SOLVABLE FUNDAMENTAL GROUPS
OF COMPACT 3-MANIFOLDS

BY

BENNY EVANS AND LOUISE MOSER

Abstract. A classification is given for groups which can occur as the fundamental group of some compact 3-manifold. In most cases we are able to determine the topological structure of a compact 3-manifold whose fundamental group is known to be solvable. Using the results obtained for solvable groups, we are able to extend some known results concerning nilpotent groups of closed 3-manifolds to the more general class of compact 3-manifolds. In the final section it is shown that each nonfinitely generated abelian group which occurs as a subgroup of the fundamental group of a 3-manifold is a subgroup of the additive group of rationals.

(1) **Introduction.** This paper is primarily concerned with the classification of those solvable groups which can occur as the fundamental group of a compact 3-manifold. We also consider the problem of determining the structure of a compact 3-manifold whose fundamental group is known to be solvable. Our results are complete except in the category of almost sufficiently large 3-manifolds and the category of 3-manifolds whose nontrivial second homotopy group is generated by projective plane boundary components.

If M is a compact, sufficiently large 3-manifold with trivial second homotopy group, and if $\pi_1(M)$ is solvable, then $\pi_1(M)$ appears in the following list of groups:

   (1) Z, $Z \oplus Z$, or $\mathscr{K}$, the fundamental group of the Klein bottle,

   (2) an extension $1 \to A \to \pi_1(M) \to Z \to 1$ where A is either $Z \oplus Z$ or $\mathscr{K}$,

   (3) a free product of two copies of $\mathscr{K}$ amalgamated along certain subgroups isomorphic with $Z \oplus Z$. These groups may be presented by $(a, b, x, y \mid bab^{-1} = a^{-1}$, $yxy^{-1} = x^{-1}, a = x^p y^{2q}, b^2 = x^r y^{2s})$ where p, q, r, s are integers such that $ps - rq = \pm 1$.

   Further, the above list is complete. That is, for each group G listed above, there is a compact sufficiently large 3-manifold M with $\pi_2(M) = 0$, and $\pi_1(M) \approx G$.

   If the restrictions that M be sufficiently large and that $\pi_2(M) = 0$ are dropped, then further groups must be added to the list. Such groups are discussed in detail in §§3, 4, 5, and 6.

   C. Thomas [16] has listed those nilpotent groups which can act as the fundamental group of a closed 3-manifold. Making use of the information we gain

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concerning solvable groups, we are able to simplify much of Thomas' work, and to extend his results to the bounded case. The only group which must be added to Thomas' list is $Z \oplus Z$.

In a remarkable paper of D. Epstein [1] those finitely generated abelian groups which occur as subgroups of the fundamental group of a 3-manifold are classified. We complete Epstein's list by proving that each nonfinitely generated abelian group which is a subgroup of the fundamental group of a 3-manifold is also a subgroup of the additive group of rationals.

(2) **Definitions and preliminaries.** If g_1 and g_2 are elements of the group G, we use the notation $[g_1, g_2]$ to denote the element $g_1^{-1} g_2^{-1} g_1 g_2$ of G. If H and K are subgroups of the group G, we use the notation $[H, K]$ to denote the subgroup of G generated by all elements $[h, k]$ of G, where $h \in H$ and $k \in K$. For a group G, we define the *nth term $G^{(n)}$ of the commutator series of G, and the nth term $G_{(n)}$ of the lower central series of G as follows:* $G_{(0)} = G^{(0)} = G$, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$, and $G_{(n+1)} = [G_{(n)}, G]$. A group G is said to be *solvable* if there is an integer n such that $G^{(n)}$ is the trivial group. A group G is said to be *nilpotent* if there is an integer n such that $G_{(n)}$ is the trivial group. Each nilpotent group is solvable. Also, each nilpotent group has nontrivial center. Each subgroup and each factor group of a nilpotent group is again nilpotent. Likewise, each subgroup and each factor group of a solvable group is a solvable group. Also each extension of a solvable group by a solvable group is again a solvable group.

We use the notation Z_p to denote the cyclic group of order p and Z shall denote the infinite cyclic group.

Free groups of rank greater than one and fundamental groups of closed 2-manifolds of Euler characteristic less than zero are not solvable. The fundamental group of the Klein bottle $\mathcal{K}$ is solvable. But $\mathcal{K}$ is not nilpotent since it has a centerless group $Z_2 * Z_2$ as a factor group.

A group G is said to be *polycyclic* if G has a sequence of subgroups $G = G_0 \supset G_1 \supset \cdots \supset G_k = 1$ such that for each i, G_{i+1} is a normal subgroup of G_i and G_i/G_{i+1} is a cyclic group. A polycyclic group G is said to be *poly-infinite-cyclic* if the series above can be chosen so that G_i/G_{i+1} is infinite cyclic for each i, $0 \leq i \leq n-1$. Each subgroup and each factor group of a polycyclic group is polycyclic. Also each extension of a polycyclic group by a polycyclic group is again polycyclic. Each subgroup of a polycyclic group is finitely generated. Finally we remark that each finitely generated, torsion free, nilpotent group is poly-infinite-cyclic.

We refer the reader to the books by Marshall Hall [3] and Magnus, Karass, and Solitar [9] for the basic notions of group theory that we shall use here.

We shall use the notation S^n to denote the standard n-dimensional sphere, and P^n shall denote n-dimensional real projective space.

In this paper we assume that all spaces and maps are in the piecewise linear category. Also all subspaces are taken to be piecewise linear subspaces. If A is a

submanifold of the manifold X, we use the notation $\mathscr{U}(A, X)$ to denote a regular neighborhood of A in X.

A 3-manifold M is said to be *irreducible* if each 2-sphere in M bounds a 3-cell in M. If M is an irreducible 3-manifold and if in addition M admits no two-sided embeddings of P^2, then we say that M is P^2-*irreducible*.

A compact 3-manifold that is homotopy equivalent to a standard 3-simplex is a *homotopy* 3-*cell*. Of course it is not known if a homotopy 3-cell is necessarily homeomorphic with a standard 3-simplex.

Compact 3-manifolds M_1, M_2 are said to be *simply equivalent* if M_2 can be obtained from M_1 by either removing the interior of a homotopy 3-cell from the interior of M_1 or by sewing a homotopy 3-cell onto a 2-sphere boundary component of M_1. Compact 3-manifolds M, M' are said to be *equivalent* if there is a finite sequence of compact 3-manifolds $M = M_0, M_1, \ldots, M_k = M'$ such that M_i is simply equivalent to M_{i+1} for each i, $0 \le i < k$. van Kampen's theorem [10] assures us that equivalent 3-manifolds have isomorphic fundamental groups.

If M is a compact 3-manifold, we have by Kneser's theorem [7] that M is equivalent to a compact 3-manifold $\hat{M}$ such that $\hat{M}$ contains no 2-sphere boundary components, and each 3-cell in $\hat{M}$ is homeomorphic with a standard 3-simplex. We shall find it convenient to adopt the convention suggested by the above remarks. That is, for each compact 3-manifold M we shall use the notation $\hat{M}$ to denote a compact 3-manifold such that $\hat{M}$ is equivalent to M, $\hat{M}$ has no 2-sphere boundary components, and each homotopy 3-cell in $\hat{M}$ is homeomorphic with a standard 3-simplex.

If X is a manifold we use bd X to denote the boundary of X and int X to denote the interior of X.

A compact 2-manifold F embedded in a 3-manifold M is *properly embedded in M if $F \cap$ bd $M =$ bd F. A compact 2-manifold properly embedded in a 3-manifold M is *compressible in M* if there exists a disk D in M such that $D \cap F =$ bd D and bd D is not contractible in F. If F is a compact 2-manifold that is properly embedded in a 3-manifold M and if F is not compressible in M, then F is said to be *incompressible in M*.

Suppose F is a compact, orientable 2-manifold with nonempty boundary. Suppose further that F is properly embedded in a 3-manifold M. If an orientation is assigned to F, then an orientation is induced on bd F. We may then consider bd F with its induced orientation as an element [bd F] of H_1(bd M). Since there are exactly two distinct orientations on F, we see that the homology class [bd F] is well defined up to a sign. If F has no boundary, we take [bd F] to denote the trivial element of H_1(bd M).

A compact 3-manifold M is *sufficiently large* if M contains a two-sided incompressible 2-manifold other than a disk or a 2-sphere.

F. Waldhausen [17] has shown that each compact, sufficiently large, orientable, irreducible 3-manifold has a *hierarchy*. That is to say that there is a sequence of

compact 3-manifolds $M = M_0 \supset M_1 \supset \cdots \supset M_k$ and a sequence of compact 2-manifolds $F_0, F_1, \ldots, F_{k-1}$ such that F_i is a 2-sided, incompressible 2-manifold properly embedded in M_i, $M_{i+1} = \mathrm{cl}\,(M_i - \mathscr{U}(F_i, M_i))$, and M_k is a collection of 3-cells. Furthermore, the hierarchy can be chosen so that [bd F_i] is not trivial in $H_1(\mathrm{bd}\ M_i)$ for each i $(0 < i \leqq k - 1)$.

We shall use I to denote the unit interval [0, 1].

Let A and B denote topological spaces. If X is a locally trivial bundle with base A and fiber B, then we shall refer to X as a B bundle over the topological space A. If F is a closed 2-manifold and if M is an I bundle over F, then we say that F is a *trivial I bundle over F* if M is homeomorphic with $F \times I$; otherwise, we say that M is a *twisted I bundle over F*.

(3) Finite groups.

By Corollary 8.7 of [1], each finite group which occurs as the fundamental group of a compact 3-manifold also occurs as the fundamental group of a closed, orientable 3-manifold. J. Milnor [11] has listed those finite groups which can occur as the fundamental group of a closed, orientable 3-manifold. The theorems of this section are proved by checking each group in the list for the appropriate property.

THEOREM 3.1. *Let M be a compact 3-manifold such that $\pi_1(M)$ is finite. Then $\pi_1(M)$ is solvable unless $\pi_1(M)$ is isomorphic with the binary dodecahedral group P_{120} or the direct sum of P_{120} with a cyclic group of order relatively prime to 120.*

THEOREM 3.2. *Let M be a compact 3-manifold such that $\pi_1(M)$ is a finite nilpotent group. Then $\pi_1(M)$ is a finite cyclic group, a generalized quaternion group $Q(2^k)$, or the direct sum of $Q(2^k)$ with a cyclic group of odd order.*

(4) The case $\pi_2(\hat{M}) = 0$ and $\pi_1(M)$ infinite.

