THE VARIATION OF SINGULAR CYCLES IN AN ALGEBRAIC FAMILY OF MORPHISMS

BY

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Abstract. (1) Let \( g: V \to W \) be a morphism of nonsingular varieties over an algebraically closed field. Under certain conditions, one can define a cycle \( S_i \) on \( V \) with \( \text{Supp} (S_i) = \{ x \mid \dim_{\kappa(x)} (\Omega^1_{V/W}(x)) \geq i \} \).

The multiplicity of a component of \( S_i \) can be computed directly from local equations for \( g \). If \( V \subset P^n \), and if \( g: V \to P^n \) is induced by projection from a suitable linear subspace of \( P^n \), then \( S_i \) is \( c_{n-i+1}(N \otimes \mathcal{O}(-1)) \), up to rational equivalence, where \( N \) is the normal bundle of \( V \) in \( P^n \).

(2) Let \( f: X \to S \) be a smooth projective morphism of noetherian schemes, where \( S \) is connected, and the fibres of \( f \) are absolutely irreducible \( r \)-dimensional varieties. For a geometric point \( \eta: \text{Spec} (k) \to S \), and a locally free sheaf \( E \) on \( X \), let \( X_\eta \) be the corresponding geometric fibre, and \( E_\eta \) the sheaf induced on \( X_\eta \). If \( E_1, \ldots, E_m \) are locally free sheaves on \( X \), and if \( i_1 + \cdots + i_m = r \), then the degree of the zero-cycle \( c_{i_1}(E_1) \cdots c_{i_m}(E_m) \) is independent of the choice of \( \eta \).

(3) The results of (1) and (2) are used to study the behavior under specialization of a closed subvariety \( V' \subset P^{2r-1} \) which is the image under generic projection of a nonsingular \( V' \subset P^n \).

1. Introduction. Let \( V \) be a nonsingular \( r \)-dimensional projective variety over an algebraically closed field \( k \). If \( V \) is projected generically onto \( V' \subset P^{2r-1} \), then \( V' \) has a singular curve with finitely many points of a type known as pinch points (cf. §5). Suppose that \( \text{char} (k) = 0 \) and that \( V \) can be specialized (along with its projective embedding) to a nonsingular variety \( V_1 \), defined over \( k_1 \), which can be projected generically onto \( V'_1 \subset P^{2r-1} \). We can ask whether \( V'_1 \) has the same number of pinch points as \( V' \). The answer is "yes" if \( \text{char} (k_1) \neq 2 \); if \( \text{char} (k_1) = 2 \), then \( V'_1 \) has half as many pinch points as \( V' \).

In this paper we develop some techniques which enable one to answer this and other enumerative questions of a similar nature. In §2, we prove a result which says roughly that a Chern polynomial of weight \( r \) is constant in a connected family of nonsingular projective varieties of dimension \( r \) (cf. Theorem 1). In §3 we recall the definition of the dependency cycle of a set of sections of a locally free sheaf on \( V \). Mattuck [6] has shown how to express the rational equivalence class of this cycle as a Chern class of \( E \). We express the multiplicities of its components as the
lengths of certain Fitting ideals (cf. Proposition 4). In the case that the set of sections is part of a Serre sequence, these results follow from Theorem 2.7 of [3]. In §4, we define, under suitable assumptions, the singular cycles $S_\ell$ of a morphism $f: V \to W^m$ of nonsingular varieties ($m \geq r$). Intuitively, $\text{Supp} (S_\ell)$ is the set of points where the kernel of the tangent map has dimension $\geq \ell$. Our definition is stated in a form which gives immediately the multiplicities of the components of $S_\ell$. Let $V \subset \mathbb{P}^n$, and let $\pi: V \to \mathbb{P}^n$ be induced by generic projection. (See Lemma 3 for the meaning of "generic" in this context.) Theorem 2 says that the rational equivalence class of $S_\ell(\pi)$ is $c_{m-\ell+1}(N \otimes \mathcal{O}(-1))$, where $N$ is the normal bundle of $V$ in $\mathbb{P}^n$. Finally, §5 gives the application of the results of §§2, 3, and 4 to the problem stated in the first paragraph. We also give a concrete example to illustrate our result.

We will deal with Chern classes constructed in the rational equivalence ring $\mathcal{A}(V)$. Our references for this topic are Grothendieck’s appendix to the Borel-Serre paper [4], and Séminaire Chevalley 1958, “Anneaux de Chow et applications.” As usual, Grothendieck’s *Eléments de géométrie algébrique* is denoted EGA.

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2. Chern classes and algebraic families. Let $f: X \to S$ be a smooth projective morphism of noetherian schemes. Assume that $S$ is connected and that all fibres of $f$ are absolutely connected. In particular, $f$ is flat, and the fibre $X_s = f^{-1}(s)$ is an absolutely nonsingular irreducible projective variety over $k(s)$ for all $s \in S$, where $k(s)$ is the residue field of $s$.

