ABSOLUTE TAUBERIAN CONSTANTS FOR CESÀRO MEANS

BY

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Abstract. This paper is concerned with introducing two inequalities of the form \( \sum_{n=0}^{\infty} |r_n - a_n| \leq KA \) and \( \sum_{n=0}^{\infty} |r_n - a_n| \leq K'B \), where \( r_n = C_n^{(k)} - C_n^{(k+1)} \), \( C_n^{(k)} \) denote the Cesàro transform of order \( k \), \( K \) and \( K' \) are absolute Tauberian constants, \( A = \sum_{n=0}^{\infty} |\Delta(na_n)| < \infty \), \( B = \sum_{n=0}^{\infty} |\Delta((1/n) \sum_{k=1}^{n} \nu_k)| < \infty \) and \( \Delta u_k = u_k - u_{k+1} \). The constants \( K, K' \) will be determined.

1. Introduction. Let \( \{s_n\} (n \geq 0) \) \( (s_n = a_0 + a_1 + \cdots + a_n) \) be a sequence of real or complex numbers. Denote by \( t_n \) a linear transform \( T \)

\[
(1.1) \quad t_n = \sum_{k} c_{n,k} s_k
\]
of \( s_k \) supposed convergent for all sufficiently large values of \( n \). In various special cases, it has been found that theorems of the following type hold. Suppose that \( p, n \) are related in an appropriate way (usually the assumption is that \( p/n \to \alpha \) as \( n \to \infty \), where \( \alpha > 0 \) is a constant). Suppose that

\[
(1.2) \quad \limsup_{n \to \infty} |na_n| < \infty.
\]

Then there is a constant \( A \) such that

\[
(1.3) \quad \limsup_{n \to \infty} |t_n - s_p| \leq A \limsup_{n \to \infty} |na_n|.
\]

There are also analogous results in which (1.1) is replaced by a sequence-to-function transformation. Usually the best possible value of the constant \( A \) has been determined.

Theorems of this type were first considered by Hadwiger [8] and have since been investigated by various authors; see for example Agnew ([1], [2]) and Jakimovski [10].

Some similar theorems have been obtained with (1.2) replaced by the weaker condition

\[
(1.4) \quad \limsup_{n \to \infty} |\gamma_n| < \infty,
\]
where we write

\( \gamma_n = \frac{1}{n+1} \sum_{v=1}^{n} va_v. \)

See, for example, Delange [5], Rajagopal [14], Meir [13] and Sherif ([15], [16]). Also, other theorems have been obtained with (1.2) replaced by a condition of Schmidt's type

\( \limsup_{n \to \infty} \max_{|p-q| \leq \lambda p^{1/2}} |s_q - s_p| \leq \lambda L \quad (\lambda > 0), \)

where \( \limsup_{n \to \infty} |n^{1/2}a_n| = L < \infty. \) See for example Anjaneyulu [3].

Denoting by \( C_n^{(k)} \) the Cesàro transform of order \( k \) so that

\[ C_n^{(k)} = \left( \frac{n+k}{n} \right)^{-1} \sum_{v=0}^{n} \binom{n-v+k}{n-v} a_v \quad (k \geq 0), \]

we introduce in this paper estimates of a new form for the absolute Cesàro summability defined by Fekete [6]. The corresponding Tauberian conditions to (1.2) and to (1.4) will be

\[ \sum |\Delta(na_n)| < \infty, \]

and

\[ \sum_n \left| \Delta \left( \frac{1}{n} \sum_{v=1}^{n-1} va_v \right) \right| < \infty \quad (2), \]

respectively, where we define \( \Delta u_k \) by

\[ \Delta u_k = u_k - u_{k+1}. \]

The estimates will be of the forms

\[ \sum |\tau_n - a_n| \leq K \sum |\Delta(na_n)|, \]

\[ \sum |\tau_n - a_n| \leq K' \sum_n \left| \Delta \left( \frac{1}{n} \sum_{v=1}^{n-1} va_v \right) \right|, \]

respectively, where

\[ \tau_n = C_n^{(k)} - C_{n-1}^{(k)}; \]

\( K \) and \( K' \) are absolute Tauberian constants.