We shall make extensive use of the following lemma which appears as a problem in §4.1 of [9]. The proof is omitted.

LEMMA 4.1. *If A and B are nontrivial groups, then $A * B$ is solvable if and only if $A \approx Z_2 \approx B$.*

THEOREM 4.2. *Let M be a compact 3-manifold such that $\pi_2(\hat{M}) = 0$. Suppose further that $\hat{M}$ has nonempty boundary. Then $\pi_1(M)$ is solvable if and only if M is equivalent to a 3-manifold from the following list:*

  (i) *a disk bundle over S^1;*
  (ii) *an I bundle over the torus;*
  (iii) *an I bundle over the Klein bottle.*

We shall require two lemmas before beginning the proof of Theorem 4.2.

LEMMA 4.3. *Let N be a compact P^2-irreducible 3-manifold with nonempty compressible boundary. Then $\pi_1(N)$ is solvable if and only if N is homeomorphic with a disk bundle over S^1.*

Proof. Let D be a disk properly embedded in N such that bd D is not contractible in bd N. Suppose D separates N into two components N_1 and N_2. As a consequence of the projective plane theorem [1], N, N_1, and N_2 are all aspherical 3-manifolds. Thus in particular Lemma 8.4 of [1] assures us that $\pi_1(N_1)$ contains no elements of finite order. By van Kampen's theorem $\pi_1(N) \approx \pi_1(N_1) * \pi_1(N_2)$. Observe that since bd D is not contractible in bd N, it follows that neither N_1 nor N_2 is simply connected. Hence by Lemma 4.1, $\pi_1(N_1) \approx Z_2 \approx \pi_1(N_2)$. This is contrary to the statement above that $\pi_1(N_1)$ contains no elements of finite order. Hence D does not separate N. Let $N^* = \mathrm{cl}\,(N - \mathscr{U}(D, N))$. By van Kampen's theorem $\pi_1(N) \approx \pi_1(N^*) * Z$. It follows that N^* is simply connected. Since N is irreducible it follows that N^* is a 3-cell. The lemma follows.

LEMMA 4.4. *Let N be a compact, orientable, irreducible 3-manifold with nonempty, incompressible boundary. Then $\pi_1(N)$ is solvable if and only if N is either $S^1 \times S^1 \times I$ or the twisted I bundle over the Klein bottle.*

Proof. Let $N = N_0 \supset N_1 \supset \cdots \supset N_k$ be a hierarchy for N where

$$N_{i+1} = \mathrm{cl}\,(N_i - \mathscr{U}(F_i, N_i)), \qquad 0 \le i < k.$$

Assume further that the homology class [bd F_i] is not trivial in $H_1(\mathrm{bd}\, N_i)$ for each i, $1 \le i < k$. For each i, bd N_i is not empty. Thus the condition that [bd F_i] is not trivial in $H_1(\mathrm{bd}\, N_i)$ insures that each N_{i+1} is connected, $0 \le i \le k-1$.

Since F_i is incompressible and two-sided in N_i, it follows from the loop theorem [14] that the inclusion induced homomorphisms $i_*: \pi_1(F_i) \to \pi_1(N_i)$ and $j_*: \pi_1(N_{i+1}) \to \pi_1(N_i)$ are monomorphisms. Hence $\pi_1(F_i)$ and $\pi_1(N_j)$ are solvable groups for each i, $0 \le i < k$, and for each j, $0 \le j \le k$.

Observe that since N_k is a 3-cell, N_{k-1} must have compressible boundary. Let r be the smallest integer such that bd N_r is compressible in N_r. Then N_r is a compact, orientable, irreducible 3-manifold with nonempty, compressible boundary. Furthermore, $\pi_1(N_r)$ is a solvable group. Hence by Lemma 4.3, N_r is homeomorphic with $D \times S^1$ where D denotes the standard 2-simplex.

Through the remainder of the proof, j shall denote an arbitrary integer such that $0 \le j < r$. Observe that since N_j is orientable and bd N_j is incompressible, it follows that each component of bd N_j is a torus. F_j is a compact 2-manifold with nonempty boundary and solvable fundamental group. Since bd N_j is incompressible, F_j is not a disk. Also F_j is two-sided and N_j is orientable so that F_j is not a möbius band. Hence F_j is an annulus.

We identify $\mathscr{U}(F_j, N_j)$ with $F_j \times [0, 1]$ in such a way that F_j is identified with $F_j \times \frac{1}{2}$. We wish to observe that the two annuli $F_j \times 0$ and $F_j \times 1$ lie in the same boundary component of N_{j+1}. If the two boundary components of F_j lie in different components of bd N_j, this is clear. If both boundary components of F_j lie in the same boundary component of N_j, then bd F_j bounds an annulus A in bd N_j. Since [bd F_j] is not trivial in $H_1(\mathrm{bd}\, N_j)$, we see that $A \cup F_j$ is a Klein bottle. But

N_j is orientable. Hence $A \cup F_j$ is one-sided in N_j. It follows that $F_j \times 0$ and $F_j \times 1$ lie in the same boundary component of N_{j+1}.

From the above remarks, we observe that N_j is obtained from N_{j+1} as follows. Let a and b be a pair of disjoint simple closed curves in the same boundary component of N_{j+1} such that neither a nor b is contractible in bd N_{j+1}. Let $h_a: F_j \times 0 \to \mathscr{U}(a, \text{bd } N_j)$ and $h_b: F_j \times 1 \to \mathscr{U}(b, \text{bd } N_{j+1})$ be homeomorphisms. Then N_j is obtained from N_{j+1} by sewing $F_j \times [0, 1]$ onto bd N_j via the homeomorphisms h_a and h_b.

As noted above, N_r is homeomorphic with $D \times S^1$ where D denotes the standard 2-simplex. Let $\beta = (\text{bd } D) \times 0$, let x_0 be a point on β, and put $\alpha = x_0 \times S^1$. According to the remarks above, N_{r-1} can be obtained from N_r by sewing $F_r \times [0, 1]$ onto bd N_r by homeomorphisms that map the annuli $F_j \times 0$, $F_j \times 1$ onto regular neighborhoods of a pair of disjoint simple closed curves a and b in bd N_r. Since bd N_r is a torus and a and b are not contractible in bd N_r, it follows that a and b are parallel. That is, up to a homotopy in bd N_r, $a = \alpha^p \beta^q$ and $b = (\alpha^p \beta^q)^{\pm 1}$ where p and q are a pair of relatively prime integers. Then van Kampen's theorem yields the following presentation for $\pi_1(N_{r-1})$: $(x, y, z \mid y = x^p, zyz^{-1} = x^{\pm p})$. Let G be the smallest normal subgroup of $\pi_1(N_{r-1})$ containing the element y. Then $\pi_1(N_{r-1})/G$ can be presented by $(x, z \mid x^p = 1)$. Observe that this is also a presentation of the group $Z * Zp$. But $\pi_1(N_{r-1})$ is solvable. Hence $p = \pm 1$. Thus we see that N_{r-1} is an I bundle over the torus or the Klein bottle. Since N_{r-1} is orientable, it follows that N_{r-1} is either $S^1 \times S^1 \times I$ or the twisted I bundle over the Klein bottle.

We shall now show that $r = 1$ and hence $N = N_{r-1}$. If $r \neq 1$ then N_{r-2} can be obtained from N_{r-1} by sewing $F_{r-2} \times [0, 1]$ onto bd (N_{r-1}) via homeomorphisms that map the annuli $F_{r-2} \times 0$, $F_{r-2} \times 1$ onto regular neighborhoods of a pair of disjoint, nontrivial curves in the same component of bd N_{r-1}.

If N_{r-1} is homeomorphic with $S^1 \times S^1 \times I$, van Kampen's theorem yields the following presentation for $\pi_1(N_{r-2})$: $(x, y, z, w \mid [x, y] = 1, \ z = x^p y^q, \ wzw^{-1} = (x^p y^q)^{\pm 1})$ where (p, q) is a pair of relatively prime integers. Let G be the smallest normal subgroup of $\pi_1(N_{r-2})$ containing the element z. Then $\pi_1(N_{r-2})/G$ can be presented by $(x, y, w \mid [x, y] = 1, x^p y^q = 1)$. Let H be the group with presentation $(x, y \mid [x, y] = 1, x^p y^q = 1)$. Observe that H is not the trivial group for any values of p and q. But $\pi_1(N_{r-2})/G$ is isomorphic with the free product $Z * H$. Hence $\pi_1(N_{r-2})/G$ is not a solvable group. It follows that if N_{r-1} is $S^1 \times S^1 \times I$, then $r = 1$ and $N = S^1 \times S^1 \times I$.

If N_{r-1} is the twisted I bundle over the Klein bottle, we obtain by van Kampen's theorem the following presentation for $\pi_1(N_{r-2})$: $(x, y, z, w \mid yxy^{-1} = x^{-1}, z = x^p y^{2q}, wzw^{-1} = (x^p y^{2q})^{\pm 1})$. Let H be the smallest normal subgroup of $\pi_1(N_{r-2})$ containing the elements x, y^2 and z. Then $\pi_1(N_{r-2})/H$ can be presented by $(y, w \mid y^2 = 1)$. But this is also a presentation for the nonsolvable group $Z * Z_2$. Hence $\pi_1(N_{r-2})$ is not solvable. It follows that $r = 1$, and N is the twisted I bundle over the Klein bottle. This completes the proof of Lemma 4.4.

Proof of Theorem 4.2. Since $\pi_2(\hat{M}) = 0$ and each homotopy 3-cell in $\hat{M}$ is homeomorphic with the standard 3-simplex, it follows that $\hat{M}$ is P^2-irreducible. If bd $\hat{M}$ is compressible in $\hat{M}$, then by Lemma 4.3, $\hat{M}$ is homeomorphic with a disk bundle over S^1. If $\hat{M}$ is orientable and bd $\hat{M}$ is incompressible, then, by Lemma 4.4, $\hat{M}$ is an I bundle over the torus or the Klein bottle. If $\hat{M}$ is nonorientable with incompressible boundary, we let M^* denote the orientable double cover of $\hat{M}$. Then M^* is equivalent to an I bundle over either the torus or the Klein bottle. But $\hat{M}$ contains no two-sided projective planes. Thus by Theorem 1 of [5] we have that $\hat{M}$ is equivalent to an I bundle over either the torus or the Klein bottle. This completes the proof of Theorem 4.2.