Let $E_1, \ldots, E_m$ be locally free sheaves on $X$. For each geometric point $\eta: \text{Spec} (k) \to S$ ($k$ is an algebraically closed field), let $X_\eta$ be the corresponding geometric fibre, and let $E_{1\eta}, \ldots, E_{m\eta}$ be the sheaves induced on $X_\eta$ by $E_1, \ldots, E_m$. We will consider the Chern classes $c_i(E_{\eta})$, which are elements of $\mathcal{A}(X_\eta)$, the Chow ring of $X_\eta$. For a rational equivalence class, $z$, of zero-cycles on $X_\eta$, we will denote by $\deg_\eta (z)$ the degree of $z$. Finally, we note that the dimension of $X_\eta$ is independent of $\eta$.

**Theorem 1.** Let $f: X \to S$ and $E_1, \ldots, E_m$ be as above. If $i_1, \ldots, i_m$ is a sequence of positive integers such that $i_1 + \cdots + i_m = \dim (X_\eta)$, and $i_j \leq \text{rk} (E_j)$, for $j = 1, \ldots, m$, then the value of

$$\deg_\eta (c_{i_1}(E_{1\eta}) \cdots c_{i_m}(E_{m\eta}))$$

is independent of the choice of the geometric point, $\eta$, of $S$.

**Remark.** The $E_i$ need not be distinct.

**Proof.** We first consider the case where the $E_i$ are all invertible, so that $c_i(E_{\eta}) = 0$ for $i > 1$. Thus let $r = \dim (X_\eta)$, and let $L_1, \ldots, L_r$ be invertible sheaves on $X$ (not necessarily distinct). Then

$$\deg (c_1(L_{1\eta}) \cdots c_1(L_{r\eta})) = (L_{1\eta} \cdots L_{r\eta}),$$

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which is the intersection number, computed on \( X_n \). Using the techniques of [7, Lecture 12], one shows

\[
(L_1 \cdots L_n)
\]

\[
= \chi(\mathcal{O}_{X_n}) - \sum_{i=1}^{r} \chi(L_{\gamma i}^{-1}) + \sum_{i < j} \chi(L_{\gamma i}^{-1} \otimes L_{\gamma j}^{-1}) - \cdots + (-1)^r \chi(L_{\gamma 1}^{-1} \otimes \cdots \otimes L_{\gamma n}^{-1}).
\]

(By induction on the dimension, one shows that our expression is correct when \( L_1 \cong \mathcal{O}_Y(D) \), with \( D \) very ample, and is linear in each variable.)

The fact that this intersection number is independent of \( \eta \) is a consequence of the following lemma.

**Lemma 1.** Let \( f: X \to S \) be a projective morphism, where \( S \) is noetherian, and let \( E \) be a coherent sheaf on \( X \) which is flat over \( S \). For a field, \( k \), and a \( k \)-valued point \( \eta: \text{Spec}(k) \to S \), let

\[
\chi(E, \eta) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X_{\eta}, E_{\eta}),
\]

where \( E_{\eta} \) is the sheaf induced on the fibre \( X_{\eta} \). If \( S \) is connected, then \( \chi(E, \eta) \) is independent of the choice of \( k \) and \( \eta \).

This lemma is a consequence of EGA III.7.9.11; cf. also 7.7.4, 7.7.12(i), and 7.9.3 of the same chapter.

We will now reduce the general case to the case just considered. We claim that there is a smooth projective morphism \( g: Y \to X \) with connected fibres such that

1. For each \( j \), \( g^*E_j \) has a filtration of locally free subsheaves

\[
E_{j1} \subset E_{j2} \subset \cdots \subset E_{j, p_j} = g^*E_j,
\]

where \( p_j = \text{rk} (E_j) \), such that the quotients \( L_{ij} = E_{ij}/E_{ij-1} \) are invertible.
2. There are invertible sheaves, \( \Lambda_1, \ldots, \Lambda_n \), on \( Y \) and positive integers, \( d_1, \ldots, d_n \), such that
   \[(a)\] \( d_1 + \cdots + d_n = d = \text{dimension of any fibre of } g, \]
   \[(b)\] \( g_n*(c_1(\Lambda_1)^{d_1} \cdots c_1(\Lambda_n)^{d_n}) = 1_{X_n} \)

for all geometric points \( \eta: \text{Spec}(k) \to S \), where \( g_n \) is obtained by base extension.

We will first use (1) and (2) to achieve the desired reduction. From (1) we have

\[
(c((g^*E_j)_n) = \prod (1 + c_1(L_{j, n})).
\]

Next, let \( z \in \mathcal{A}(X_n) \), and set \( y_n = (c_1(\Lambda_1)^{d_1} \cdots c_1(\Lambda_n)^{d_n}) \). Using the projection formula (cf. [1, p. 3-17]), and (2), we obtain

\[
z = z \cdot 1_{X_n} = z \cdot g_n*(y_n) = g_n*(g_n^*(z) \cdot y_n).
\]

In particular, this implies that

\[
\deg_n(z) = \deg_n (g_n^*(z) \cdot y_n)
\]
if \( z \) is the class of a zero-cycle on \( X_n \). Note that the expression on the right is the degree of a zero-cycle on \( Y_n \).