It has been proved by Hyslop [9] that, if \( \sum a_n \) is absolutely Abel summable and if (1.8) holds, then \( \sum a_n \) is absolutely convergent. Since absolute summability \( |C, k| \) implies absolute Abel summability(3), this theorem includes the result:

(3) \( \Delta u_k \) can be stated in the form that the sequence \( (na_n) \) is absolutely summable \( |C, 1| \).

(4) See Fekete [7].
(A) absolute summability \(|C, k| together with (1.8) implies absolute convergence.  
A fortiori, it includes the result:  
(B) absolute summability \(|C, k| together with (1.7) implies absolute convergence.  
It will be noted that, just as the Tauberian constant theorems already cited  
include the familiar “o” Tauberian theorems, so Theorems 3.1 and 2.1 of  
the present paper include (A) and (B) respectively.  
I have much pleasure in expressing my gratitude to Professor B. Kuttner for his  
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2. Theorem 2.1. Suppose that (1.7) holds. Then, whether  
\[
\sum |\tau_n| < \infty,
\]
holds or not, (1.10) holds, where for \(k \geq 0\),  
\[
K = \Gamma'(k+1)/\Gamma(k+1) + \gamma,
\]
(\(\gamma\) is Euler’s constant).

This result is the best possible in the sense that (1.10) becomes false if \(K\) is replaced  
by any smaller constant.

For the proof of Theorem 2.1, we require the following lemmas.

Lemma 2.1. Let  
\[
A_n = \sum_v \alpha_{n,v} b_v.
\]
Suppose that  
\[
\sum_n |\alpha_{n,v}| \text{ is bounded}^{(4)}.\]
Let  
\[
K = \sup_v \sum_n |\alpha_{n,v}|.
\]
Then  
\[
\sum_n |A_n| \leq K \sum_v |b_v|
\]
and this constant is the best possible in the sense that (2.6) becomes false if \(K\) is re-  
placed by any smaller constant.

Proof. \(\sum |A_n| \leq \sum_n \sum_v |\alpha_{n,v} b_v| = \sum_v |b_v| \sum_n |\alpha_{n,v}| \leq K \sum_v |b_v|\). On the other hand,  
given \(\epsilon > 0\), there is a \(v_0\) say such that \(\sum_n |\alpha_{n,v_0}| > K - \epsilon\). The conclusion follows on  
taking \(b_{v_0} = 1\); \(b_v = 0\) (\(v \neq v_0\)).

\(^{(4)}\) It has been shown by Mears [12], K. Knopp and G. G. Lorentz [11] that for the transformation (2.3) to transform every absolutely convergent series into an absolutely convergent  
series, it is necessary and sufficient that (2.4) holds.
Lemma 2.2. Let
\begin{equation}
(2.7) \quad \beta > \alpha + 1.
\end{equation}

Then
\begin{equation}
(2.8) \quad \sum_{n=\nu}^{\infty} \binom{n-\nu+\alpha}{n-\nu} \binom{n+\beta}{n} = \beta \cdot \Gamma(\nu+1)\Gamma(\beta-1-\alpha)/\Gamma(\nu+\beta-\alpha).
\end{equation}

Proof. The left-hand side of (2.8) is equal to
\begin{equation}
(2.9) \quad \sum_{n=\nu}^{\infty} \binom{n-\nu+\alpha}{n-\nu} \binom{n+\beta}{n} = \left(\frac{\nu+\beta}{\nu}\right)^{\nu} \frac{\Gamma(\nu+\beta+1)}{\Gamma(\nu+\beta-\alpha)} \cdot F\left(1+\alpha; (\nu+1); (\nu+\beta+1); 1\right),
\end{equation}

with the notation of Chapter I of Bailey’s tract [4]. By Gauss’ theorem of §1.3 of Bailey [4], (2.9) is equal to
\begin{equation}
\left[\frac{\nu+\beta}{\nu}\right]^{\nu} \frac{\Gamma(\nu+\beta+1)\Gamma(\beta-1-\alpha)}{\Gamma(\nu+\beta-\alpha)\Gamma(\beta)}
\end{equation}

from which the right-hand side of (2.8) is established.