THEOREM 4.5. *Let M be a closed, sufficiently large 3-manifold with $\pi_2(M) = 0$. Then $\pi_1(M)$ is solvable if and only if M is equivalent to a 3-manifold from the following list*:

  (i) *a torus bundle over S^1*;

  (ii) *a Klein bottle bundle over S^1*;

  (iii) *the union of two twisted I bundles over the Klein bottle sewn together along their boundaries.*

Proof. Since $\pi_2(M) = 0$, it follows that the associated manifold $\hat{M}$ is P^2-irreducible. Since $\hat{M}$ is sufficiently large, there is a two-sided incompressible 2-manifold F embedded in M. Furthermore, F is not a 2-sphere. By the loop theorem [14], the homomorphism $i_*: \pi_1(F) \to \pi_1(\hat{M})$ induced by inclusion is a monomorphism. Hence $\pi_1(F)$ is a solvable group. Also F is not a 2-sphere, and since F is two-sided, F is not a projective plane. Hence F is either a torus or a Klein bottle.

Case 1. F separates M. If $\hat{M}$ is nonorientable then F can be chosen so that it does not separate $\hat{M}$. Thus in this case we may assume that $\hat{M}$ is orientable. Hence F is homeomorphic with $S^1 \times S^1$. The 2-manifold F separates $\hat{M}$ into two components whose closures we denote by M_1 and M_2. Then M_1 and M_2 are compact, P^2-irreducible 3-manifolds with nonempty, incompressible boundary. Furthermore, the groups $\pi_1(M_1)$ and $\pi_1(M_2)$ are solvable. It follows then from Lemma 4.4 that M_1 and M_2 are each homeomorphic with an I bundle over either the torus or the Klein bottle. Observe that M_1 and M_2 are orientable and that each manifold has connected boundary. It follows that M_1 and M_2 are both homeomorphic with a twisted I bundle over the Klein bottle.

Case 2. F does not separate $\hat{M}$. Let $M_1 = \mathrm{cl}\,(\hat{M} - \mathscr{U}(F, \hat{M}))$. Then M_1 is a compact P^2-irreducible 3-manifold with nonempty, incompressible boundary. Furthermore, $\pi_1(M_1)$ is a solvable group. Thus M_1 is homeomorphic with an I bundle over either the torus or the Klein bottle. Observe that bd M_1 is not connected. It follows that M_1 is homeomorphic with a product $G \times [0, 1]$ where G is either $S^1 \times S^1$ or the Klein bottle. Hence M is either a torus or Klein bottle bundle over S^1.

It is straightforward to show that the fundamental groups of the manifolds (i) and (ii) are solvable.

The groups (iii) contain a normal subgroup $Z \oplus Z$ which is the fundamental group of the torus separating the I bundles over the Klein bottles. The factor group obtained by moding out this normal subgroup is the solvable group $Z_2 * Z_2$. Hence the groups (iii) are solvable. This completes the proof of Theorem 4.5.

Let M be a closed 3-manifold with $\pi_2(M) = 0$, $\pi_1(M)$ solvable and $\beta_1(M) > 0$. Then there exists an incompressible, two-sided 2-manifold embedded in M which does not separate M. Hence we see from the proof of Theorem 4.5 that M is equivalent to a torus or Klein bottle bundle over S^1.

COROLLARY 4.6. *The following is a complete list of all solvable groups G which occur as the fundamental group of a compact, sufficiently large 3-manifold with trivial second homotopy group*:

    (i) *$Z, Z \oplus Z$, or $\mathcal{K}$, the fundamental group of the Klein bottle*;

    (ii) *an extension $1 \to A \to G \to Z \to 1$ where A is either $Z \oplus Z$ or $\mathcal{K}$*;

    (iii) *a free product of two copies of $\mathcal{K}$ amalgamated along certain subgroups isomorphic with $Z \oplus Z$. In this case G has the presentation $(a, b, x, y \mid bab^{-1} = a^{-1}$, $yxy^{-1} = x^{-1}$, $a = x^p y^{2q}$, $b^2 = x^r y^{2s})$ where p, q, r, s are integers such that $ps - rq = \pm 1$.*

Proof. If G is the fundamental group of a compact, sufficiently large 3-manifold with trivial second homotopy group, then, by Theorems 4.2 and 4.5, G is among the groups listed above. Thus we need only show that each group G in the list above actually occurs as the fundamental group of a compact, sufficiently large 3-manifold M_G with trivial second homotopy group. If G is $Z, Z \oplus Z$, or $\mathcal{K}$, this is straightforward. Let F denote either $S^1 \times S^1$ or the Klein bottle, and suppose G is an extension of $\pi_1(F)$ by Z. The generator of the group Z acts on $\pi_1(F)$ by conjugation to induce an automorphism $\theta: \pi_1(F) \to \pi_1(F)$. By Baer's theorem [12] there exists a homeomorphism $h: F \to F$ such that the induced automorphism $h_*: \pi_1(F) \to \pi_1(F)$ is θ. Let $M_1 = F \times [0, 1]$. Let M_G be the identification space obtained from M_1 by identifying $(x, 0)$ with $(h(x), 1)$ for each x in F. Then by van Kampen's theorem, $\pi_1(M_G) \approx G$.

Let G be a group with presentation $(a, b, x, y \mid bab^{-1} = a^{-1}, yxy^{-1} = x^{-1}, a = x^p y^{2q}, b^2 = x^r y^{2s})$, $ps - rq = \pm 1$. We construct M_G as follows. Let M_1 and M_2 denote twisted I bundles over the Klein bottle. Then bd M_1 and bd M_2 are tori. Hence $\pi_1(\text{bd } M_1) \approx Z \oplus Z \approx \pi_1(\text{bd } M_2)$. Let $h:$ bd $M_1 \to$ bd M_2 be a homeomorphism such that the induced isomorphism $h_*: Z \oplus Z \to Z \oplus Z$ has matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Let M_G be the adjunction space $M_1 \cup_h M_2$. By van Kampen's theorem $\pi_1(M_G) \approx G$.

LEMMA 4.7. *Let M be a compact 3-manifold with infinite, solvable fundamental group. Then M has a finite sheeted, regular covering space $(\tilde{M}, p)$ such that $\beta_1(\tilde{M}) > 0$.*

Proof. Let $G = \pi_1(M)$. Since G is solvable, there exists an integer k such that $G^{(k)} = 1$. Since G is infinite and $G^{(k)} = 1$, there is an integer i such that $G^{(i+1)}$ is of infinite index in $G^{(i)}$. Let j be the smallest such integer. Let $(\tilde{M}, p)$ be the covering

space of M associated with the normal subgroup $G^{(j)}$ of $\pi_1(M)$. Then $(\tilde{M}, p)$ is a finite sheeted regular covering space of M. Furthermore $H_1(\tilde{M}) \approx G^{(j)}/G^{(j+1)}$ is infinite.

We use the notation $GL_n(Z)$ to denote the multiplicative group of invertible, $n \times n$ matrices with entries from the ring of integers.

THEOREM 4.8. *Let M be a closed, nonsufficiently large 3-manifold with $\pi_2(M) = 0$. Suppose further that $\pi_1(M)$ is an infinite, solvable group. Then M has a finite sheeted, regular covering space $(\tilde{M}, p)$ such that $\tilde{M}$ is equivalent to a torus bundle over S^1. Furthermore, $(\tilde{M}, p)$ can be chosen so that the group of covering transformations of $(\tilde{M}, p)$ is a finite, solvable subgroup of $GL_n(Z)$ with $n \leq 3$.*

Proof. Observe first of all that since M is not sufficiently large, M is an orientable 3-manifold. By Lemma 4.7, M has a finite sheeted, regular covering space $(\tilde{M}, p)$ such that $\beta_1(\tilde{M}) > 0$. Since $\tilde{M}$ is orientable, we see by the remarks following Theorem 4.5 that $\tilde{M}$ is equivalent to a torus bundle over S^1.

We have the exact sequence $1 \to \pi_1(\tilde{M}) \to \pi_1(M) \xrightarrow{f} Q \to 1$ where Q denotes the finite group of covering transformations of $(\tilde{M}, p)$. We assume that among all finite sheeted, regular covering spaces (N, p') of M such that $\beta_1(N) > 0$, $(\tilde{M}, p)$ is chosen so that the group Q of covering transformations of $(\tilde{M}, p)$ has minimum order.

Since $\tilde{M}$ is compact, $H_1(\tilde{M})$ is a finitely generated, abelian group. Hence there exists a positive integer α such that $\alpha H_1(\tilde{M}) = \{\alpha x \mid x \in H_1(\tilde{M})\}$ is torsion free. Furthermore, since $\tilde{M}$ is equivalent to a torus bundle over S^1, $\alpha H_1(\tilde{M})$ is isomorphic with Z, $Z \oplus Z$ or $Z \oplus Z \oplus Z$. Let r be the rank of $\alpha H_1(\tilde{M})$. Then $r \leq 3$.

Each covering transformation q in Q induces an automorphism

$$q_*: H_1(\tilde{M}) \to H_1(\tilde{M}).$$

For $q \in Q$, let $\hat{q}$ denote the restriction of q_* to $\alpha H_1(\tilde{M})$. Let $\psi: Q \to GL_r(Z)$ be the homomorphism defined by $\psi(q) = \hat{q}$ for each q in Q. We shall show that the assumption of minimum rank for Q insures that ψ is a monomorphism.