We apply (**) in the case \( z = c_1(E_1) \cdots c_m(E_m) \) and use (*) to express \( g_z^*(z) - y_n \) as a polynomial of degree \( r + d \) in the \( c_1(\Lambda_{1,n}) \) and the \( c_1(L_{1,n}) \). In this way, we reduce the question to the case of the theorem already proved.

We now prove the existence of \( Y \). In the case \( m = 1 \), write \( E = E_1 \), and take \( Y = \text{Flag}(E) \), the flag bundle of \( E \) over \( X \) (cf. [1, pp. 4–18 and 4–19]). Thus, if \( p = \text{rk}(E) \), we have a sequence of morphisms

\[
Y = P_{p-1} \to P_{p-2} \to \cdots \to P_1 \to P_0 = X,
\]
such that \( P_i = \mathcal{P}(E) \), and \( P_{i+1} = \mathcal{P}(F_i) \), for \( i \geq 1 \), \( F_i \) being defined by the exactness of

\[
0 \to F_i \to q_i^*(F_{i-1}) \to \mathcal{O}_{P_i}(1) \to 0,
\]
where \( q_i : P_i \to P_{i-1} \), and \( F_0 = E \). Now, it is known that

\[
q_i^*((c_1(\mathcal{O}_{P_i}(1)))^e) = 0, \quad e < p - i,
\]

\[
= 1, \quad e = p - i
\]
(cf. [1, p. 4–13]). Letting \( g_i : Y \to P_i \) be the composition \( P_{p-1} \to \cdots \to P_i \), we set \( \Lambda_i = g_i^*(\mathcal{O}_{P_i}(1)) \) and \( d_i = p - i \). Writing \( g = g_0 \) and using (***), and the projection formula repeatedly, we find

\[
g_{n*}(c_1(\Lambda_{1,n})^{p-1} \cdots c_1(\Lambda_{p-1,n})) = 1_{X_n}.
\]

In the case \( m > 1 \), we proceed by induction on \( m \). Suppose that \( q : Z \to X \) and the invertible sheaves \( \Lambda_1, \ldots, \Lambda_s \) on \( Z \) have properties (1) and (2) relative to \( E_1, \ldots, E_m \). Let \( Y = \text{Flag}(q^*E_m) \), and let \( r : Y \to Z \). We pull back \( \Lambda_1, \ldots, \Lambda_s \) to \( r^*\Lambda_1, \ldots, r^*\Lambda_s \) on \( Y \) and form \( p - 1 \) other invertible sheaves on \( Y \) by the process used in the case \( m = 1 \), where \( p = \text{rk}(E_m) \). Setting \( g = q \circ r \), one uses the projection formula and the fact that \( g_{n*} = q_{n*} \circ r_{n*} \) to check that the sheaves constructed on \( Y \) have property (2).

3. Dependency cycles. In this section, \( V \) will be a nonsingular quasi-projective variety defined over an algebraically closed field \( k \), and \( E \) will be a locally free sheaf on \( V \), of rank \( p \leq \dim(V) \). We will recall the definition of the dependency cycle of a set of sections of \( E \) (cf. Mattuck [6, §4]), and we will give an expression for the multiplicities of the components of this cycle.

Let \( \hat{E} = \mathcal{P}(\mathcal{O}_V \oplus E^*) \) (cf. EGA II, §4), where \( E^* = \mathcal{H}_{\text{ext}}(E, \mathcal{O}_V) \), and let \( \pi : \hat{E} \to V \) be the natural projection. Let \( \sigma_0 : V \to \hat{E} \) correspond to the surjection \( \mathcal{O}_V \oplus E^* \to \mathcal{O}_V \) which restricts to the identity on \( \mathcal{O}_V \) and to the zero map on \( E^* \). Thus \( V \cong \sigma_0(V) \).

Let \( s \in \Gamma(V, E) \), and let \( \sigma : V \to \hat{E} \) correspond to the surjection \( \mathcal{O}_V \oplus E^* \to \mathcal{O}_V \) which restricts to the identity on \( \mathcal{O}_V \) and to the map dual to \( s : \mathcal{O}_V \to E \) on \( E^* \).