We are now in position to prove Theorem 2.1. It is clear that \( \tau_0 = a_0 \). But for \( n \geq 1 \),
\begin{equation}
\tau_n = \left[\frac{n+k}{n}\right]^{\nu} \sum_{\nu=1}^{n} \binom{n-\nu+k}{n-\nu} a_{\nu} - \left[\frac{n-1-\nu+k}{n-\nu-1}\right]^{\nu-1} \sum_{\nu=1}^{n-1} \binom{n-1-\nu+k}{n-1-\nu} a_{\nu}
\end{equation}

or
\begin{equation}
\tau_n = \left[\frac{n+k}{n}\right]^{\nu} \sum_{\nu=1}^{n} \binom{n-\nu+k-1}{n-\nu} \Delta(na_{\nu})
\end{equation}

Also
\begin{equation}
(2.12) \quad a_n = \frac{1}{n} na_n = -\frac{1}{n} \sum_{\nu=0}^{n-1} \Delta(na_{\nu}).
\end{equation}

Thus, it follows from (2.11) and (2.12) that
\begin{equation}
(2.13) \quad \tau_n - a_n = \sum_{\nu=0}^{n-1} \Delta(na_{\nu}) \left[1 - \left(\frac{n-\nu-1+k}{n-\nu-1}\right) \binom{n+k}{n}\right].
\end{equation}
Now, (2.13) is a transformation of the type considered in Lemma 2.1 and, for \( n \geq 1 \),
\[
\alpha_{n, \nu} = 0 \quad \text{for } \nu > n,
\]
\[
= (1/n)[ -((n-\nu-1+k)/(n+k)) ] \quad \text{for } \nu \leq n-1.
\]

Thus, the conditions of Lemma 2.1 are satisfied with
\[
(2.15) \quad K = \sup_{\nu} S_{\nu}
\]
where
\[
(2.16) \quad S_{\nu} = \sum_{n=\nu+1}^{\infty} \frac{1}{n} \left( 1 - \left( \frac{n-\nu-1+k}{n-\nu} \right) \right) / \left( \frac{n+k}{n} \right) - \frac{1}{\nu}
\]
provided that \( S_{\nu} \) is bounded; next, we note that, since \( k > 0 \), \( 0 < (\frac{n-\nu-1+k}{n+k}) < (\frac{n+k}{n}) \), so that we may omit the modulus sign in (2.16). We now have
\[
(2.17) \quad S_{\nu} - S_{\nu-1} = \frac{1}{k+1} \sum_{n=\nu}^{\infty} \left( \frac{n-\nu+k-1}{n-\nu} \right) - \frac{1}{\nu}
\]
Replacing \( n \) by \( n+1 \), we thus get
\[
(2.18) \quad S_{\nu} - S_{\nu-1} = \frac{1}{k+1} \sum_{n=\nu}^{\infty} \left( \frac{n-\nu+k}{n+1-\nu} \right) / \left( \frac{n+k+1}{n} \right) - \frac{1}{\nu}
\]
Applying Lemma 2.2, we find that
\[
(2.19) \quad S_{\nu} - S_{\nu-1} = \Gamma(\nu)/\Gamma(\nu+1) - 1/\nu = 0.
\]
Thus,
\[
(2.20) \quad K = S_0 = \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - \left( \frac{n-1+k}{n-1} \right) \right) / \left( \frac{n+k}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - \frac{n}{n+k} \right).
\]
The conclusion thus follows from (2.20) and §12.16 of Whittaker and Watson [17].