Let Q' denote the kernel of the homomorphism ψ. We have the following exact commutative diagram.

$$
\begin{array}{ccccccccc}
& & 1 & & 1 & & & & \\
& & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(\tilde{M}) & \longrightarrow & f^{-1}(Q') & \xrightarrow{f} & Q' & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\tilde{M}) & \longrightarrow & \pi_1(M) & \xrightarrow{f} & Q & \longrightarrow & 1 \\
& & & & \downarrow & & \downarrow & & \\
& & & & Q/Q' & = & Q/Q' & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 1 & & 1 & &
\end{array}
$$

Let (N, p') denote the regular covering space of M associated with the normal subgroup $f^{-1}(Q')$ of $\pi_1(M)$. From the spectral sequence of a covering map [8] we obtain an exact sequence of terms of low degree. In particular we have the following exact sequence. $H_2(Q') \to H_1(\tilde{M}) \otimes_{Q'} Z \to H_1(N) \to H_1(Q')$. We wish to show that $H_1(N)$ is infinite. Since Q' is finite, it follows that $H_2(Q')$ is also finite. Hence in order to show that $H_1(N)$ is infinite, we need only show that the tensor product $H_1(\tilde{M}) \otimes_{Q'} Z$ is infinite. Let K denote the subgroup of $H_1(\tilde{M})$ generated by all elements of the form $x - q_*(x)$ where $x \in H_1(\tilde{M})$ and $q \in Q'$. Then $H_1(\tilde{M}) \otimes_{Q'} Z$ is isomorphic with $H_1(\tilde{M})/K$. Consider the subgroup $(\alpha H_1(\tilde{M})) \cap K$ of $H_1(\tilde{M})$. If $z \in (\alpha H_1(\tilde{M})) \cap K$, then there exist elements $x, y_1, \ldots, y_k$ of $H_1(\tilde{M})$, and there exist elements $q_1, \ldots, q_k$ of Q' such that $\alpha x = \sum_{i=1}^{k} y_i - q_{i_*}(y_i)$. Then $\alpha^2 x = \sum_{i=1}^{k} \alpha y_i - q_{i_*}(\alpha y_i) = \sum_{i=1}^{k} \alpha y_i - \hat{q}_i(\alpha y_i)$. But $\hat{q}_i = \psi(q_i)$ is the identity map since q_i is an element of the kernel of ψ. Hence $\alpha^2 x = 0$. But $\alpha H_1(\tilde{M})$ is torsion free. Thus $z = \alpha x = 0$. It follows that $(\alpha H_1(\tilde{M})) \cap K = 0$. Hence the restriction of the natural projection of $H_1(\tilde{M})$ onto $H_1(\tilde{M})/K$ to $\alpha H_1(\tilde{M})$ is a monomorphism. Thus $H_1(\tilde{M}) \otimes_{Q'} Z$ is infinite. It follows that $H_1(N)$ is also infinite.

Thus (N, p') is a finite sheeted, regular covering space of M such that $\beta_1(N) > 0$. The group of covering transformations of (N, p') is isomorphic with Q/Q'. By the minimality assumption for $(\tilde{M}, p)$, the order of Q is not greater than the order of Q/Q'. It follows that Q' is the trivial group. Hence ψ is a monomorphism and the proof of Theorem 4.8 is complete.

COROLLARY 4.9. *Let G be the fundamental group of a compact, nonsufficiently large 3-manifold with trivial second homotopy group. Suppose further that G is an infinite solvable group. Then G is a nonsplit extension of the fundamental group of an orientable torus bundle over S^1 by a finite solvable subgroup of $\mathrm{GL}_r(Z)$, $r \leq 3$.*

Looking closely at the proofs of this section, we have the following corollary.

COROLLARY 4.10. *Let M be a compact sufficiently large 3-manifold with trivial second homotopy group. Then $\pi_1(M)$ is solvable if and only if $\pi_1(M)$ contains no free subgroup of rank two.*

(5) **The case $\pi_2(\hat{M}) \neq 0$.** Let G be a split extension of a group H by a group K. Then each element k of K acts by conjugation on H to induce an automorphism θ_k on H. We say that K acts *without fixed points* on H if for each k in K and each h in H, $\theta_k(h) = h$ if and only if $k = 1$ or $h = 1$.

THEOREM 5.1. *Suppose M is a compact 3-manifold such that $\pi_2(\hat{M}) \neq 0$. If $\pi_1(M)$ is an infinite, solvable group, then precisely one of the following is true:*

(i) *$\pi_1(M) \approx Z$, $Z \oplus Z_2$, or $Z_2 * Z_2$.*

(ii) *$\pi_2(\hat{M})$ is generated as a $\pi_1(\hat{M})$ module by the projective plane boundary components of $\hat{M}$, and $\pi_1(M)$ is a split extension of G by Z_2 where G is the fundamental group of a compact, orientable, irreducible, 3-manifold with solvable fundamental group. Furthermore, Z_2 acts without fixed points on G.*

If M is a compact 3-manifold which falls into the second category of Theorem 5.1, then M can be obtained as follows. Let N be a compact, orientable 3-manifold with $\pi_2(N) = 0$ such that $\pi_1(N)$ is an infinite, solvable group. Let $f: N \to N$ be an involution whose fixed point set is a finite collection of points $x_1, \ldots, x_k$. Let M_1 be the orbit space $\mathrm{cl}\,(N - \bigcup_{i=1}^{k} \mathscr{U}(x_i, N))/f$. Then M is obtained from M_1 by removing the interiors of a finite (possibly empty) collection of 3-cells.

Numerous examples of such 3-manifolds exist. For example we can construct an involution σ on $N = S^1 \times S^1 \times [0, 1]$ which fixes exactly four points as follows. Let C denote the complex plane. The 2-manifold $S^1 \times S^1$ is naturally embedded in $C \times C$ as the set of ordered pairs $(e^{2\pi i \alpha}, e^{2\pi i \beta})$ such that $0 \le \alpha \le 1$, $0 \le \beta \le 1$. We define the involution $\tau: S^1 \times S^1 \to S^1 \times S^1$ by

$$\tau(e^{2\pi i \alpha}, e^{2\pi i \beta}) = (e^{2\pi i(1-\alpha)}, e^{2\pi i(1-\beta)}).$$

The involution τ fixes exactly four points; namely $x_1 = (1, 1)$, $x_2 = (1, e^{2\pi i/2})$, $x_3 = (e^{2\pi i/2}, 1)$, $x_4 = (e^{2\pi i/2}, e^{2\pi i/2})$. Define $\sigma: N \to N$ by $\sigma(x, t) = (\tau(x), 1 - t)$. Let M be constructed from N as described above. Then $\pi_1(M)$ has presentation $(x, y, t \mid [x, y] = 1, txt^{-1} = x^{-1}, tyt^{-1} = y^{-1}, t^2 = 1)$. Observe that since Z_2 acts without fixed points on $\pi_1(N) \approx Z \oplus Z$, it follows that $\pi_1(M)$ is the only split extension of $Z \oplus Z$ by Z_2 which occurs as the fundamental group of a compact 3-manifold.

Proof of Theorem 5.1. Since $\pi_2(\hat{M}) \ne 0$, we have by the projective plane theorem [1] that $\hat{M}$ contains either a noncontractible 2-sphere or a two-sided projective plane.

Case 1. *There exists a noncontractible 2-sphere S in $\hat{M}$.* If S separates $\hat{M}$ into two components M_1 and M_2, then by van Kampen's theorem $\pi_1(\hat{M}) \approx \pi_1(M_1) * \pi_1(M_2)$. Since S is not contractible in $\hat{M}$ and $\hat{M}$ has no 2-sphere boundary components, it follows that neither M_1 nor M_2 is simply connected. Hence $\pi_1(M_1) \approx Z_2 \approx \pi_1(M_2)$. Thus $\pi_1(M) \approx \pi_1(\hat{M}) \approx Z_2 * Z_2$.

If s does not separate $\hat{M}$, then van Kampen's theorem allows us to write $\pi_1(\hat{M}) \approx Z * H$ where H is the fundamental group of $\hat{M} - S$. Since $\pi_1(\hat{M})$ is solvable, it follows that H is the trivial group. Hence $\pi_1(M) \approx \pi_1(\hat{M}) \approx Z$.

Case 2. *Each 2-sphere in $\hat{M}$ is contractible in $\hat{M}$.* Let $f_1, \ldots, f_r$ be maps of S^2 into the projective plane boundary components of $\hat{M}$ (if any exist) such that f_i identifies antipodal points for each i, $1 \le i \le r$. Let Λ denote the $\pi_1(M)$-submodule of $\pi_2(\hat{M})$ generated by $f_1, \ldots, f_r$.

Case 2A. $\Lambda \ne \pi_2(M)$. By the projective plane theorem [1] there is a map $f: S^2 \to \hat{M}$ such that $f \notin \Lambda$ and $f(S^2)$ is either a 2-sphere or a two-sided projective plane. Since each 2-sphere in $\hat{M}$ is contractible, it follows that $f(S^2)$ is a two-sided projective plane. Let $(\tilde{M}, P)$ denote the orientable double cover of $\hat{M}$, and let $\tilde{f}: S^2 \to \tilde{M}$ be a lifting of f. Let $\tau: \tilde{M} \to \tilde{M}$ denote the nontrivial covering transformation of $(\tilde{M}, P)$. Suppose $f(S^2)$ does not separate $\hat{M}$. Let

$$M_1 = \mathrm{cl}\,(\hat{M} - \mathscr{U}(f(S^2), \hat{M})).$$

Suppose $\tilde{f}(S^2)$ separates $\tilde{M}$. Then since $\hat{M} - f(S^2)$ is connected, it follows that τ interchanges the components of $\tilde{M} - \tilde{f}(S^2)$. Let S^* denote a 2-sphere in $\tilde{M}$ such that S^* is parallel to $\tilde{f}(S^2)$. Then since $\tau(S^*) \cap S^* = \varnothing$, it follows that $P|_{S^*}$ is an injection. Hence $P(S^*)$ is a noncontractible 2-sphere in $\hat{M}$. Since each 2-sphere in $\hat{M}$ is contractible, we have established a contradiction to the assumption that $\tilde{f}(S^2)$ separates $\tilde{M}$. Let $\tilde{M}_1 = P^{-1}(M_1)$. Then $\pi_1(\tilde{M}) \approx Z * \pi_1(\tilde{M}_1)$. It follows that $\pi_1(\tilde{M}_1) = 1$. Since $\tilde{M}_1$ is a connected, two-sheeted covering space of M_1, it follows that $\pi_1(M_1) \approx Z_2$. Then by Theorem 5.1 of [2], M_1 is homotopy equivalent with $P^2 \times [0, 1]$. It follows that $\pi_1(M) \approx \pi_1(\hat{M}) \approx Z \oplus Z_2$.

Suppose $f(S^2)$ separates $\hat{M}$ into two components whose closures we denote by M_1 and M_2. Let $\tilde{M}_1 = P^{-1}(M_1)$, and let $\tilde{M}_2 = P^{-1}(M_2)$. Since $\tilde{f}(S^2)$ is not in $P_*^{-1}(\Lambda)$, it follows that neither $\tilde{M}_1$ nor $\tilde{M}_2$ is simply connected. Thus since $\pi_1(\tilde{M}) \approx \pi_1(\tilde{M}_1) * \pi_1(\tilde{M}_2)$, we have that $\pi_1(\tilde{M}_1) \approx Z_2 \approx \pi_1(\tilde{M}_2)$. But $P|\tilde{M}_1 : \tilde{M}_1 \to M_1$ is a two-sheeted covering map. Hence $\pi_1(M_1)$ is an extension of Z_2 by Z_2. This is not possible since as proved in [1], the only finite group which can occur as the fundamental group of a compact, nonorientable 3-manifold is Z_2.