Under the assumption that \( \sigma^{-1}(\sigma_0(V)) = \{ x \in V \mid s(x) = 0 \} \) is of pure codimension \( p \) in \( V \), we say that the cycle of zeros of \( s \) is defined; this cycle is defined to be \( \sigma^*(\sigma_0(V)) \) and is denoted \( s^*(0) \).
Let us write $s^*(0) = \sum \mu_Z Z$; the sum ranges over the irreducible components of $\sigma^{-1}(\sigma_0(X))$. For a fixed $Z$, let $x$ be the generic point of $Z$, and let $U$ be a neighborhood of $x$ on which $E$ is free. On $U$, we can write $s = \sum_{i=1}^p f_i e_i$, where $f_i \in \Gamma(U, \mathcal{O}_V)$, and $\{e_1, \ldots, e_p\}$ is a basis of $\Gamma(U, E)$.

**Proposition 1.** With the above notation, $\mu_Z = \ell_A(A/(f_1, \ldots, f_p))$, where $A = \mathcal{O}_{V,x}$, and $\ell_A(M)$ denotes the length of the $A$-module $M$.

**Proof.** Since $\sigma_0(V)$ and $\sigma(V)$ are locally complete intersections on $\tilde{E}$, one can check that $\ell_A(A/(f_1, \ldots, f_p))$ is the multiplicity of the corresponding component of $\sigma_0(V) \cdot \sigma(V)$. Q.E.D.

**Proposition 2.** If $s \in \Gamma(V, E)$ is such that $s^*(0)$ is defined, then $s^*(0) = \operatorname{cp}_s(E)$ in $\mathcal{A}(V)$.

The proof can be modeled directly after one given by Grothendieck [4, Theorem 2].

Let $q \leq p$, and let $s_1, \ldots, s_q \in \Gamma(V, E)$. Now, $s_1, \ldots, s_q$ define a map of $\mathcal{O}_V$-modules, $\phi: \mathcal{O}_V^q \to E$; by duality we get $\phi^*: E^* \to \mathcal{O}_V^q$. Let $P = V \times \mathbb{P}^{q-1}$, and let $\pi: P \to V$ and $\rho: P \to \mathbb{P}^{q-1}$ be the two projections. Then $\phi^*$ gives rise to a map of $\mathcal{O}_P$-modules: $\pi^* \mathcal{O}_E^* \to \rho^* \mathcal{O}(1)$. Tensoring with $\mathcal{O}(-1)$ and dualizing, we obtain $s \in \Gamma(P, \pi^* \mathcal{O} \otimes \mathcal{O}(1))$. We will make two assumptions:

1. $s^*(0)$ is defined;
2. $\pi$ sends every irreducible component of $s^*(0)$ to a subvariety of codimension $p-q+1$ in $V$.

When these assumptions are satisfied, we define the dependency cycle, $D(\Sigma)$ of the set $\Sigma = \{s_1, \ldots, s_q\}$ to be $\pi_* (s^*(0))$.

**Proposition 3.** Let $\Sigma \subseteq \Gamma(V, E)$ and $s \in \Gamma(P, \pi^* \mathcal{O} \otimes \mathcal{O}(1))$ be as above. If $D(\Sigma)$ is defined, then $D(\Sigma) = \operatorname{cp}_{p-q+1}(E)$ in $\mathcal{A}(V)$.

For a proof, see Mattuck [6, Theorem 2].

We will now give a local description of the section $s \in \Gamma(P, \pi^* \mathcal{O} \otimes \mathcal{O}(1))$. Let $U = \text{Spec}(A) \subseteq V$ be an affine open set on which $E$ is free, and let $\{e_1, \ldots, e_p\}$ be a basis of $\Gamma(U, E)$. Thus, $s_i = \sum_{j=1}^p f_i e_j$, where $f_i \in \Gamma(U, \mathcal{O}_V)$, for $i=1, \ldots, q$. Let $T_1, \ldots, T_q$ be a basis of $\Gamma(\pi^{-1}U, \mathcal{O}(1))$, and let $U_m$ be the open subset where $T_m \neq 0$, for $1 \leq m \leq q$. It is easy to verify:

\[
(*) \quad s|_{U_m} = \left( \sum_{i=1}^q \sum_{j=1}^p (t_i|_{T_m}) f_i e_j \right) \otimes T_m,
\]

where $T_i|_{U_m} = (t_i|_{T_m}) \cdot (T_m|_{U_m})$. This implies

\[
(**) \quad s|_{(\pi^{-1}U)} = \sum_{i,j} f_{ij} (e_j \otimes T_i).
\]

Relation $(*)$ also implies that $x \in \text{Supp}(D(\Sigma))$ iff $s_1, \ldots, s_q$ become dependent in $E \otimes \mathbb{k}(x)$. 

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Assume that $D(\Sigma)$ is defined, and let $D(\Sigma) = \sum \mu_Z \cdot Z$. For a fixed $Z$, let $x$ be a generic point of $Z$, and choose $U \subseteq V$, as above, so that $x \in U$.

