that, under inclusion

$$H_1(W - X; \mathbb{Z}_p) \to H_1(Z^3; \mathbb{Z}_p)$$

is zero.

Let $V = f^{-1}(W)$, and let $[\sigma] \in H_1(V - f^{-1}(X); \mathbb{Z}_p)$ where we can assume that $\sigma$ is a finite, pairwise disjoint collection of (oriented, if $p \neq 2$) simple closed curves such that $f(\sigma)$ is polyhedral in $Z^3$. Let $F^3$ be a regular neighborhood of $f(\sigma)$ in $(\text{Int } Z^3) - X$. We can triangulate $Z^3$ so that $F^3$ and $f(\sigma)$ are subcomplexes of the triangulation. Then the homeomorphism $f^{-1}|(\text{Bd } Z^3 \cup F^3)$ induces a triangulation of $f^{-1}(\text{Bd } Z^3 \cup F^3)$. Since each of the finite number of components of $f^{-1}(Z^3)$ is a 3-manifold with boundary, by Theorem 5 of [2] there is a triangulation of $f^{-1}(Z^3)$ which is compatible with the above triangulation of $f^{-1}(\text{Bd } Z^3 \cup F^3)$. Using the relative simplicial approximation theorem, there is a piecewise-linear, nondegenerate map $g$ from $f^{-1}(Z^3)$ onto $Z^3$ such that

$$g|f^{-1}(\text{Bd } Z^3 \cup F^3) = f|f^{-1}(\text{Bd } Z^3 \cup F^3),$$
$$g^{-1}(\text{Bd } Z^3 \cup F^3) = f^{-1}(\text{Bd } Z^3 \cup F^3).$$

By subdividing we can assume that $g$ is simplicial.

At this point we divide the remainder of the first half of the proof into three cases: Case 1 ($p = 0$), Case 2 ($p = 2$), and Case 3 ($p > 2$).

**Case 1 ($p = 0$).** Since $f(\sigma) \subset W - X$, $[f(\sigma)] = 0$ in $H_1(Z^3; \mathbb{Z})$. Thus $f(\sigma)$ must bound a 2-complex $L^2$ in $Z^3$ where each component of $L^2$ is an orientable, two-sided 2-manifold with boundary. We can adjust $L^2$ slightly so that it is in general position mod $f(\sigma)$ with respect to our last triangulation of $Z^3$. Then $g^{-1}(L^2)$ will be a 2-complex in $f^{-1}(Z^3) \subset U$, where each component of $g^{-1}(L^2)$ is a two-sided 2-manifold with boundary. Thus, since $M^3$ is orientable, each component of $g^{-1}(L^2)$ is orientable. Since $\sigma$ bounds $g^{-1}(L^2)$, $[\sigma] = 0$ in $H_1(U; \mathbb{Z})$, and the inclusion-induced homomorphism $H_1(V - f^{-1}(X); \mathbb{Z}) \to H_1(U; \mathbb{Z})$ is trivial.
Case 2 \((p=2)\). The proof is essentially the same as Case 1, except that \(L^2\) and \(g^{-1}(L^2)\) may not be orientable.

Case 3 \((p > 2)\). Note that
\[
H_1(Z^3; \mathbb{Z})/G \cong H_1(Z^3; \mathbb{Z}) \otimes \mathbb{Z}_p \cong H_1(Z^3; \mathbb{Z}_p)
\]
where \(G\) is the subgroup of \(H_1(Z^3; \mathbb{Z})\) generated by elements of the form \(p[y]\) where \([y] \in H_1(Z^3; \mathbb{Z})\). Since \([f(\alpha)] = 0\) in \(H_1(Z^3; \mathbb{Z}_p)\), there is a 1-cycle \([\tau] \in H_1(Z^3; \mathbb{Z})\) so that \([f(\alpha)] = p[\tau]\) in \(H_1(Z^3; \mathbb{Z})\). We can assume that \(\tau\) is a finite, pairwise disjoint collection of polyhedral, oriented, simple closed curves which are in general position with respect to our last triangulation of \(Z^3\). Then \(g^{-1}(\tau)\) is a finite, pairwise disjoint collection of simple closed curves in \(f^{-1}(Z^3)\). We can find a regular neighborhood \(T^3\) of \(\tau\) so close to \(\tau\) that \(g^{-1}(T^3)\) is a regular neighborhood of \(g^{-1}(\tau)\). We can find a 1-cycle \([\delta] \in H_1(Bd T^3; \mathbb{Z})\) so that \([f(\alpha)] = p[\delta]\) in \(H_1(Z^3 - \text{Int } T^3; \mathbb{Z})\). We can assume that \(\delta\) is a finite collection of mutually exclusive, oriented, simple closed curves on \(Bd T^3\). Then there is a 2-complex \(L^2 \subset Z^3 - \text{Int } T^3\) where each component of \(L^2\) is a two-sided, orientable, 2-manifold, and where \(Bd L^2 = f(\epsilon) \cup \delta\) (homologically \(f(\alpha) - \delta\)). We can assume that \(L^2\) is in general position mod \(f(\alpha)\) with respect to our last triangulation of \(Z^3\). Then \(g^{-1}(L^2)\) will be a 2-complex where each component of \(g^{-1}(L^2)\) is a two-sided 2-manifold with boundary. Thus \(g^{-1}(L^2)\) is orientable.

Since \(L^2\) is two-sided in \(Z^3\), \(\delta\) is two-sided in \(Bd T^3\). Thus \(g^{-1}(\delta)\) is two-sided in \(g^{-1}(Bd T^3)\), and using this two-sidedness, we can induce an orientation of \(g^{-1}(\delta)\) which is consistent with that on \(g^{-1}(L^2)\). Thus \([g^{-1}(\delta)] = [\alpha]\) in \(H_1(f^{-1}(Z^3); \mathbb{Z})\).