Case 2B. $\Lambda = \pi_2(\tilde{M})$. Let $\tilde{f}_1, \ldots, \tilde{f}_r$ be liftings of the generators $f_1, \ldots, f_r$ of Λ to the covering space $\tilde{M}$. The map $P_* : \pi_2(\tilde{M}) \to \pi_2(M)$ is an isomorphism of abelian groups. Thus since for each i, $1 \leq i \leq r$, $P^{-1}f(S^2)$ is connected, it follows that $\pi_2(\tilde{M})$ is generated as a $\pi_1(\tilde{M})$ module by $\tilde{f}_1, \ldots, \tilde{f}_r$. Hence $\pi_2(\tilde{M}^{\wedge}) = 0$. It follows that $\tilde{M}^{\wedge}$ is an irreducible, orientable, compact 3-manifold with infinite, solvable fundamental group.

Let $G = \pi_1(\tilde{M})$. Then $\pi_1(\hat{M})$ is an extension of G by Z_2. The group G is the fundamental group of an aspherical 3-manifold. Hence by Lemma 8.4 of [1], G is torsion free. But $\hat{M}$ has at least one projective plane boundary component. Hence $\pi_1(\hat{M})$ contains a nontrivial element t of order 2. It follows that the above extension is split by a homomorphism that maps the generator of Z_2 onto t.

The element t acts on G by conjugation to define an automorphism $\theta : G \to G$. Suppose there exists a nontrivial element g in G such that $\theta(g) = g$. Then $tgt^{-1} = g$. Since G is torsion free, it follows that t and g generate a subgroup H of $\pi_1(\hat{M})$ such that H is isomorphic with $Z \oplus Z_2$. Then by Theorem 9.5 of [1], we can write $\hat{M}$ as a connected sum $\hat{M} = Q \# R$ where $\pi_1(Q) \approx Z \oplus Z_2$. By van Kampen's theorem, $\pi_1(\hat{M}) \approx \pi_1(Q) * \pi_1(R)$. Since $\pi_1(\hat{M})$ is solvable, it follows that $\pi_1(\hat{M}) \approx Z \oplus Z_2$. Then $\pi_1(\tilde{M}) \approx Z$. Since $\tilde{M}^{\wedge}$ is irreducible and orientable, it follows that $\tilde{M}^{\wedge}$ is homeomorphic with a product $D \times S^1$ where D denotes the standard 2-simplex. Thus $\hat{M}$ has a boundary component T such that T is isomorphic with either $S^1 \times S^1$ or the Klein bottle. Since $\pi_1(\hat{M}) \approx Z \oplus Z_2$, we see that T is compressible in $\hat{M}$. Let D be a disk in M such that $D \cap$ bd $\hat{M} = D \cap T =$ bd D, and bd D is not contractible in T. If D separated $\hat{M}$, van Kampen's theorem would allow us to write the fundamental group $Z \oplus Z_2$ of $\hat{M}$ as a nontrivial free product. Thus D does not separate $\hat{M}$. Then by van Kampen's theorem $Z \oplus Z_2 \approx \pi_1(\hat{M}) \approx Z * K$ where K is the fundamental group of $\hat{M} - D$. This contradiction assures

us that Z_2 acts on G without fixed points. The proof of Theorem 5.1 is complete.

We are now able to prove a general theorem about solvable fundamental groups of compact 3-manifolds.

THEOREM 5.2. *If M is a compact 3-manifold then $\pi_1(M)$ is solvable if and only if $\pi_1(M)$ is polycyclic.*

Proof. Let M denote a compact 3-manifold with solvable fundamental group. A finite group G is solvable if and only if G is polycyclic. Thus we need only consider the case that $\pi_1(M)$ is infinite.

Case 1. $\pi_2(\hat{M}) = 0$. If $\hat{M}$ has nonempty boundary, then $\pi_1(M)$ is either Z, $Z \oplus Z$, or the fundamental group of the Klein bottle. All these groups are polycyclic. If $\hat{M}$ is closed, then we let $\tilde{M}$ denote the orientable double cover of $\hat{M}$ if $\hat{M}$ is nonorientable; otherwise, let $\tilde{M} = \hat{M}$. Then $\tilde{M}$ has a finite sheeted, regular covering space M^* such that M^* is equivalent to a torus bundle over S^1. Since $\pi_1(M^*)$ is polycyclic, it follows that $\pi_1(\tilde{M})$ is polycyclic. Hence $\pi_1(M) \approx \pi_1(\hat{M})$ is also polycyclic.

Case 2. $\pi_2(\hat{M}) \neq 0$. In this case $\pi_1(M)$ is either an extension of a polycyclic group by Z_2, or $\pi_1(M)$ is isomorphic with one of the polycyclic groups Z, $Z \oplus Z_2$ or $Z_2 * Z_2$.

(6) **Seifert fiber spaces.** In this section we consider again the category of closed, P^2-irreducible 3-manifolds which are not sufficiently large but which have solvable, infinite fundamental groups. All such 3-manifolds known to the authors are Seifert fiber spaces. We therefore devote this section to the classification of those Seifert fiber spaces with solvable fundamental groups.

A *Seifert fiber space* M is constructed as follows. Let F be a closed 2-manifold and let M_1 denote a locally trivial S^1 bundle over F. Let $P: M_1 \to F$ be the bundle projection. We shall refer to F as the Seifert surface of M. Choose disjoint disks $D_1, \ldots, D_k$ in F and set $M_2 = \mathrm{cl}\,(M_1 - \bigcup_{i=1}^{k} P^{-1}(D_i))$. Note that $P^{-1}(D_i)$ is a solid torus for each i, $1 \leq i \leq k$. We identify $P^{-1}(D_i)$ with $D_i \times S^1$. Put $q_i = (\mathrm{bd}\, D_i) \times 0$ and $h_i = x_i \times S^1$ where x_i is a point on q_i. We complete M by sewing solid tori $D_i' \times S^1$ onto $q_i \times h_i$ in such a way that the curve $(\mathrm{bd}\, D_i') \times 0$ is sewn to the curve $q_i^{\alpha_i} h_i^{\beta_i}$ where (α_i, β_i) is a pair of relatively prime integers. If y_i is the center of the disk D_i, we refer to the simple closed curve $P^{-1}(y_i)$ as a *singular fiber of type* (α_i, β_i). Observe that if $\alpha_i = \pm 1$ then M can be refibered in such a way that y_i no longer occurs as a singular fiber of M. Thus in the sequel, we lose no generality in assuming that $|\alpha_i| > 1$ for each i, $1 \leq i \leq k$.

If F is orientable of genus g, van Kampen's theorem yields the following presentation for $\pi_1(M)$: $(a_i, b_i, q_j, h \mid \prod_{i=1}^{g} [a_i, b_i] = h^b \prod_{i=1}^{k} q_i, [h, q_j] = 1, a_i h a_i^{-1} = h^{\varepsilon_i}, b_i h b_i^{-1} = h^{\lambda_i}, q_j^{\alpha_j} h^{\beta_j} = 1)$, $1 \leq i \leq g$, $1 \leq j \leq k$, $\varepsilon_i, \lambda_i = \pm 1$. In case F is not orientable, $\pi_1(M)$ may be presented as $(c_i, q_j, h \mid \prod_{i=1}^{n} c_i^2 = h^b \prod_{i=1}^{k} q_i, [h, q_j] = 1, c_i h c_i^{-1} = h^{\varepsilon_i}, q_j^{\alpha_j} h^{\beta_j} = 1)$, $1 \leq i \leq n$, $1 \leq j \leq k$.

A Seifert fiber space M with nonorientable Seifert surface F and k singular fibers of types $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$ has a double cover $\tilde{M}$, a Seifert fiber space with Seifert surface $\tilde{F}$, the orientable double cover of F, and $2k$ singular fibers of types $(\alpha_1, \beta_1), (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k), (\alpha_k, \beta_k)$. Hence we need only consider the case where M has orientable Seifert surface. Furthermore, $\pi_1(M)$ is an extension of the normal, infinite cyclic subgroup generated by h.

Hence we have reduced the problem to classification of the planar discontinuous groups $G(g, k | \alpha_1, \ldots, \alpha_k)$ with presentation $(a_i, b_i, q_j \mid \prod_{i=1}^{g} [a_i, b_i] = \prod_{i=1}^{k} q_i, q_j^{\alpha_j} = 1)$, $1 \leqq i \leqq g$, $1 \leqq j \leqq k$.

LEMMA 6.1. *If $g \geqq 2$, or if $g = 1$ and $k \geqq 1$, then $G(g, k | \alpha_1, \ldots, \alpha_k)$ is nonsolvable.*

Proof. If H is the smallest normal subgroup of $G(g, k | \alpha_1, \ldots, \alpha_k)$ generated by $q_1, \ldots, q_k$, then G/H can be presented by $(a_i, b_i \mid \prod_{i=1}^{g} [a_i, b_i] = 1)$, $1 \leqq i \leqq g$. This is also a presentation of the fundamental group of a closed 2-manifold of genus g. Thus if $g \geqq 2$, $G(g, k | \alpha_1, \ldots, \alpha_k)$ is not solvable.

If $g = 1$ and $k \geqq 1$, then $G(g, k | \alpha_1, \ldots, \alpha_k)$ has a factor group H with presentation $(a, b \mid [a, b]^{\alpha_1} = 1)$. The factor group H is obtained by factoring out the smallest normal subgroup of $G(g, k | \alpha_1, \ldots, \alpha_k)$ containing the elements $q_2, \ldots, q_k$. Let T denote a torus with the interior of a disk removed. Let K denote the 2-complex obtained by sewing the boundary of a disk D onto bd T in such a way that bd D wraps α_1 times around bd T. Then by van Kampen's theorem $\pi_1(K) \approx H$. K has a two-sheeted covering space $\tilde{K}$ obtained as follows. Let $\tilde{T}$ denote a torus with the interiors of two disks removed. $\tilde{K}$ is completed by sewing a disk onto each boundary component of $\tilde{T}$ so that each disk wraps α_1 times around the boundary component onto which it is sewn. Then $\pi_1(\tilde{K})$ is a subgroup of $\pi_1(K)$. By van Kampen's theorem, $\pi_1(\tilde{K})$ has presentation $(a, b, t_1, t_2 \mid [a, b] = t_1 t_2, t_1^{\alpha_1} = t_2^{\alpha_1} = 1)$. $\pi_1(\tilde{K})$ has a factor group with presentation $(a, t_1 \mid t_1^{\alpha_1} = 1)$. But this is also a presentation for the nonsolvable group $Z * Z_{\alpha_1}$. This completes the proof of Lemma 6.1.