3. Theorem 3.1. Suppose that (1.8) holds. Then whether (2.1) holds or not, (1.11) holds, where
\[
K' = K \quad \text{for } k \geq 1,
\]
\[
= -K+2 \quad \text{for } 0 < k < 1.
\]

Proof. Write
\[
(3.2) \quad \phi_n = -\Delta \left( \frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_{\nu} \right).
\]
Let
\[
(3.3) \quad u_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_{\nu}, \quad \phi_n = u_n - u_{n-1}.
\]
Then,
\[ n\alpha_n = (n+1)u_n - nu_{n-1} = u_n + n\phi_n = \sum_{\mu=1}^{n} \phi_{\mu} + n\phi_n, \]
i.e.
\[ a_n = \frac{1}{n} \sum_{\mu=1}^{n} \phi_{\mu} + \phi_n. \]

Using (3.5), it follows from (2.10) that
\[ \tau_n = \left[ n\binom{n+k}{n} \right]^{-1} \sum_{\nu=1}^{n} \binom{n-\nu+k-1}{n-\nu} \left\{ \sum_{\mu=1}^{\nu} \phi_{\mu} + n\phi_{\nu} \right\} = A + B \quad (\text{say}). \]
But
\[ A = \left[ n\binom{n+k}{n} \right]^{-1} \sum_{\nu=1}^{n} \phi_{\mu} \sum_{\nu=1}^{n} \binom{n-\nu+k-1}{n-\nu} = \left[ n\binom{n+k}{n} \right]^{-1} \sum_{\mu=1}^{n} \phi_{\mu} \binom{n-\mu+k}{n-\mu}. \]
Thus,
\[ \tau_n = \left[ n\binom{n+k}{n} \right]^{-1} \sum_{\nu=1}^{n} \left\{ \binom{n-\nu+k}{n-\nu} + \binom{n-\nu+k-1}{n-\nu} \right\} \phi_{\nu}. \]

It follows from (3.5) and (3.7) that
\[ \alpha_n - \tau_n = \frac{1}{n} \sum_{\nu=1}^{n} \left\{ 1 - \left( \frac{n-\nu+k}{n-\nu} \right) + \left( \frac{n-\nu+k-1}{n-\nu} \right) \right\} \phi_{\nu} + \phi_n \]
\[ = \sum_{\nu=1}^{n} \alpha_{n,\nu} \phi_{\nu} \quad (\text{say}), \]
where
\[ \alpha_{n,\nu} = 0 \quad \text{for } \nu > n, \]
\[ = \frac{1}{n} \left[ 1 - \left( \binom{n-\nu+k}{n-\nu} + \binom{n-\nu+k-1}{n-\nu} \right) \right] \quad \text{for } 0 < \nu < n, \]
\[ = \frac{1}{n} \left[ 1 - (n+1) \binom{n+\nu+k}{n} \right] + 1 \quad \text{for } \nu = n. \]

Thus, the conditions of Lemma 2.1 are satisfied with
\[ K' = \sup_{\nu} \psi_{\nu}, \]
where
\[ \psi_{\nu} = \sum_{\nu=1}^{\nu} |\alpha_{n,\nu}|. \]
Write
\[ b_{n,\nu} = \left( 1 + \frac{nk}{n-\nu+k} \right) \binom{n-\nu+k}{n-\nu} / \binom{n+k}{n} \quad (0 \leq \nu \leq n), \]
so that
\[ \alpha_{n,\nu} = (1/n)(1 - b_{n,\nu}) \quad \text{for } 0 \leq \nu \leq n-1, \]
\[ = (1/n)(1 - b_{n,n}) + 1 \quad \text{for } \nu = n. \]
Then

\[ \psi_v = \sum_{n=v}^{\infty} \left| \frac{1}{n} (1-b_{n,v}) + 1 \right| \]