Let \(\alpha\) be a meridional curve on \(Bd T^3\) which is in general position with respect to \(\delta\). Then \(\alpha\) will intersect \(\delta\) algebraically \(\pm p\) times. Since the two-sidedness of \(\delta\) is preserved by \(g^{-1}\), each component of \(g^{-1}(\alpha)\) which is a meridional curve must intersect \(g^{-1}(\delta)\) algebraically \(\pm p\) times. Thus, \([g^{-1}(\delta)] = p[g^{-1}(\tau)]\) in \(H_1(T^3; \mathbb{Z})\).

Therefore, \([\alpha] = p[g^{-1}(\tau)]\) in \(H_1(Z^3; \mathbb{Z})\), and the inclusion-induced homomorphism \(H_1(V - X; \mathbb{Z}_p) \rightarrow H_1(U; \mathbb{Z}_p)\) is trivial. This completes Case 3.

By Theorem 2, we can find a compact polyhedron \(H_0^3\), where each component of \(H_0^3\) is a 3-manifold with nonempty boundary, and where \(H_0^3\) has the following structure: it is obtained from a compact polyhedron \(Q_0^3\), each component of which is a 3-manifold whose boundary consists entirely of 2-spheres, by adding to \(Bd Q_0^3\) a finite number of (solid, possibly nonorientable) 1-handles.

We can also assume that each 1-handle is attached to only one boundary component of \(Bd Q_0^3\) since we can add 1-handles to \(Bd Q_0^3\) which join different components of \(Bd Q_0^3\) without destroying the property that \(Bd Q_0^3\) consists entirely of 2-spheres.

We claim that each component of \(Bd Q_0^3\) separates \(M^3\). For suppose that \(S_0\) is a component of \(Bd Q_0^3\) that does not separate \(M^3\). Then there is a polyhedral simple closed curve \(J\) which intersects \(S_0\) at exactly one point which is a piercing point. It is easy to see that we can choose \(J\) so that it does not intersect any of the 1-handles.
which are added to $Q_0^3$ to obtain $H_0^3$. Let $S_1$ be the component of $\text{Bd } H_0^3$ which is obtained from $S_0$ by adding handles. Then $J$ intersects $S_1$ only in the same piercing point. Since $f^{-1}|f(\text{Bd } H_0^3)$ is a homeomorphism, $f(J)$ is a loop in $N^3$ which intersects $f(S_1)$ in exactly one piercing point. Thus $f(S_1)$ does not separate $N^3$. But $f(S_1)$ is a 2-sided surface in $N^3$, so $f(S_1)$ must separate $N^3$. This is a contradiction, so $S_0$ does separate $M^3$.

Let $Q^3$ be the closure of the “inside” complementary domains of the “outermost” boundary components of $Q_0^3$. (Here, “inside” and “outermost” are relative to $\text{Bd } M^3$, which is connected.) Thus we have “filled in the holes” in $Q_0^3$ to obtain $Q^3$, and each component of $Q^3$ has connected boundary. We define $H^3$ to be $Q^3$ union the 1-handles of $H_0^3 - Q_0^3$ which are not already contained in $Q_3$.

There are properly embedded polyhedral disks $B_1^3, \ldots, B_r^3$ in $H^3$ such that the 1-handles which are added to $Q^3$ to obtain $H^3$ are regular neighborhoods of $B_1^3, \ldots, B_r^3$ in $H^3$. Let these 1-handles be $N(B_1^3), \ldots, N(B_r^3)$. Since $S_i \subset f^{-1}(X) \subset \text{Int } H^3$, each component of $f(H^3)$ is a 3-manifold with boundary in $\text{Int } N^3$. Each $B_i^3$ is mapped properly into $f(H^3)$ by $f$, and furthermore, $f|B_i^3$ has no singularities near $\text{Bd } B_i^3$. So by Dehn’s Lemma, there exist nonsingular properly embedded polyhedral disks $D_1^3, \ldots, D_r^3$ in $f(H^3)$ with $\text{Bd } D_i^3 = f(\text{Bd } B_i^3)$. By a cutting and pasting argument, we can choose $D_1^3, \ldots, D_r^3$ to be disjoint. We can also find disjoint regular neighborhoods $N(D_1^3), \ldots, N(D_r^3)$ of $D_1^3, \ldots, D_r^3$ in $f(H^3)$ so that $\quad f(N(B_i^3) \cap \text{Bd } H^3) = N(D_i^3) \cap \text{Bd } f(H^3).$

For each $i$, there is a homeomorphism $h_i: N(B_i^3) \rightarrow N(D_i^3)$ such that $\quad h_i| (\text{Bd } H^3 \cap N(B_i^3)) = f| (\text{Bd } H^3 \cap N(B_i^3)).$

We define a homeomorphism $\quad h: M^3 - \text{Int } Q^3 \rightarrow (N^3 - \text{Int } f(H^3)) \cup \left( \bigcup_{i=1}^r N(D_i^3) \right)$

by $h|(M^3 - \text{Int } H^3) = f|(M^3 - \text{Int } H^3)$, and by $h|N(B_i^3) = h_i$ for each $i = 1, \ldots, r$.

Then $h(\text{Bd } Q^3)$ is a finite disjoint collection of 2-spheres in $N^3$ each of which bounds a $Z_p$-homology 3-cell. Furthermore, these homology 3-cells are disjoint since each component of $h(\text{Bd } Q^3)$ is outermost in the sense that it can be joined to $\text{Bd } N^3$ with an arc which misses $h(\text{Bd } Q^3)$ except at one end point.

Let $K_1^3, \ldots, K_r^3$ be these homology 3-cells, and let $Q_1^3, \ldots, Q_m^3$ be the corresponding components of $Q^3$ so that $h^{-1}(\text{Bd } K_i^3) = \text{Bd } Q_i^3$. Each $Q_i^3$ is a 3-manifold with 2-sphere boundary. Then $h$ is a homeomorphism from $M^3 - (\bigcup_{i=1}^r Q_i^3)$ onto $N^3 - (\bigcup_{i=1}^m K_i^3)$. Thus we obtain $N^3$ from $M^3$ by cutting out the $Q_i^3$'s and replacing each with the corresponding $K_i^3$.