**Proposition 4.** With the above notation and assumptions, $\mu_Z = \ell_A(A/I)$, where $A = \mathcal{O}_V_{x, x}$, and $I$ is the ideal in $A$ generated by the $q \times q$ minors of the $q \times p$ matrix $(f_{ij})$. Thus, $I$ is the $0$th Fitting ideal of Coker $(\phi_x)$, where $\phi: \mathcal{O}_x^q \to E$ is defined by $s_{11}, \ldots, s_q$.

**Proof.** Every $q \times q$ minor of the matrix $(f_{ij})$ vanishes along $W = Z \cap U$. Assumption (2) implies that some $(q-1) \times (q-1)$ minor of $(f_{ij})$ is nonzero at $x$. We may assume that this minor is nonzero at all points of $U$. Thus, some subset of $\Sigma$, say $\{s_1, \ldots, s_{q-1}\}$, is a subset of a basis of $\Gamma(U, E)$.

Let $J = \Gamma(U, \mathcal{O}_U)$ be generated by the $q \times q$ minors of the matrix $(f_{ij})$. Since $E|U$ is free, $J$ is the $(p-q)$th Fitting ideal of Coker $(\Gamma(U, \mathcal{O}_U)^q \to \Gamma(U, E))$. Hence $J$ is independent of the choice of basis of $\Gamma(U, E)$ (cf. Fitting [2, Hauptsatz]), and we may assume that $e_i = s_i$, for $i = 1, \ldots, q-1$. Therefore $J$ is generated by the $p-q+1$ elements $f_{q+1}, \ldots, f_{qp}$. Moreover, the relation $(**)$ becomes

$$s|(\pi^{-1}U) = \sum_{j=1}^{q-1} (e_i \otimes T_j + f_{q}(e_j \otimes T_q)) + \sum_{j=q}^{p} f_{q}(e_j \otimes T_q).$$

Thus, $\text{Supp } (s^*(0) \cap U) \subseteq U_q$. Now, $(*)$ becomes

$$s|U_q = \sum_{j=1}^{q-1} (t_j + f_{q})(e_j \otimes T_q) + \sum_{j=q}^{p} f_{q}(e_j \otimes T_q),$$

where we have set $t_q = 1$. Let $Z'$ be the unique component of $s^*(0)$ lying above $Z$, let $y$ be the generic point of $Z'$, and let $B = \mathcal{O}_{E^y, y}$. If $\mu_{Z'}$ is the multiplicity of $Z'$ in $s^*(0)$, Proposition 1 implies that $\mu_{Z'} = \ell_B(B/I^*)$, where $I^* = (t_1 + f_{q+1}, \ldots, t_{q-1} + f_{q,q-1}, f_{q+1}, \ldots, f_{pq})B$.

Further, we have isomorphisms

$$B/I^* \cong (A/I)[T_1, \ldots, T_{q-1}]/(T_1 + f_{q+1}, \ldots, T_{q-1} + f_{q,q-1}) \cong A/I.$$ 

This implies that (i) $A$ and $B$ have the same residue field, and (ii) $\ell_A(A/I) = \ell_B(B/I^*)$. Now, (i) implies that $\pi_* (Z') = Z$. Using (ii), we obtain $\mu_Z = \ell_A(A/I)$. Q.E.D.

**4. The cycles $S_i$.** Let $f: V^r \to W^m$ be a morphism of nonsingular varieties over the algebraically closed field $k$, where $m \geq r$. Let $\Omega^1_{V/W}$ be the sheaf of relative differentials, and let $S_i \subseteq V$ be the closed subset $\{x | \dim k(x) (\Omega^1_{V/W} \otimes k(x)) \geq i\}$, for each $i \geq 1$. We will say that $S_i$ has the proper codimension if every irreducible component has codimension $i(m-r+i)$. If $S_i$ has the proper codimension, we define the cycle $S_i$ by

$$S_i = \sum \nu_Z \cdot Z;$$
the sum ranges over all components, and \( v_z = \ell_{e_x}(\mathcal{O}_x/J) \), where \( x \) is the generic point of \( Z \), and \( J \) is the \((i-1)\)st Fitting ideal of \((\Omega^1_{V/W})_x \). We must check that \( v_z \) is finite. We have an exact sequence of \( \mathcal{O}_x \)-modules

\[
\Omega^1_{\mathcal{E}_y/k} \otimes \mathcal{O}_x \xrightarrow{\phi} \Omega^1_{\mathcal{E}_y/k} \to \Omega^1_{\mathcal{E}_y/e_y} \to 0,
\]

where \( y = f(x) \). The first two terms in this sequence are free of rank \( m \) and \( r \) respectively. If we choose suitable bases for these modules, then \( \phi \) is described by an \( m \times r \) matrix, and \( J \) is generated by its minors of rank \( r-i+1 \). Since \( Z \) is the only component of \( S_i \) containing \( x \), the maximal ideal \( m_X \) is the only associated prime of \( J \). Therefore \( v_z \) is finite. We will call the cycles \( S_i \) the singular cycles of \( f \).