If A and B are groups, and if $\phi : H \rightarrow K$ is an isomorphism between the subgroups H of A and K of B, then we denote the free product of A and B amalgamated along the subgroups H and K by $*(A, B, H, K, \phi)$.

LEMMA 6.2. *Let $G = *(A, B, H, K, \phi)$, where $A \neq H$ and $B \neq K$. Then G is solvable if and only if H is solvable and H is of index two in A and K is of index two in B.*

Proof. If H is of index two in A and K is of index two in B, then observe that $G = *(A, B, H, K, \phi)$ is an extension of H by the solvable group $Z_2 * Z_2$. Thus G is solvable if and only if H is.

The statement of the sufficiency of the lemma is a special case of a lemma of R. J. Gregorac [2]. Gregorac has shown that if $G = *(A, B, H, K, \phi)$ satisfies some nontrivial identity relation, then H is of index two in A and K is of index two in B. Since each solvable group must satisfy some commutator identity (e.g. a group that is solvable of length two must satisfy the identity $[[X, Y], [Z, W]] = 1$), we may apply Gregorac's lemma above to obtain our result.

THEOREM 6.3. $G(0, k|\alpha_1, \ldots, \alpha_k)$ *is solvable in exactly the following cases*:

(1) $k=4$ *and* $\alpha_1=\alpha_2=\alpha_3=\alpha_4=2$;

(2) $k=3$ *and* $(\alpha_1, \alpha_2, \alpha_3)=(2, 2, \alpha_3)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 6)$, $(2, 4, 4)$, or $(3, 3, 3)$;

(3) $k \leqq 2$.

LEMMA A. *If* $G(0, k|\alpha_1, \ldots, \alpha_k)$ *is solvable, then* $k \leqq 4$. *If* $k=4$, *then* $\alpha_1=\alpha_2=\alpha_3 =\alpha_4=2$.

Proof. Since $k \geqq 4$, we may write $G=G(0, 1|\alpha_1, \ldots, \alpha_k)$ as $G=*(A, B, H, K, \phi)$ where A is the group presented by $(q_1, q_2 \mid q_i^{\alpha_i}=1)$, B is the group presented by $(q_3, \ldots, q_k \mid q_i^{\alpha_i}=1)$, H is the subgroup of A generated by $q_1 q_2$, K is the subgroup of B generated by $q_3 \cdots q_k$, and ϕ is the isomorphism of H onto K that maps $q_1 q_2$ onto $(q_3 \cdots q_k)^{-1}$. According to Lemma 6.2 if G is to be solvable, then H is of index two in A and K is of index two in B. Thus A and B are each extensions of Z by Z_2. In particular both A and B are solvable groups. But A and B are free products of finite groups and hence by Lemma 4.1 can only be solvable if $k=4$ and $\alpha_1=\alpha_2 =\alpha_3=\alpha_4=2$.

LEMMA B. *If* $\alpha_1, \alpha_2, \alpha_3$ *are pairwise relatively prime, then* $G(0, 3|\alpha_1, \alpha_2, \alpha_3)$ *is nonsolvable.*

Proof. If we abelianize $G(0, 3|\alpha_1, \alpha_2, \alpha_3)$, we have that $q_1^{\alpha_1}=1$, $q_2^{\alpha_2}=1$, $q_1^{\alpha_3}q_2^{\alpha_3}=1$. Then $q_1^{\alpha_2 \alpha_3}=q_1^{\alpha_2 \alpha_3}q_2^{\alpha_2 \alpha_3}=(q_1^{\alpha_3}q_2^{\alpha_3})^{\alpha_2}=1$. Since α_1 and $\alpha_2\alpha_3$ are relatively prime, we have $q_1=1$. Similarly $q_2=q_3=1$. Hence $G(0, 3|\alpha_1, \alpha_2, \alpha_3)$ is in fact perfect.

We are now prepared to narrow our list of solvable groups $G(0, k|\alpha_1, \ldots, \alpha_k)$ to those given by Theorem 6.3. According to Lemma A, we need only consider the case $k=3$. Further, by Lemma B we need only consider groups $G(0, 3|\alpha_1, \alpha_2, \alpha_3)$ when there exists a prime p that divides both α_2 and α_3. In such a case, we can construct a homomorphism of $G(0, 3|\alpha_1, \alpha_2, \alpha_3)$ onto Z_p by mapping q_1 onto the identity element of Z_p, q_2 onto the generator of Z_p, and q_3 onto the inverse of the generator of Z_p. This map has kernel K which may be presented by $(x_i, q_2', q_3' \mid$ $(\prod_{i=0}^{p-1} x_i)q_2'q_3'=1, x_i^{\alpha_1}=1, q_2'^{\alpha_2'}=1, q_3'^{\alpha_3'}=1)$, $0 \leqq i < p$, $\alpha_2'=\alpha_2/p$, $\alpha_3'=\alpha_3/p$ under the map $x_i \mapsto q_2^i q_1 q_2^{-i}$, $0 \leqq i < p$, $q_2' \mapsto q_2^p$, and $q_3' \mapsto q_3^p$. Observe that K is isomorphic with $G(0, p+2|\alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_2/p, \alpha_3/p)$.

(1) $G(0, 3|\alpha, pq, pr)$, $q \neq 1$, $r \neq 1$, $(\alpha, p, q, r) \neq (2, 2, 2, 2)$. This group has a non-solvable subgroup $G(0, p+2|\alpha, \alpha, \ldots, \alpha, q, r)$.

(2) $G(0, 3|\alpha, p, p)$, $p \geqq 5$ or $p=3$ and $\alpha \geqq 4$. If $p \geqq 5$, then the subgroup $G(0, p|\alpha, \alpha, \ldots, \alpha)$ is not solvable. If $p=3$ and $\alpha \geqq 4$ then the subgroup $G(0, 3|\alpha, \alpha)$ is not solvable by the above if α is a prime or by (1) if $\alpha=p'q$, $p'a$ prime and $q \neq 1$.

(3) $G(0, 3|\alpha, p, pr)$, $r \neq 1$, $p \geqq 5$ has a subgroup $G(0, p+1|\alpha, \alpha, \ldots, \alpha, r)$ which is not solvable.

(4) $G(0, 3|\alpha, 3, 3r)$, $r \neq 1$ and $\alpha \neq 2$ or $r \neq 2$. The subgroup $G(0, 4|\alpha, \alpha, \alpha, r)$ is not solvable.

(5) $G(0, 3|\alpha, 2, 2r)$, $r \neq 1$ and $\alpha \geq 5$ or $\alpha = 4$, $r \geq 3$, or $\alpha = 3$, $r \geq 4$. The subgroup $G(0, 3|\alpha, \alpha, r)$ is not solvable. If $\alpha \geq 5$ this follows from (2) if α is prime and by (1) if α is composite. If $\alpha = 4$ and $r \geq 3$, this follows from (1). If $\alpha = 3$ and $r \geq 4$ this follows from (4).

The only groups remaining to be considered are those listed in Theorem 6.3. It is straightforward to show that each of these groups is solvable. We have the following theorem.

THEOREM 6.4. *Let M be a Seifert fiber space such that $\pi_1(M)$ is solvable. Then M has Seifert surface F either the torus, the Klein bottle, the projective plane, or the 2-sphere. Furthermore, if F is the torus or the Klein bottle then M has no singular fibers. If F is the projective plane, then M has no singular fibers, one singular fiber of arbitrary type, or two singular fibers of types $(2, \beta_1)$, $(2, \beta_2)$. If F is the 2-sphere then M has zero, one, or two singular fibers of arbitrary types, three singular fibers of types $(2, \beta_1)$, $(2, \beta_2)$, (α_3, β_3); $(2, \beta_1)$, $(3, \beta_2)$, $(3, \beta_3)$; $(2, \beta_1)$, $(3, \beta_2)$, $(4, \beta_3)$; $(2, \beta_1)$, $(3, \beta_2)$, $(6, \beta_3)$; $(2, \beta_1)$, $(4, \beta_2)$, $(4, \beta_3)$; $(3, \beta_1)$, $(3, \beta_2)$, $(3, \beta_3)$; or four singular fibers of types $(2, \beta_1)$, $(2, \beta_2)$, $(2, \beta_3)$, $(2, \beta_4)$.*

In particular, we wish to call attention to the Seifert fiber spaces M with Seifert surface a 2-sphere with three singular fibers of types $(2, \beta_1)$, $(3, \beta_2)$, $(6, \beta_3)$; $(2, \beta_1)$, $(4, \beta_2)$, $(4, \beta_3)$; or $(3, \beta_1)$, $(3, \beta_2)$, $(3, \beta_3)$ with the additional restriction that $\beta_1 \alpha_2 \alpha_3 + \alpha_1 \beta_2 \alpha_3 + \alpha_1 \alpha_2 \beta_3 + b\alpha_1 \alpha_2 \alpha_3 \neq 0$. These manifolds have infinite, solvable fundamental groups. However, in [18] Waldhausen has shown that these manifolds are not sufficiently large. These manifolds constitute the complete collection of compact, nonsufficiently large, P^2-irreducible 3-manifolds with infinite, solvable fundamental groups known to the authors.

(7) **Nilpotent groups.** In [16] Thomas classifies those nilpotent groups which occur as the fundamental group of a closed 3-manifold. Here we provide a somewhat simpler proof of the results of Thomas and extend them to the bounded case.

If M is a torus bundle over S^1, then we have an exact sequence $1 \to Z \oplus Z \to \pi_1(M) \to Z \to 1$. The generator of the infinite cyclic quotient group induces an automorphism $\theta_M : Z \oplus Z \to Z \oplus Z$. The automorphism θ_M determines the fundamental group of M up to isomorphism. Each torus bundle over S^1 is a P^2-irreducible, sufficiently large 3-manifold. Hence by a theorem of W. Heil [4], if N is any other torus bundle over S^1, and if $\theta_M = \theta_N$, then M is homeomorphic with N.