**Remark.** If we define $*Q_i^3$ to be the closed 3-manifold obtained from $Q_i^3$ by sewing a 3-cell onto $\text{Bd } Q_i^3$, and if we define $*K_i^3$ to be the closed 3-manifold obtained from $K_i^3$ in the same way, then $\quad M^3 \# *K_1^3 \# \cdots \# *K_r^3 \cong N^3 \# *Q_1^3 \# \cdots \# *Q_m^3.$
We should also note that we have shown that for any open set \( U \) in \( M^3 \) which contains \( X \), then \( f^{-1}(X) \) has a polyhedral neighborhood \( H^3 \subseteq U \) where each component of \( H^3 \) is formed by adding 1-handles to a 3-manifold with 2-sphere boundary. Furthermore, we have shown that these 1-handles are attached in an orientable fashion to the 2-sphere boundary.

**Corollary 2.** Let \( M^3 \) and \( N^3 \) be compact 3-manifolds, possibly with boundary. Let \( X \) be a compact proper set in \( \text{Int } N^3 \) with the following property: For each open set \( U \subseteq \text{Int } N^3 \) with \( X \subseteq U \), there is an open set \( V, X \subseteq V \subseteq U \), such that under inclusion \( H_1(V-X; Z_p) \to H_1(V; Z_p) \) is zero. Suppose also that \( X \) has a polyhedral neighborhood each component of which is an orientable, irreducible 3-manifold with boundary. If there is a boundary preserving map \( f \) from \( M^3 \) onto \( N^3 \) such that \( f(S_f) \subseteq X \), then \( M^3 \) can be obtained from \( N^3 \) by removing the interiors of a finite number of 3-manifolds each of which is bounded by a 2-sphere, and by replacing each by a 3-cell.

**Proof.** By using Theorem 2 and the fact that \( X \) has a polyhedral neighborhood each component of which is an irreducible 3-manifold with boundary, we see that \( X \) has a polyhedral neighborhood each component of which is a cube-with-handles. Thus we can assume that \( N^3 \) is a cube-with-handles. The remainder of the proof of Theorem 3 now goes through with the weaker hypothesis on \( X \).

**Theorem 4.** Let \( M^3 \) and \( N^3 \) be 3-manifolds, possibly with boundary, and let \( f: M^3 \to N^3 \) be an onto, compact, boundary preserving mapping from \( M^3 \) onto \( N^3 \) such that \( f(S_f) \subseteq X \) where \( X \) is a closed 0-dimensional set in \( N^3 \). Then \( f \) is monotone, and \( \{ x \in N^3 : f^{-1}(x) \) is not cellular in \( M^3 \} \) is a locally finite subset of \( N^3 \).

**Proof.** Let \( x \in X \), and let \( U \) be an arbitrarily small open 3-cell containing \( x \). Then there is a polyhedral 3-manifold with boundary \( K^3 \) so that \( x \in \text{Int } K^3 \subseteq K^3 \subseteq U \) and so that \( \text{Bd } K^3 \cap X = \emptyset \). In fact, using Theorem 2 of [12] and the fact that \( U \) is irreducible, we can see that \( K^3 \) can be chosen to be a cube-with-handles. Then \( f^{-1}(K^3) \) is a connected neighborhood of \( f^{-1}(x) \) which can be chosen "arbitrarily close" to \( f^{-1}(x) \). Thus \( f \) is monotone.

We can cover \( X \) with the interiors of a locally finite collection of mutually exclusive collection of cubes-with-handles. Thus, in order to prove the theorem, it suffices to consider the case where \( N^3 \) is a cube-with-handles, and where \( M^3 \) is a compact 3-manifold with connected boundary. In this case, we will prove that all but a finite number of point inverses of \( f \) are cellular.

The set \( X \) is strongly 1-acyclic over \( Z_2 \) in \( N^3 \), and thus by the remark following the proof of Theorem 3, we have \( f^{-1}(X) = \bigcap_{i=1}^{\infty} H^3_i \), where \( H^3_i \) is a 3-manifold with connected boundary, and where \( H^3_i \subseteq \text{Int } H^3_{i-1} \). We can assume that \( H^3_i \) is obtained from a compact polyhedron \( Q_i^3 \) where each component of \( Q_i^3 \) is a 3-manifold with 2-sphere boundary, by adding to \( \text{Bd } Q_i^3 \) a finite number of (orientable, solid) 1-handles. We also have that each 1-cycle in \( \text{Bd } H^3_i \) bounds in \( \text{Int } H^3_{i-1} \). We have assumed that \( M^3 \) is compact and that \( H_1(M^3; Z_2) \) is finitely generated;
so it is easy to show that there is an integer $N$ so that there are not more than $N$ disjoint 3-manifolds with 2-sphere boundary and nontrivial $\mathbb{Z}_2$-homology in $\text{Int} \, M^3$. Therefore, all but at most $N$ components of $f^{-1}(X)$ are the intersection of a decreasing sequence of $\mathbb{Z}_2$-homology cubes-with-handles.

If $Z_3^j$ is a $\mathbb{Z}_2$-homology cube-with-handles, the inclusion-induced homomorphism $H_1(\text{Bd} \, Z_3^j; \mathbb{Z}_2) \to H_1(Z_3^j; \mathbb{Z}_2)$ is onto. Thus, if $Z_3^j \subset \text{Int} \, Z_3^{j-1}$ where $Z_3^{j-1}$ is another $\mathbb{Z}_2$-homology cube-with-handles, and if each 1-cycle in $\text{Bd} \, Z_3^j \, \mathbb{Z}_2$-bounds in $\text{Int} \, Z_3^{j-1}$, then the inclusion-induced homomorphism $H_1(Z_3^j; \mathbb{Z}_2) \to H_1(Z_3^{j-1}; \mathbb{Z}_2)$ is trivial. Therefore, each component of $f^{-1}(X)$ which is the intersection of $\mathbb{Z}_2$-homology cubes-with-handles must be strongly 1-acyclic over $\mathbb{Z}_2$. This shows that at most a finite number of point inverses of $f$ are not strongly 1-acyclic over $\mathbb{Z}_2$.