**Lemma 2.** Suppose that \( S_i \) is of the proper codimension. Let \( Z \) be a component of \( S_i \), with \( Z \nsubseteq S_2 \), and with generic point \( x \). Then \( v_z = \ell_{e_x}((\Omega^1_{V/W})_x) \).

**Proof.** Since \((\Omega^1_{V/W})_x \) is generated by one element, \((\Omega^1_{V/W})_x \cong \mathcal{O}_x/J \), where \( J \) is the 0th Fitting ideal. Q.E.D.

If \( x \) is a closed point of \( V \), then \( x \in S_i \) iff \( \dim_k (m_x/(m_x\mathcal{O}_x + m_y^2)) \geq i \), where \( y = f(x) \), and \( m_x \) and \( m_y \) are the maximal ideals of \( \mathcal{O}_x \) and \( \mathcal{O}_y \). In particular, suppose that \( V \subset P^n \) and that \( f: V \to P^m \) is induced by projection from an \((n-m-1)\)-subspace \( L \subset P^n \) such that \( L \cap V = \emptyset \). For a closed point \( x \in V \), it follows that \( x \in S_i \) iff \( \dim (L \cap t_{v,x}) \geq i-1 \), where \( t_{v,x} \) is the \( r \)-subspace of \( P^n \) tangent to \( V \) at \( x \). (This can be checked using the techniques of the proof of Proposition 3 of [8].)

**Lemma 3.** Let \( l \leq m < n \), and let \( V' \subset P^n \) be nonsingular. Then there is a dense open subset of the Grassmann variety \( G = G(n, n-m-1) \) consisting of linear subspaces \( L \subset P^n \) such that \( L \cap V = \emptyset \), and \( S_l(\pi) \) is purely of codimension \( i(m-r+i) \) for all \( i \), where \( \pi: V \to P^m \) is induced by projection from \( L \).

**Proof.** For each \( i \), consider the correspondence \( Z_i \subset V \times G \) consisting of all \((x, L)\) such that \( \dim (L \cap t_{v,x}) \geq i-1 \). By a counting of constants which uses standard facts about Schubert cycles on \( G \) (cf. [5, Chapter XIV, §2]), one finds that \( \dim (Z_i) = \dim (G) + r - i(m-r+i) \). Q.E.D.

Henceforth we will fix an \((n-m-1)\)-subspace \( L \subset P = P^n \), such that \( L \cap V = \emptyset \), and \( S_i = S_l(\pi) \) is of the proper codimension for all \( i \), where \( \pi: V \to P^m \) is induced by projection from \( L \). We will also fix a basis, \( \{T_0, \ldots, T_n\} \) of \( \Gamma(P, \mathcal{O}(1)) \), such that \( L \) is given by \( T_0 = \cdots = T_m = 0 \).

**Theorem 2.** Let \( V, L, \) and \( \pi \) be as above. Then

\[ S_1 = c_{m-r+1}(N \otimes \mathcal{O}_y(-1)) \text{ in } \mathcal{A}(V), \]

where \( N \) is the normal bundle of \( V \) in \( P^n \), viz., \( N = (I/I^2)^* = \mathcal{H}om_{\mathcal{O}_y}(I/I^2, \mathcal{O}_y) \), where \( I = \text{Ker} (\mathcal{O}_p \to \mathcal{O}_y) \).

**Proof.** For \( j = 0, \ldots, n \), let \( T_j \) be as above, and let \( U_j \subset P \) be the open set \( \{x \mid T_j(x) \neq 0\} \). Choose \( t_0 = 1, t_1, \ldots, t_n \in k(P) \) such that \( T_i = (t_i/t_j)T_j \) on \( U_j \). Thus,
\( \Gamma(U_j, \mathcal{O}_P) \) is the polynomial ring \( k[t_0/t_j, \ldots, t_n/t_j] \). For \( 0 \leq i \leq n \) and \( i \neq j \), let \( D_{ij} \) be the derivation of \( \Gamma(U_j, \mathcal{O}_P) \) given by \( D_{ij}(t_k/t_j) = \delta_{ij} \) (Kronecker delta), and let \( D_{jj} = -\sum_{i \neq j} (t_i/t_j) D_{ij} \). We can extend these derivations to derivations of the function field \( k(P) = k(t_1, \ldots, t_n) \) and check that \( D_{ij}(t_i/t_n) D_{ijn} \) for all \( i, j, h \).

For each \((i, j)\), \( D_{ij} \) induces an element \( \bar{D}_{ij} \in \Gamma(U_j, (I/I^2)^*) \). We define

\[
s_{ij} = \bar{D}_{ij} \otimes T^{-1} \in \Gamma(U_j, N \otimes \mathcal{O}(-1)).
\]

Since \( \bar{D}_{ij}|U_j \cap U_h = (t_j/t_i) \bar{D}_{ih}|U_j \cap U_h \), the various \( s_{ij} \) (for each \( i \)) fit together to give sections \( s_0, \ldots, s_n \in \Gamma(V, N \otimes \mathcal{O}(-1)) \).