THEOREM 7.1. *Let M be a compact 3-manifold such that $\pi_2(\hat{M}) = 0$. Then $\pi_1(M)$ is an infinite, nilpotent group if and only if M is equivalent to a manifold from the following list:*

  (i) *a disk bundle over S^1;*

  (ii) *an I bundle over $S^1 \times S^1$;*

  (iii) *a torus bundle N over S^1 with θ_N of the form $\begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}$, Y an integer.*

Proof. Since $\pi_2(\hat{M}) = 0$, it follows that $\hat{M}$ is P^2-irreducible. If $\hat{M}$ has nonempty boundary, then since $\pi_1(M)$ is solvable we may apply Theorem 4.2. Eliminating the nonnilpotent fundamental group of the Klein bottle from the list of Theorem 4.2, we have that $\hat{M}$ is equivalent to either a disk bundle over S^1 or an I bundle over the torus.

If $\hat{M}$ is closed, we could again refer to the work in §4 on solvable groups; however, an easier approach is available. Since $\hat{M}$ is aspherical, it follows from Lemma 8.4 of [1] that $\pi_1(\hat{M})$ is torsion free. Hence $\pi_1(\hat{M})$ is a finitely generated, torsion free, nilpotent group. It follows that $\pi_1(\hat{M})$ is poly-infinite-cyclic. Thus there is a map f of $\pi_1(\hat{M})$ onto the infinite cyclic group. Furthermore, since $\pi_1(\hat{M})$ is polycyclic, it follows that each subgroup of $\pi_1(\hat{M})$ is finitely generated. In particular the kernel of the map f is finitely generated. It follows then from a theorem of Stallings [15] that $\hat{M}$ is homeomorphic with a closed 2-manifold bundle over S^1. Since $\pi_1(\hat{M})$ is nilpotent, $\hat{M}$ is a torus bundle over S^1.

We refer to Lemma 2 of Thomas' paper [16] for a proof that $\theta_{\hat{M}}$ is of the form $\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$.

THEOREM 7.2. *Let M be a compact 3-manifold such that $\pi_2(\hat{M}) \neq 0$. Then $\pi_1(M)$ is an infinite, nilpotent group if and only if $\pi_1(M)$ is either Z or $Z \oplus Z_2$.*

Proof. If $\pi_2(\hat{M})$ is not generated as a $\pi_1(\hat{M})$ module by its projective plane boundary components, then by Theorem 5.1, $\pi_1(M)$ is either Z, $Z \oplus Z_2$ or $Z_2 * Z_2$. Since $Z_2 * Z_2$ is centerless, it is not nilpotent.

If $\pi_2(\hat{M})$ is generated by the projective plane boundary components of $\hat{M}$, then by Theorem 5.1, $\pi_1(M)$ is a split extension of a nontrivial group G by Z_2. Furthermore, Z_2 acts without fixed points on G. Suppose the extension is split by a homomorphism that maps the generator of Z_2 onto an element t of $\pi_1(M)$. Then each element of $\pi_1(M)$ can be written in the form gt^ε where $g \in G$ and ε is either 0 or 1. Since $\pi_1(M)$ is nilpotent, there is a nontrivial element $g_0 t^\varepsilon$ in the center of $\pi_1(M)$. Since Z_2 acts without fixed points on G, $\varepsilon \neq 0$. For otherwise $g_0 t^\varepsilon$ does not commute with t. Hence $g_0 t^\varepsilon = g_0 t$. Then $t^{-1} g_0 t = t^{-1}(g_0 t) = (g_0 t) t^{-1} = g_0$. Since Z_2 acts without fixed points on G, it follows that $g_0 = 1$. But then $g_0 t^\varepsilon = t$ and t does not commute with any nontrivial element of G. Hence $\pi_1(M)$ is a centerless group. This contradiction completes the proof of Theorem 7.2.

(8) Abelian groups. In [1] Epstein lists those finitely generated abelian groups which can occur as the fundamental group of a (possibly noncompact) 3-manifold. In this section we prove that if M is a 3-manifold with nonfinitely generated, abelian fundamental group, then $\pi_1(M)$ is a subgroup of the additive group of rationals. The authors wish to thank William Jaco for considerable help in proving the theorems of this section.

An abelian group G is said to be of *local rank one* if each finitely generated subgroup of G is cyclic. The primary algebraic tool that we shall require is the following theorem which is proved in [6].

THEOREM 8.1. *If G is a torsion free abelian group of local rank one, then G is a subgroup of the additive group of rationals.*

LEMMA 8.2. *If M is a 3-manifold with nonfinitely generated abelian group then $\pi_1(M)$ is torsion free.*

Proof. Suppose $\pi_1(M)$ contains a nontrivial element x of order k, and let H denote the subgroup $\pi_1(M)$ generated by x. Then H is isomorphic with Z_k. If k is not two, then Theorem 8.2 of [1] allows us to write M as a connected sum $M = Q \# R$ where $\pi_1(R)$ is a nontrivial finite group. By van Kampen's theorem $\pi_1(M) \approx \pi_1(Q) * \pi_1(R)$. Then since $\pi_1(M)$ is abelian we have that $\pi_1(Q) = 0$ and $\pi_1(M) = \pi_1(R)$. This of course is contrary to the assumption that $\pi_1(M)$ is not finitely generated.

Thus we have that each element of $\pi_1(M)$ is either of infinite order or has order 2. Let y denote a nontrivial element of $\pi_1(M)$ other than the torsion element x above, and let K denote the subgroup of $\pi_1(M)$ generated by the elements x and y. If y were of finite order, then as noted above we would have $y^2 = 1$ and so K would be isomorphic with $Z_2 \oplus Z_2$. But Theorem 9.1 of [1] assures us that $Z_2 \oplus Z_2$ is not a subgroup of the fundamental group of any 3-manifold. Thus y must be of infinite order and so K is isomorphic with $Z \oplus Z_2$. Then by Theorem 9.5 of [1] M can be written as a connected sum $M = Q \# R$ with $\pi_1(R) \approx Z \oplus Z_2$. Since $\pi_1(M)$ is not a free product, we have $\pi_1(M) \approx Z \oplus Z_2$. This is contrary to the assumption that $\pi_1(M)$ is not finitely generated.

This completes the proof of Lemma 8.2.

We are now ready to prove the main theorem of this section.

THEOREM 8.3. *If M is a 3-manifold and if $\pi_1(M)$ is a nonfinitely generated abelian group then $\pi_1(M)$ is a subgroup of the additive group of rationals.*

Proof. In view of Theorem 8.1 and Lemma 8.2, we need only prove that $\pi_1(M)$ is of local rank one. Furthermore, it suffices to prove the theorem in the case that M is orientable. For if M is nonorientable we consider the orientable double cover $\tilde{M}$ of M. Assuming the theorem known for orientable manifolds, we have the exact sequence $1 \to \pi_1(\tilde{M}) \to \pi_1(M) \to Z_2 \to 1$ where $\pi_1(M)$ is torsion free and $\pi_1(\tilde{M})$ is of local rank one. It is then a simple exercise to show that $\pi_1(M)$ is also of local rank one.

Thus we need only establish that if M is an orientable 3-manifold with nonfinitely generated, abelian fundamental group then $\pi_1(M)$ is of local rank one. The proof is by contradiction. Thus we assume that M is an orientable 3-manifold with nonfinitely generated abelian fundamental group and $\pi_1(M)$ is not of local rank one. Then $\pi_1(M)$ contains a subgroup H isomorphic with $Z \oplus Z$. Let (N, P) denote the covering space of M associated with the subgroup H of $\pi_1(M)$ and let C denote the group of covering transformations of (N, P).

LEMMA A. *There exists an embedding* $f: S^1 \times S^1 \to N$ *such that* $f_*: \pi_1(S^1 \times S^1)$ $\to \pi_1(N)$ *is an isomorphism.*

Proof. Let $g_1: S^1 \times S^1 \to N$ be a map such that $g_{1_*}: \pi_1(S^1 \times S^1) \to \pi_1(N)$ is an isomorphism, and let K_1 denote a regular neighborhood of $g_1(S^1 \times S^1)$ in N. If the boundary of K_1 is compressible in N, then there is a disk D in N such that $D \cap$ bd K_1 = bd D and bd D is a nontrivial curve in bd K_1. If $D \cap$ int $K_1 = \varnothing$, then we set $g_2 \equiv g_1$, and $K_2 = K_1 \cup \mathscr{U}(D, N)$. Otherwise we have $D \subset K_1$. We change g_1 by a small isotopy if necessary so that g_1 is a general position map with respect to the disk D. Then $g_1^{-1}(g_1(S^1 \times S^1) \cap D)$ consists of a collection of mutually exclusive simple closed curves on $S^1 \times S^1$. If α is a component of $g_1^{-1}(g_1(S^1 \times S^1) \cap D)$ then $g_1(\alpha) \subset D$ so that $g_1(\alpha)$ is contractible in N. Since g_{1_*} is an isomorphism, it follows that α is contractible on $S^1 \times S^1$. Thus each simple closed curve α in $g_1^{-1}(g_1(S^1 \times S^1) \cap D)$ bounds a disk D_α on $S^1 \times S^1$. Choose new base points if necessary so that the base point of $S^1 \times S^1$ does not lie on any disk D_α for α a component of $g_1^{-1}(g_1(S^1 \times S^1) \cap D)$. g_{1_*} remains an isomorphism. Now let α denote a fixed component of $g_1^{-1}(g_1(S^1 \times S^1) \cap D)$. Define $g_2: S^1 \times S^1 \to N$ as follows. We set $g_2|_{\mathrm{cl}\,(S^1 \times S^1 - D_\alpha)} = g_1|_{\mathrm{cl}\,(S^1 \times S^1 - D_\alpha)}$, and $g_2|_{D_\alpha}$ is a contraction of $g_2(\alpha)$ in D. We then change g_2 by a small homotopy so that $g_2(D_\alpha)$ lies on the proper side of a regular neighborhood of D. Now $g_1 \equiv g_2$ outside a regular neighborhood of D_α, and each loop l in $S^1 \times S^1$ is homotopic relative to the base point to a loop l' in $S^1 \times S^1$ such that $l' \cap D_\alpha = \varnothing$. It follows that $g_{2_*}: \pi_1(S^1 \times S^1) \to \pi_1(N)$ is an isomorphism. Continuing in this fashion we see that it is possible to construct a map $g_r: S^1 \times S^1 \to N$ such that $g_{r_*}: \pi_1(S^1 \times S^1) \to \pi_1(N)$ is an isomorphism, $g_r(S^1 \times S^1) \subset K_1$, and $D \cap g_r(S^1 \times S^1) = \varnothing$. Then we set $K_2 = \mathrm{cl}\,(K_1 - \mathscr{U}(D, N))$.