We can now apply Theorem 1 which implies that only a finite number of the strongly 1-acyclic over $\mathbb{Z}_2$ point inverses of $f$ are not cellular.

**IV. Maps almost all of whose point inverses are strongly 1-acyclic over $\mathbb{Z}_p$.**

**Lemma 4.** Let $f: M \to N$ be a compact map from a metric space $M$ onto a metric space $N$. Let $X$ be a closed set in $N$. Let $G$ be a decomposition of $M$ defined by

$$G = \{ f^{-1}(y) : y \in X \} \cup \{ x \in M : f(x) \notin X \}.$$ 

Let $Q = M/G$ and let $\pi: M \to Q = M/G$ be the projection map for the decomposition $G$. Let $p: Q \to N$ be defined so as to make the following diagram commute:

$$
\begin{array}{ccc}
M & \xrightarrow{\pi} & Q \\
f \downarrow & & \downarrow p \\
N & & \\
\end{array}
$$

Then

1. $G$ is upper semicontinuous and hence $\pi$ is continuous and compact.
2. The decomposition $\{ p^{-1}(y) : y \in N \}$ is upper semicontinuous and hence $p$ is continuous and compact.

**Proof.** Lemma 4 follows from the fact that $\{ f^{-1}(y) : y \in N \}$ is an upper semicontinuous decomposition of $M$.

**Lemma 5.** Let $p: Q \to N^3$ be a compact, monotone map from a metric space $Q$ onto a 3-manifold $N^3$, possibly with boundary. Let $X$ be a closed set in $N^3$ containing $\text{Bd} \, N^3$. Suppose that $p|p^{-1}(X)$ is a homeomorphism, and that $W = Q - p^{-1}(X)$ is an open 3-manifold. If $p^{-1}(x)$ is cellular for all $x \in N^3 - X$, then there is a homeomorphism $h: N^3 \to Q$ such that $h|X = p^{-1}|X$.

The proof of Lemma 9 is the same as the proof of Theorem 1 of [1].

Suppose $f: M^3 \to N^3$ is a mapping. We let $A^p_f = \{ x \in M^3 : f^{-1}(f(x))$ is either not connected or is not strongly 1-acyclic over $\mathbb{Z}_p \}$. 

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Theorem 5. Let \( p \) denote 0 or a prime, and let \( M^3 \) and \( N^3 \) be compact 3-manifolds, possibly with boundary, where \( M^3 \) is orientable if \( p \neq 2 \). Let \( Y \) be a compact set in \( \text{Int } N^3 \) each component of which is strongly acyclic over \( \mathbb{Z}_p \). Let \( f: M^3 \to N^3 \) be an onto, boundary preserving map such that \( f(A_p^3) \subseteq Y \). Then \( N^3 \) can be obtained from \( M^3 \) by cutting out of \( M^3 \) a finite number of polyhedral 3-manifolds, each bounded by a 2-sphere, and replacing each by a \( \mathbb{Z}_p \)-homology 3-cell.

Proof. By Theorem 1 there are only a finite number of points \( x_1, x_2, \ldots, x_n \) in \( N^3 - Y \) whose inverses under \( f \) are not cellular in \( M^3 \). Let

\[
X = Y \cup \{x_1, x_2, \ldots, x_n\} \cup \text{Bd } N^3.
\]

We use this \( X \) to define \( Q, \pi: M^3 \to Q, \) and \( p: Q \to N^3 \) as in Lemma 4. Since \( \pi \left( M^3 - f^{-1}(X) \right) \) is a homeomorphism from \( M^3 - f^{-1}(X) \) onto \( W = Q - p^{-1}(X) \), \( W \) is an open 3-manifold. And since \( p | p^{-1}(X) \) is one-to-one and continuous, \( p | p^{-1}(X) \) is a homeomorphism. Therefore, by Lemma 5, there is a homeomorphism \( h: N^3 \to Q \). In particular, \( Q \) is a 3-manifold \( Q^3 \). Let

\[
X' = Y \cup \{x_1, \ldots, x_n\}.
\]

Then \( \pi(S_a) \subseteq p^{-1}(X') = h(X') \), and \( X' \) is strongly acyclic over \( \mathbb{Z}_p \), so the map \( \pi \) satisfies the hypotheses of Theorem 3.

Theorem 6. Let \( p \) denote 0 or a prime, and let \( M^3 \) and \( N^3 \) be 3-manifolds, possibly with boundary, where \( M^3 \) is orientable if \( p > 2 \). Let \( Y \) be a closed 0-dimensional set in \( \text{Int } N^3 \), and let \( f: M^3 \to N^3 \) be an onto, compact, boundary preserving map such that \( f(A_p^3) \subseteq Y \). Then \( \{x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3 \} \) is a locally finite subset of \( N^3 \).

Proof. By Corollary 1, the set \( \{x \in N^3 - Y : f^{-1}(x) \text{ is not cellular in } M^3 \} \) is a locally finite subset of \( N^3 \).