Let \( x \) be any closed point of \( V \). Since \( L \cap \mathcal{V} = \emptyset \), we have \( T_q(x) \neq 0 \) for some \( q \leq m \). We may assume \( q = 0 \); thus \( x \in U_0 \). Let \( g_1, \ldots, g_{n-r} \) generate \( I \) in a neighborhood of \( x \), and let \( \sigma_i : I/I^2 \rightarrow \mathcal{O}_V \) be given locally by \( \sigma_i([g_j]) = \delta_{ij} \). With these notations, we find that

\[
s_i = \sum_{j=1}^{n-r} (\partial g_j/\partial t_i)(\sigma_j \otimes T_0^{-1})
\]

in some neighborhood of \( x \). (The \( g_j \) are polynomials in \( t_1, \ldots, t_n \).) In particular, it follows that \( s_{m+1}, \ldots, s_n \) become dependent in \((N \otimes \mathcal{O}(-1)) \otimes k(x) \) iff \( x \in S_1 \). To see this, we note that \( L \cap t_{V,x} = \emptyset \) iff there are elements \( b_{m+1}, \ldots, b_n \in k \), not all zero, such that \( \sum_{i=m+1}^{n} b_i (\partial g_i/\partial t_i)(x) = 0 \), for \( i = 1, \ldots, n-r \). Since \( S_1 \) is purely of codimension \( m-r+1 = (n-r)-(n-m)+1 \), it follows that \( D(\Sigma) \) is defined \((\Sigma = \{s_{m+1}, \ldots, s_n\}) \), and \( \text{Supp} (D(\Sigma)) = \text{Supp} (S_1) \). Proposition 4 implies that \( D(\Sigma) = \sum \mu_Z Z \), where \( \mu_Z = \ell_{e_\Sigma} (\mathcal{O}_X/J' \), with \( x \) the generic point of \( Z \), and \( J' \) the ideal in \( \mathcal{O}_X \) generated by the minors of order \((n-m)-1 \) of the \((n-r) \times (n-m)-1 \) matrix \((\partial g_i/\partial t_j)_{1 \leq i \leq n-r, m+1 \leq j \leq n} \). Finally, Proposition 3 implies that \( D(\Sigma) = c_{m-r+1}(N \otimes \mathcal{O}_V(-1)) \) in \( \mathcal{A}(V) \).

Since \( S_1 \) and \( D(\Sigma) \) have the same irreducible components, it will follow that \( S_1 = D(\Sigma) \) if we can show that the ideal \( J' \) defined above is the 0th Fitting ideal of \( \Omega_{x,\mathcal{V}}^1(y = \pi(x)) \). This will complete the proof. Thus, let \( U = P^n - L = U_0 \cup \cdots \cup U_m \). The projection \( \pi : P^n - L \rightarrow P^n \) has the property that \( \pi|U_i \) looks like the projection \( A^n \times A^{n-m} \rightarrow A^n \). Hence in the exact sequence of \( \mathcal{O}_V \)-modules

\[
I/I^2 \xrightarrow{\delta} \Omega_{B/P^n} \otimes \mathcal{O}_V \xrightarrow{\varepsilon} \Omega_{V/P^n}^1 \xrightarrow{\varepsilon} 0,
\]

the first two terms are free. Moreover, if \( x \in U_0 \), and if \( g_1, \ldots, g_{n-r} \) generate \( I_x \), then \( \delta \) is given by

\[
\delta([g_i]) = \sum_{j=m+1}^{n} (\partial g_j/\partial t_i)(dt_i \otimes 1),
\]

for \( i = 1, \ldots, n-r \). We conclude that \( J' \) is the 0th Fitting ideal of \( \Omega_{V/P^n}^1 \). Q.E.D.

**Corollary 1.** With the assumptions of Theorem 2,

\[
S_k = \sum_{j=0}^{m+1} \binom{m+1}{j} \gamma_{q-j} h^j \text{ in } \mathcal{A}(V),
\]
where $h$ is the divisor class of a hyperplane section, \((1 - \gamma_1 + \gamma_2 - \cdots + (-1)^r\gamma_r) = c(\Omega_h^1)^{-1}$, and \(q = m - r + 1\).

**Proof.** With $P = P^n$, let $D_{ij}$ be the derivations of $k(P)$ defined above. Let $\Theta = (\Omega_h^1)^*$, and let $s_0, \ldots, s_n \in \Gamma(P, \Theta \otimes \mathcal{O}_P(-1))$ satisfy $s_i | U_j = D_{ij} \otimes T_j^{-1}$. The $s_i$ give a map $\psi: \mathcal{O}_P^{n+1} \rightarrow \Theta \otimes \mathcal{O}_P(-1)$ and thus an exact sequence

\[ 0 \rightarrow \Omega_h^1 \rightarrow \mathcal{O}_P(-1)^{n+1} \rightarrow \mathcal{O}_P \rightarrow 0. \]

This gives $c(\Omega_h^1) = (1 - h)^{n+1}$.