We note that, for \(0 \leq v \leq n-1\), \(b_{n,v+1}/b_{n,v} = h_{n,v}/g_{n,v}\) where

\[ h_{n,v} = (n-v)[n-v-1+(v+2)k], \quad g_{n,v} = (n-v-1+k)(n-v+(v+1)k]. \]

It is easily verified that \(h_{n,v}-g_{n,v} = k(1-k)(v+1)\). Thus, for fixed \(n\), \(b_{n,v}\) is an increasing function of \(v\) if \(k < 1\) and decreasing if \(k > 1\). But

\[ b_{n,0} = 1. \]

Then, if \(k \geq 1\), \(b_{n,v} \leq 1\). Also, if \(0 < k < 1\), \(b_{n,v} > 1\).

(i) \(k \geq 1\). Since \(b_{n,v} \leq 1\), we can omit the modulus sign in (3.11). We thus get

\[ \psi_v = \sum_{n=v}^{\infty} \frac{1}{n} (1-b_{n,v}) + 1. \]

We deduce that

\[ \psi_v = S_{v-1} - M_v + 1, \]

where

\[ M_v = \sum_{n=v}^{\infty} \sum_{k=1}^{n-v} \binom{n-v+k}{n-v} \binom{n+k}{n} (n-v+k) \]

\[ = \frac{\nu}{k+1} \sum_{n=v}^{\infty} \binom{n-v+k-1}{n-v} \binom{n+k}{n}. \]

Replacing \(n\) by \(n+1\), we thus get

\[ M_v = \frac{\nu}{k+1} \sum_{n=v}^{\infty} \binom{n-v+k}{n+1-v} \binom{n+k}{n}. \]

Now, using Lemma 2.2, we have

\[ M_v = 1. \]

Combining (2.15), (2.19), (3.9), (3.14) and (3.16), the result clearly follows.

(ii) \(0 < k < 1\). Since \(b_{n,v} > 1\), then

\[ \alpha_{n,v} < 0 \quad \text{for} \quad 1 \leq v < n-1. \]

But

\[ \alpha_{n,n} = \frac{1}{n} \left[ 1-(n+1) \binom{n+k}{n} \right] +1 = \frac{(n+1)}{n} \left[ 1-\binom{n+k}{n} \right] > 0. \]

Hence, it follows from (3.10), (3.17) and (3.18) that

\[ \psi_v = - \sum_{n=v+1}^{\infty} \alpha_{n,v} + \alpha_{v,v} = - \sum_{n=v}^{\infty} \alpha_{n,v} + 2\alpha_{v,v}. \]
The argument given in case (i) shows that $\sum_{n=1}^{\infty} \alpha_{n,\gamma} = S_{\gamma}$. Thus,

\[(3.19) \quad \psi_{\gamma} = -S_{\gamma} + 2\alpha_{\gamma,\gamma}.\]

Now, using (3.18), we find that

\[(3.20) \quad \alpha_{\gamma,\gamma} = \frac{(\nu+1)}{\nu} \left(1 - \frac{1}{\binom{\nu+k}{\nu}}\right).\]

But, since $k < 1$, $(\nu^k) < (\nu+1)$. Hence

\[(3.21) \quad 1 - \frac{1}{\binom{\nu+k}{\nu}} < 1 - \frac{1}{(\nu+1)}.\]

It thus follows from (3.20) and (3.21) that

\[(3.22) \quad \lim_{\gamma \to \infty} \alpha_{\gamma,\gamma} < 1.\]

Since, $\lim_{\gamma \to \infty} \alpha_{\gamma,\gamma}$, it follows from (3.22) that

\[(3.23) \quad \sup_{\gamma} \alpha_{\gamma,\gamma} = 1.\]

Combining (3.9), (3.19) and (3.23), the final conclusion holds.

**References**


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