Let

\[
X = Y \cup \text{Bd } N^3 \cup \{x \in N^3 - Y : f^{-1}(x) \text{ is not cellular}\}.
\]

Let \( Q, \pi: M^3 \to Q, p: Q \to N^3, \) and \( h: N^3 \to Q \) be defined as in Lemmas 4 and 5. Let

\[
X' = Y \cup \{x \in N^3 - Y : f^{-1}(x) \text{ is not cellular}\}.
\]

Then \( \pi(S_a) \subseteq p^{-1}(X') = h(X') \), and thus \( \pi(S_a) \) is contained in a closed 0-dimensional set in \( Q \). Theorem 4 can be applied to the map \( \pi: M^3 \to Q^3 \) to say that

\[
\{y \in Q^3 : \pi^{-1}(y) \text{ is not cellular in } M^3 \}
\]

is a locally finite subset of \( Q^3 \). The image under \( p \) (or \( h^{-1} \)) of this set is

\[
\{x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3 \}
\]

which must then be a locally finite subset of \( N^3 \).
V. Further applications. The following lemma is a slight generalization of Lemma 5 of [13]. While the proof of Lemma 5 of [13] suffices to prove our Lemma 6, a proof is included here for completeness and since part of the proof will be needed to prove Theorem 7.

**Lemma 6.** Let \( M^3 \) and \( N^3 \) be 3-manifolds. Let \( f: M^3 \to N^3 \) be a compact, monotone mapping so that \( f(S_t) \) is 0-dimensional. Let \( x \in N^3 \). If there is an open set \( U \) containing \( f^{-1}(x) \) so that the inclusion-induced homomorphism from \( H_1(U; Z) \) into \( H_1(M^3; Z) \) is trivial, then \( f^{-1}(x) \) is strongly 1-acyclic over \( Z \).

**Proof.** Let \( B^3 \) be an open 3-cell in \( N^3 \) with compact closure so that \( x \in B^3 \) and \( W = f^{-1}(B^3) \) is contained in \( U \). Let \( K_1, K_2, K_3, \ldots \) be a locally finite collection of compact sets in \( W \) so that \( \bigcup_{i=1}^{\infty} K_i = W \) and each \( K_i \) is contained in an open 3-cell \( B_i^3 \subset W \). Let

\[
\epsilon_i = \inf \{ \rho(x, y) : x \in K_i \text{ and } y \in W - B_i^3 \}
\]

where \( \rho \) is a metric on \( M^3 \). Let

\[
C_i = \{ x \in N^3 : \text{diam} (f^{-1}(x)) \geq \epsilon_i \text{ and } f^{-1}(x) \cap K_i \neq \emptyset \}.
\]

It is easy to see that each \( C_i \) is a closed set. Let \( C = \bigcup_{i=1}^{\infty} C_i \).

We will show that \( \{ f(K_i) \} \) is a locally finite collection in \( B^3 \). Let \( x_0 \in B^3 \) and let \( V \) be a neighborhood of \( x_0 \) in \( B^3 \) with compact closure. Since \( f \) is a compact map, \( f^{-1}(V) \) has compact closure. Since \( \{ K_i \} \) is a locally finite collection in \( W \), \( f^{-1}(V) \) intersects only a finite number of the \( K_i \)'s, and thus \( V \) intersects only a finite number of the \( f(K_i) \)'s. Using the fact the \( \{ f(K_i) \} \) is a locally finite collection, we see that \( C \) is a closed 0-dimensional subset of \( B^3 \).

Consider the following commutative diagram where the horizontal maps are induced by inclusion, and the vertical maps are induced by \( f \).

\[
\begin{array}{ccc}
H_1(W - f^{-1}(C); Z) & \xrightarrow{\alpha} & H_1(W; Z) \\
\downarrow & & \downarrow \\
H_1(B^3 - C; Z) & \longrightarrow & H_1(B^3; Z)
\end{array}
\]

First, we claim that \( \alpha \) is an epimorphism. Let \([\delta] \in H_1(W; Z)\) where \( \delta \) is a simple closed curve. Let \( O \) be an open set in \( B^3 \) so that \( f(\delta) \subset O \) and \( (\text{Bd } O) \cap C = \emptyset \). By applying Lemma 2 of [13], we see that \( \delta \) is homologous in \( f^{-1}(O) \) to a 1-cycle in \( f^{-1}(O) - f^{-1}(O \cap C) \subset W - f^{-1}(C) \).

Finally, we claim that \( \alpha \) is the zero homomorphism. Let \([\tau] \in H_1(W - f^{-1}(C); Z)\) where \( \tau \) is a simple closed curve. We can also suppose that \( f(\tau) \) is a simple closed curve, and that \( f(\tau) \) bounds an orientable surface \( S \) in \( B^3 - C \). By our choice of the \( \epsilon_i \)'s, for each \( y \in B^3 - C \), there is an open set \( V_y \) so that \( f^{-1}(V_y) \) is contractible in \( W \). Let \( \mathcal{V} = \{ V_y : y \in B^3 - C \} \). We can find a triangulation \( T \) of \( S \) which is so fine that
for each 2-simplex $\sigma \in T$, there is a $V_\sigma \in \mathcal{V}$ so that $\sigma \subset V_\sigma$. Using the fact that $f$ is monotone, we can find a map $h$ from the 1-skeleton of $T$ into $W - f^{-1}(C)$ so that, if $\sigma$ is a 2-simplex of $T$, $h(\partial \sigma) \subset f^{-1}(V_\sigma)$. (See the proof of Theorem 2.1 of [15] for details.) We can also suppose that $hf | \tau$ is the identity. Since each $V_\sigma$ is contractible in $W$, $h$ can be extended to a map $H$ which takes the surface $S$ into $W$ and which takes $\partial S$ onto $\tau$. Thus, $a[\sigma] = 0$ in $H_1(W; \mathbb{Z})$.

**Theorem 7.** Let $M^3$ and $N^3$ be 3-manifolds, possibly with boundary. Let $f$ be a compact, monotone, boundary preserving mapping from $M^3$ onto $N^3$ such that $f(S_f)$ is 0-dimensional. Then $\{x \in N^3 : f^{-1}(x)$ is not cellular $\}$ is a locally finite subset of $N^3$.