If bd K_2 is compressible in N, we repeat the above process to construct K_3. Eventually we arrive at a map $g_s: S^1 \times S^1 \to N$ and a compact manifold $K_t \subset N$ such that $g_{s_*}: \pi_1(S^1 \times S^1) \to \pi_1(N)$ is an isomorphism, $g_s(S^1 \times S^1) \subset K_t$, and bd K_t is incompressible in N.

Define $g_s': S^1 \times S^1 \to K_t$ by $g_s'(x) = g_s(x)$. We have the following commutative diagram where i_* is induced by inclusion.

$$
\begin{array}{ccc}
\pi_1(S^1 \times S^1) & \xrightarrow{\;g_{s_*}'\;} & \pi_1(K_t) \\
 & \searrow{\scriptstyle g_{s_*}} & \downarrow{\scriptstyle i_*} \\
 & & \pi_1(N)
\end{array}
$$

Now g_{s_*}' and i_* are injections and g_{s_*} is an isomorphism. Hence g_{s_*}' and i_* are also isomorphisms. Thus K_t is a compact 3-manifold with $\pi_1(K_t) = Z \oplus Z$. It then follows from a theorem of Heil [4] that K_t is equivalent to an I bundle over $S^1 \times S^1$. Since N is orientable we have that in fact K_t is equivalent to $S^1 \times S^1 \times I$. Let $f: S^1 \times S^1$

$\to K_t$ be an embedding such that $f_* \colon \pi_1(S^1 \times S^1) \to \pi_1(K_t)$ is an isomorphism. Then f is the map required by Lemma A.

Let $f(S^1 \times S^1) = F$. Since $i_* \colon \pi_1(F) \to \pi_1(N)$ is an isomorphism F must separate N.

LEMMA B. *Let τ be a covering transformation of (N, p) such that $F \cap \tau(F) = \varnothing$. Then F and $\tau(F)$ cobound a manifold K_τ such that K_τ is homeomorphic to $F \times [0, 1] \# R$ where R is a possibly noncompact, simply connected 3-manifold.*

Proof. Since F and $\tau(F)$ both separate N, F and $\tau(F)$ cobound a manifold K_τ. Since F and $\tau(F)$ are incompressible in N, we have that $j_* \colon \pi_1(K_\tau) \to \pi_1(N)$ is a monomorphism. We have the following commutative diagram of maps induced by inclusion.

$$\pi_1(F) \xrightarrow{\ i_*\ } \pi_1(K_\tau)$$
$$k_* \searrow \qquad \downarrow j_*$$
$$\pi_1(N)$$
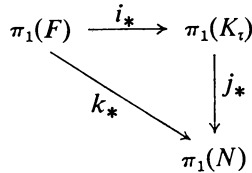
Since each map is a monomorphism and k_* is an isomorphism, we have that i_* is also an isomorphism. Thus each loop in $\tau(F)$ is freely homotopic in K_τ to a loop in F. Then proceeding closely along the lines of Lemma 5.1 of [17] we are able to construct a pair of annuli A, B such that bd $A = a_1 \cup a_2$, bd $B = b_1 \cup b_2$ with the following properties:

(1) A meets B transversely in a single arc k with endpoints k_1, k_2,

(2) $k_1 = a_1 \cap b_1$, $k_2 = a_2 \cap b_2$,

(3) a_1, b_1 is a pair of nontrivial simple closed curves in $\tau(F)$ that meet in a single transverse intersection point k_1,

(4) a_2, b_2 is a pair of nontrivial simple closed curves in F that meet in a single transverse intersection point k_2.

Let X denote a regular neighborhood of $F \cup \tau(F) \cup A \cup B$ in K_τ. We note that bd X is a 2-sphere in K_τ. Since $\pi_1(K_\tau)$ is not a nontrivial free product, we must have that bd X bounds a simply connected manifold R^* in K_τ. Lemma B follows.

LEMMA C. *C, the group of covering transformations of (N, p), contains an element of infinite order.*

Proof. Since C is an infinite group, there is a covering transformation τ in C such that $F \cap \tau(F) = \varnothing$. Let F separate N into components N_1 and N_2. With no loss of generality we assume that $\tau(F) \subset N_1$. We have either $\tau(N_1) = \mathrm{cl}\,(N_1 - K_\tau)$ or $\tau(N_2) = \mathrm{cl}\,(N_1 - K_\tau)$. If $\tau(N_1) = \mathrm{cl}\,(N_1 - K_\tau)$, then $\cdots \subsetneqq \tau^2(N_1) \subsetneqq \tau(N_1) \subsetneqq N_1$. Hence τ would be of infinite order. Thus we assume that $\tau(N_1) = N_2 \cup K_\tau$, and $\tau(N_2) = \mathrm{cl}\,(N_1 - K_\tau)$. By Lemma B we can write $K_\tau = K \# R$ where K is compact and R is simply connected. Since for each $\sigma \in C$, $\sigma(F)$ is incompressible in N, it is not possible that $\sigma(F) \subset R$ for any $\sigma \in C$. Thus if $\sigma \in C$, then $\sigma(F) \cap K_\tau \neq \varnothing$ if and only if $\sigma(F) \cap K \neq \varnothing$. Since K is compact we have that there are only finitely many

covering transformations σ such that $\sigma(F) \cap K_\tau \neq \varnothing$. Thus we are able to find some $\lambda \in C$ such that $\lambda(F) \cap K_\tau = \varnothing$. If $\lambda(F) \subset N_2$ we put $\lambda_1 = \tau\lambda$; otherwise, $\lambda_1 = \lambda$. Thus we have $\lambda_1 \in C$ such that $\lambda_1(F) \subset N_1 - K_\tau$. We have either $\lambda_1(N_1) = \mathrm{cl}\ (N_1 - K_{\lambda_1})$ or $\lambda_1(N_2) = \mathrm{cl}\ (N_1 - K_{\lambda_1})$. If $\lambda_1(N_1) = \mathrm{cl}\ (N_1 - K_{\lambda_1})$, then $\cdots \subsetneqq \lambda_1^2(N_1) \subsetneqq \lambda_1(N_1) \subsetneqq N_1$. Hence λ_1 is of infinite order. We are left with the case that $\lambda_1(N_2) = \mathrm{cl}\ (N_1 - K_{\lambda_1})$ $\subsetneqq \mathrm{cl}\ (N_1 - K_\tau) = \tau(N_2)$. Then $\cdots \subsetneqq (\tau^{-1}\lambda_1)^2(N_2) \subsetneqq (\tau^{-1}\lambda_1)(N_2) \subsetneqq N_2$. Hence $\tau^{-1}\lambda_1$ is of infinite order. This completes the proof of Lemma C.

The supposition that $\pi_1(M)$ is not of local rank one has now led us to the following situation. H is a subgroup of $\pi_1(M)$, and H is isomorphic with $Z \oplus Z$. Furthermore, since $\pi_1(M)/H$ is isomorphic with the group of covering transformations of (N, p), we have that $\pi_1(M)/H$ contains an element of infinite order. It follows that $\pi_1(M)$ contains a subgroup K isomorphic with $Z \oplus Z \oplus Z$. We note that $Z \oplus Z \oplus Z$ is the fundamental group of the closed, aspherical, 3-manifold $S^1 \times S^1 \times S^1$. Then Theorem 8.8 of [1] allows us to write M as a connected sum, $M = Q \# R$ where K is of finite index in $\pi_1(R)$. By van Kampen's theorem, $\pi_1(M) \approx \pi_1(Q) * \pi_1(R)$. But $\pi_1(M)$ is not a nontrivial free product. It follows that $\pi_1(M) \approx \pi_1(R)$ contrary to the assumption that $\pi_1(M)$ is not finitely generated. Thus we have in fact that $\pi_1(M)$ is a torsion free, abelian group of local rank one. It follows that $\pi_1(M)$ is a subgroup of the additive group of rationals.

We do not know if a nonfinitely generated abelian group can occur as a subgroup of the fundamental group of a compact 3-manifold. However each subgroup of the additive group of rationals is the fundamental group of some 3-manifold. In order to show this, it clearly suffices to construct a 3-manifold with fundamental group the additive group of rationals. The example that we shall provide seems to be well known, but we include it for the sake of completeness. The example is a noncompact 3-manifold M which is the complement of a solenoid in S^3. We construct M as follows.

Let k_1 be an unknotted simple closed curve in S^3, and let T_1^1 denote a regular neighborhood of k_1. Let $T_1^2 = \mathrm{cl}\ (S^3 - T_1^1)$. Let x_1 denote the generator of $\pi_1(T_1^1)$ and y_1 denote the generator of $\pi_1(T_1^2)$. Beginning with T_1^1 and T_1^2 we construct two sequences of solid tori $T_1^2 \subset T_2^2 \subset \cdots$, $T_1^1 \supset T_2^1 \supset \cdots$ where T_{n+1}^1 and T_{n+1}^2 are constructed as follows. Let k_{n+1} be a simple closed curve in the interior of T_n^1 such that k_{n+1} is unknotted in S^3, and k_{n+1} is homotopic in T_n^1 to x_n^{n+1} where x_n is the generator of $\pi_1(T_n^1)$. Let T_{n+1}^1 denote a regular neighborhood of k_{n+1} in T_n^1, and put $T_{n+1}^2 = \mathrm{cl}\ (S^3 - T_{n+1}^1)$. Let x_{n+1} be the generator of $\pi_1(T_{n+1}^1)$, and let y_{n+1} be the generator of $\pi_1(T_{n+1}^2)$.

Set $M = \bigcup_{n=1}^\infty T_n^2$. Using van Kampen's theorem we obtain a presentation for $\pi_1(M)$ as a direct limit of infinite cyclic groups. $\pi_1(M)$ is presented by

$$(y_1, y_2, y_3, \ldots \mid y_1 = y_2^2, y_2 = y_3^3, \ldots).$$

Then $\pi_1(M)$ is isomorphic with the additive group of rationals under a map which takes y_n onto $1/n!$.

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DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TEXAS 77001

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

Current address (Benny Evans): Institute for Advanced Study, Princeton, New Jersey 08540

Current address (Louise Moser): Department of Mathematics, California State College at Hayward, Hayward, California 94542