**Proof.** By a procedure similar to the first part of the proof of Lemma 6, we can find a closed set $C \subset f(S_f) \subset N^3$ so that, if $x \notin C$, then there is an open set $U_x$ where $f^{-1}(x) \subset U_x$ and $U_x$ is contractible in $M^3$. By Lemma 6, if $x \in N^3 - C$, then $f^{-1}(x)$ is strongly 1-acyclic over $\mathbb{Z}$. Thus $f(A^0) \subset C$, and $C$ is a closed 0-dimensional set. Theorem 7 now follows from Theorem 6.

Let $f: M^3 \to N^3$ be an onto, compact, boundary preserving map as before. Many of our earlier results have shown that $\{x \in N^3 : f^{-1}(x)$ is not cellular in $M^3 \}$ is a locally finite subset of $N^3$. The following three corollaries concern mappings of this type.

**Corollary 3.** Let $M^3$ and $N^3$ be 3-manifolds, possibly with boundary. Let $f: M^3 \to N^3$ be a compact, monotone, boundary preserving mapping such that $\{x \in N^3 : f^{-1}(x)$ is not cellular $\}$ is a locally finite subset of $N^3$. Then

(i) For each $x \in N^3$ and each open set $U$ containing $f^{-1}(x)$, there is an open set $V$ with $f^{-1}(x) \subset V \subset U$, such that $V - f^{-1}(x)$ is homeomorphic to $S^2 \times (0, 1)$.

(ii) $N^3$ can be obtained from $M^3$ by cutting out of $M^3$ a locally finite collection of mutually exclusive, polyhedral 3-manifolds, each with 2-sphere boundary, and replacing each by a 3-cell.

**Proof.** (i) If $f^{-1}(x)$ is cellular, this follows from Theorem 1 of [3].

Let $x_1, x_2, x_3, \ldots$ be the points in $N^3$ such that $f^{-1}(x_i)$ is not cellular for $i = 1, 2, 3, \ldots$. Let $X = \{x_1, x_2, x_3, \ldots \} \cup \text{Bd } N^3$. Let the 3-manifold $Q^3$, the maps $\pi: M^3 \to Q^3$, $p: Q^3 \to N^3$, and the homeomorphism $h: N^3 \to Q^3$ be defined as in Lemmas 4 and 5. It will be sufficient to show that $f^{-1}(x_i)$ has the required neighborhood. We are given an open set $U \supset f^{-1}(x_i)$. Let $U'$ be an open set in $M^3$ so that $f^{-1}(x_i) \subset U' \subset U$ and $U' \cap f^{-1}(x_i) = \emptyset$ for $i \geq 2$. Then $h^{-1}(U')$ is an open set containing $x_i$ in $N^3$. Let $W$ be an open 3-cell so that $x_i \subset W \subset h^{-1}(U')$. Let $V = \pi^{-1} h(W)$. Then $V - f^{-1}(x_i)$ is homeomorphic by $\pi^{-1} h$ to $W - \{x_i\}$ which is homeomorphic to $S^2 \times (0, 1)$.

(ii) As in part (i) let $x_1, x_2, x_3, \ldots$ be the points of $N^3$ whose inverses are not cellular. We can find pairwise disjoint closed neighborhoods $K_1, K_2, K_3, \ldots$ of $f^{-1}(x_1), f^{-1}(x_2), f^{-1}(x_3), \ldots$ respectively so that $K_i - f^{-1}(x_i)$ is homeomorphic to $S^2 \times (0, 1]$. Then each $K_i$ is a 3-manifold with 2-sphere boundary, and $\pi | K_i$ is a
boundary preserving map of $K_i$ onto a 3-cell. Furthermore, $\pi|_{M^3 - \bigcup_{i=1}^{n} K_i}$ is a homeomorphism. Thus $Q^3$ can be obtained by cutting $K_1, K_2, K_3, \ldots$ out of $M^3$, and replacing each by a 3-cell.

**Corollary 4.** Let $M^3$ and $N^3$ be compact 3-manifolds, possibly with boundary. Let $f: M^3 \to N^3$ be a boundary preserving, onto map such that $\{x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3\}$ is a finite set. If $M^3$ is homeomorphic to $N^3$, then $f^{-1}(x)$ is cellular for every $x \in N^3$.

**Proof.** By Corollary 3, part (ii), there are closed 3-manifolds $*K^3_0, \ldots, *K^3_n$ such that

$$M^3 = N^3 \# *K^3_0 \# \cdots \# *K^3_n.$$  

By a corollary to the Grushko-Neumann Theorem (see p. 192 of [10]), the rank of $\pi_1(M^3)$ is equal to the sum of the ranks of $\pi_1(N^3), \pi_1(K^3_0), \ldots, \pi_1(K^3_n)$. Therefore

$$\pi_1(*K^3_0) = \cdots = \pi_1(*K^3_n) = 1,$$

and each $*K^3_i$ ($i=0, \ldots, n$) is a homotopy 3-sphere.

If $M^3$ is closed and orientable, we use the unique decomposition theorem of Milnor [14] to show that $*K^3_0, \ldots, *K^3_n$ are all 3-spheres. This shows that $f^{-1}(x)$ is cellular for every $x \in N^3$.

If $M^3$ is orientable with boundary, we can sew a cube-with-handles onto each boundary component of $M^3$ to obtain a closed manifold $M^3_0$. The homeomorphism from $M^3$ to $N^3$ induces a similar sewing of cubes-with-handles onto $\text{Bd } N^3$ to give a closed 3-manifold $N^3_0$ which is homeomorphic to $M^3_0$. We have

$$M^3 = N^3 \# *K^3_0 \# \cdots \# *K^3_n$$

and the argument for the closed orientable case applies.

If $M^3$ is nonorientable, we apply the previous argument to the orientable double covering of $M^3$.

**Corollary 5.** Let $M^3$ and $N^3$ be compact (i.e., closed) 3-manifolds. Let $f: M^3 \to N^3$ be an onto map such that $\{x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3\}$ is finite, and let $g: N^3 \to M^3$ be an onto map such that $\{x \in M^3 : g^{-1}(x) \text{ is not cellular in } N^3\}$ is finite. Then $M^3$ is homeomorphic to $N^3$.

**Proof.** By Corollary 2, we have $M^3 = N^3 \# *K^3_0 \# \cdots \# *K^3_n$ and $N^3 = M^3 \# *Q^3_0 \# \cdots \# *Q^3_n$. By a corollary to the Grushko-Neumann Theorem (p. 192 of [10]) we see that $*K^3_0, \ldots, *K^3_n, *Q^3_0, \ldots, *Q^3_n$ are all homotopy 3-spheres. This implies that all of the point inverses of $f$ and $g$ have property $UV\infty$. Then Corollary 5 follows from Corollary 2.3 of [11].

**VI. On Haken’s finiteness theorem.** In [5], Wolfgang Haken stated a finiteness theorem for incompressible surfaces in a compact 3-manifold $M^3$. We are interested here only in the special case of the theorem where the surfaces are closed: this
case is stated as Theorem C. Some difficulties arise with Haken's proof in the case where $M^3$ is not irreducible. Haken's proof is correct and can be simplified considerably in the case where $M^3$ is irreducible. We give here an argument due to John Hempel to show that the finiteness theorem holds in the case where $M^3$ may not be irreducible. Haken intended to prove Kneser's Theorem [7] as a special case of the finiteness theorem; our argument uses Kneser's Theorem. The previous results of this paper depend on the finiteness theorem directly through Theorem 2 of [12].

In this section we will be working in the piecewise-linear category. A surface is a 2-manifold. If $F^2$ is a surface in a 3-manifold $M^3$, and if $F^2$ is not a 2-sphere, then $F^2$ is incompressible in $M^3$ if every simple closed curve in $F^2$ that bounds an (open) disk in $M^3 - F^2$ also bounds a disk in $F^2$. A 2-sphere is incompressible in $M^3$ if it does not bound a 3-cell in $M^3$. A 3-manifold $M^3$ is irreducible if every 2-sphere in $M^3$ bounds a 3-cell in $M^3$.

Two surfaces $F^2_0$ and $F^2_1$ in a 3-manifold $M^3$ are parallel in $M^3$ if there is an embedding $\alpha: F^2_0 \times [0, 1] \to M^3$ such that $\alpha_0: F^2_0 \to M^3$ is the inclusion map, and $\alpha_1: F^2_0 \to M^3$ takes $F^2_0$ homeomorphically onto $F^2_1$. If $F^2_1, \ldots, F^2_n$ are disjoint surfaces in a 3-manifold $M^3$, and if $L^3$ is the closure of a complementary domain of $M^3 - \bigcup_{i=1}^n F^2_i$, then $L^3$ is a parallelity component if, for some $1 \leq i \leq n$, there is a homeomorphism $h: F^2_i \times [0, 1] \to L^3$ such that $h_0: F^2_i \to L^3$ is the inclusion map, and $h_1: F^2_i \to L^3$ takes $F^2_i$ homeomorphically onto $F^2_j$ for some $1 \leq j \leq n, j \neq i$.

If $C^3$ is a 3-manifold, possibly with boundary, we define $\hat{C}^3$ to be the 3-manifold, possibly with boundary, obtained from $C^3$ by capping off each 2-sphere boundary component of $C^3$ with a 3-cell.

If $B^3$ is a 3-cell, and if $B^3_1, \ldots, B^3_k$ are disjoint polyhedral 3-cells in $\text{Int } B^3$, then we call the manifold-with-boundary $B^3 - (\bigcup_{i=1}^k \text{Int } B^3_i)$ a punctured 3-cell.

**Lemma A.** If $F^2$ is an incompressible surface in the product $M^2 \times [0, 1]$, where $M^2$ is a compact 2-manifold, then $F^2$ is parallel to $M^2 \times \{0\}$ and $M^2 \times \{1\}$.

This lemma is stated and proved by Haken on pp. 91–96 of [5].

**Lemma B.** If $C^3$ is a 3-manifold, possibly with boundary, and $\hat{C}^3$ is irreducible, then the finiteness theorem holds for $C^3$. In other words, there is an integer $n = n(C^3)$ such that if $F^2_1, \ldots, F^2_{n+1}$ are $n+1$ disjoint incompressible polyhedral surfaces in $C^3$, then two of these surfaces are parallel.

**Proof.** We have assumed the finiteness theorem for irreducible 3-manifolds, so there is an integer $n(\hat{C}^3)$ such that if there are more than $n(\hat{C}^3)$ disjoint incompressible surfaces in $\hat{C}^3$, then two of them are parallel. There are disjoint 3-cells $B^3_1, \ldots, B^3_k$ such that $C^3 = C^3 - \bigcup_{i=1}^k \text{Int } B^3_i$. Let $n = n(C^3) = n(\hat{C}^3) + 2k$. Let $F^2_1, \ldots, F^2_{n+1}$ be $n+1$ disjoint incompressible surfaces in $C^3$. Then $n-k+1$ of these surfaces are irreducible in $\hat{C}^3$. There are $k+1$ distinct pairs from $F^2_1, \ldots, F^2_{n+1}$ which are parallel in $\hat{C}^3$. (We say that the pair $(F^2_i, F^2_j)$ is distinct from the pair...
THEOREM C. Let $M^3$ be a compact 3-manifold, possibly with boundary. Then there is an integer $n_0 = n(M^3)$ such that if $F_1, \ldots, F_{n_0+1}$ are $n_0+1$ disjoint polyhedral-incompressible surfaces in $M^3$, then two of these surfaces are parallel.

Proof. Let $\Sigma = \{S_1^2, \ldots, S_k^2\}$ be a disjoint collection of 2-spheres in $M^3$. Let $N_1^3, \ldots, N_k^3$ be disjoint regular neighborhoods of $S_1^2, \ldots, S_k^2$ respectively. Let $C_1^3, \ldots, C_k^3$ be the components of $\text{Cl}(M^3 - \bigcup_{i=1}^k N_i^3)$. (The $C_i^3$'s are determined up to homeomorphism by the $S_i^2$'s and do not depend on the choice of the $N_i^3$'s. Note that $k$ may not equal $l$ since some of the $S_i^2$'s may not separate $M^3$.) We will call $\Sigma$ a complete system of 2-spheres in $M^3$ if $C_1^3, \ldots, C_k^3$ are each irreducible.

We will let $n(M^3, \Sigma) = \sum_{i=1}^k n(C_i^3)$ where $n(C_i^3) = n(C_i^3)$ is defined in Lemma B.

Kneser's Theorem [7] shows that there is a complete system $\Sigma_0$ of 2-spheres in $M^3$. We will assume $\Sigma_0$ is a fixed complete system and we will let $n_0 = n(M^3, \Sigma_0)$.

Let $F_1^2, \ldots, F_{n_0+1}^2$ be disjoint incompressible surfaces in $M^3$. Let $F^2 = \bigcup_{i=1}^{n_0+1} F_i^2$. Suppose $\Sigma = \{S_1^2, \ldots, S_k^2\}$ is a complete system of 2-spheres in $M^3$, each of which is in general position with respect to $F^2$, and suppose that $n(M^3, \Sigma) = n_0$. Let $m(M^3, \Sigma, F^2)$ be the number of components of $(\bigcup_{i=1}^k S_i^2) \cap F^2$. (Each of these components is a simple closed curve.) We can suppose $m(M^3, \Sigma, F^2)$ is minimal over all such complete systems of 2-spheres in $M^3$. Theorem C will be proved if $m(M^3, \Sigma, F^2)$ is zero. For then there will be more than $n(C_i^3)$ of the surfaces $F_1^2, \ldots, F_{n_0+1}^2$ in one of the components $C_i^3$, and two of these surfaces must be parallel in $C_i^3$ by Lemma B. (Let $N_1^3, \ldots, N_k^3$ and $C_1^3, \ldots, C_k^3$ be defined as before.)

So we suppose that $m(M^3, \Sigma, F^2) > 0$. Any simple closed curve of $(\bigcup_{i=1}^k S_i^2) \cap F^2$ must bound a disk in $F^2$, since $F^2$ is incompressible. Therefore, we can choose an "innermost" (on $F^2$) simple closed curve $J$ of $(\bigcup_{i=1}^k S_i^2) \cap F^2$; suppose $J \subset S_r^2 \cap F_r^2$ for some $r = 1, \ldots, l$ and $s = 1, \ldots, n_0+1$. Let $D^2$ be the disk that $J$ bounds in $F_r^2$. Then $D^2$ is contained in some $C_q^3$ (where $q = 1, \ldots, k$) except for a regular neighborhood of $\partial D^2$.

Let $E_1^2$ and $E_2^2$ be the two disks bounded by $J$ in $S_r^2$. We can push each of the 2-spheres $E_1^2 \cup D^2$ and $E_2^2 \cup D^2$ to one side so that they each miss $D^2$, and so that they are each contained in $C_q^3$. Then one of these 2-spheres must be in the boundary of a punctured cube $P^3$ in $C_q^3$ since $C_q^3$ is irreducible. Let $S_i^2$ be the 2-sphere that is not in the boundary of $P^3$, and let $\Sigma' = \{S_1^2, \ldots, S_{i-1}^2, S_{i+1}^2, \ldots, S_k^2\}$. We will show that $\Sigma'$ is a complete system of 2-spheres in $M^3$, that $n(M^3, \Sigma') = n_0$, and that $m(M^3, \Sigma', F^2) < m(M^3, \Sigma, F^2)$.

Let $C_i^3 (i = 1, \ldots, k)$ be the component of $\text{Cl}(M^3 - \bigcup_{i=1}^k N_i^3)$ on the "other side" of $S_i^2$. (If $S_i^2$ does not separate $M^3$, then $C_i^3$ may equal $C_i^3$.) If we choose a small regular neighborhood $N_i^3$ of $S_i^2$ (so that $N_i^3 \cap D^2 = \emptyset$) and let $N_i^3 = N_i^3$ for

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we can define \( C_i^0 \) and \( C_i^3 \) to be components of \( \text{Cl}(M^3 - \bigcup_{i=1}^n N_i^3) \). A subdisk \( D^2 \) of \( D^3 \) is a spanning disk of \( C_i^2 \) and if we remove the interior of a regular neighborhood of \( D^2 \), this separates \( C_i^2 \) into two components, one homeomorphic to \( C_i^0 \), and the other homeomorphic to the punctured cube \( P^3 \). Thus \( C_i^2 \) is homeomorphic to \( C_i^0 \). Furthermore, \( C_i^3 \) is homeomorphic to the manifold obtained by sewing \( P^3 \) to \( C_i^3 \) along a disk on the boundary of each. Thus \( C_i^3 \) is homeomorphic to \( C_i^0 \). We also have \( n(C_i^0) + n(C_i^0) = n(C_i^3) + n(C_i^3) \) since the 2-sphere boundary components of \( C_i^3 \cap P^3 \) which were removed from \( C_i^3 \) to obtain \( C_i^0 \) were added to \( C_i^0 \) to obtain \( C_i^3 \). Thus \( n(M^3, \Sigma^0) = n_0 \).

Since \( S^2 \cap D^2 = \emptyset \), \( m(M^3, \Sigma', F^2) < m(M^3, \Sigma, F^2) \), and this contradicts our assumption that \( m(M^3, \Sigma, F^2) \) was minimal.

**Bibliography**