On $V$ we have an exact sequence

\[ 0 \rightarrow (I/I^2) \rightarrow \Omega_h^1 \otimes \mathcal{O}_V \rightarrow \Omega_h^1 \rightarrow 0. \]

Thus $c(N^*) = j^*(c(\Omega_h^1) \cdot c(\Omega_h^1)^{-1})$, where $j: V \rightarrow P$. The rest of the computation will be omitted.

**Corollary 2.** If $m = r = 1$, and if the curve $V \subset P^n$ has genus $g$ and degree $d$, then

\[ 2g - 2 = -2d + \sum_{x \in S_1(x)} e_x((\Omega_{x/p}^1) \chi). \]

This is a special case of Hurwitz' formula for the genus change under a morphism of curves. For a proof, use the exact sequence

\[ 0 \rightarrow I/I^2 \rightarrow \Omega_h^1 \otimes \mathcal{O}_V \rightarrow \Omega_h^1 \rightarrow 0. \]

(with $j: V \rightarrow P^n$) to show that $\deg (N \otimes \mathcal{O}(-1)) = 2g + 2d - 2$. Lemma 2 shows that the summation on the right side of the formula also is $\deg (N \otimes \mathcal{O}(-1))$.

**Generalization.** It is also possible to express the cycles $S_i$, $i > 1$, in terms of Chern classes. Thus let $G' = G'(n, n - r - 1)$ be the Grassmannian which parameterizes $(n - r)$-quotients of rank $n + 1$ free sheaves, and let $\Phi: \mathcal{O}_P^{n+1} \rightarrow E$ be the universal surjection. There is a morphism $\psi: V \rightarrow G'$ such that $\Phi$ pulls back to $\psi: \mathcal{O}_P^{n+1} \rightarrow N \otimes \mathcal{O}_V(-1)$. The Chern classes of $E$ can be expressed in terms of Schubert cycles on $G'$. On the other hand, it seems clear that the Schubert cycles which pull back to the cycles $S_i$ can be expressed in terms of the Fitting ideals of $\text{Coker} (\Phi)$. Thus, one should obtain formulas which are similar to formula (10) on p. 357 of [5].

5. **Enumeration of pinch points.** Let $V^r$ be a nonsingular projective variety over an algebraically closed field $k$. One can find a projective embedding $V \subset P^n$ such that there is a finite morphism $\pi: V \rightarrow P^{2r-1}$ induced by projection from an $(n - 2r)$-subspace $L \subset P^n$ satisfying

(I) $S_i(\pi)$ is purely 0-dimensional, and $\pi|S_i(\pi)$ is injective. Moreover, if $r \geq 2$, then $\pi^{-1}(\pi(x)) = \{x\}$ for all $x \in S_i(\pi)$.

(II) If $x \in S_i(\pi)$ and $y = \pi(x)$, then $\hat{\mathcal{O}}_{V,x}$ and $\hat{\mathcal{O}}_{P^{2r-1},y}$ can be identified with formal power series rings $B = k[[t_1, \ldots, t_r]]$ and $A = k[[t_1, \ldots, t_{2r-1}]]$ so that $\pi$ induces the
homomorphism \( f: A \rightarrow B \) given by

\[
\begin{align*}
f(t_i) &= t_i & \text{for } i = 1, \ldots, r-1, \\
f(t_i) &= t_{i-r+1}t_r & \text{for } i = r, \ldots, 2r-2, \\
f(t_{2r-1}) &= t_r^2 + t_r^3.
\end{align*}
\]

(If \( \text{char} \ (k) \neq 2 \) we can replace \( t_r^2 + t_r^3 \) by \( t_r^2 \).)

In fact, if \( r \geq 2 \), a suitable embedding may be found by replacing any given embedding by the embedding determined by hypersurface sections of degree \( d \geq 2 \), and Theorem 3 of [8] states that if \( L \) is chosen generically, then \( \pi \) has the following properties which imply (I) and (II). If \( V' = \pi(V) \), then \( V' \) is birational to \( V \), \( \text{Sing} \ (V') \) is purely of dimension 1, and \( V' \) has singular branches at only finitely many closed points \( y \in V' \). If \( V' \) has a singular branch at \( y \), then \( \partial V - y \) is isomorphic to \( f(A) \), where \( f: A \rightarrow B \) is as above. (Recall that if \( \{x\} = \pi^{-1}(y) \), then \( x \in S_1(\pi) \) iff \( y \) has a singular branch at \( y \).)

If \( r = 1 \), and \( V \) is of genus \( g \), one embeds \( V \) by using a complete linear system of degree \( \geq 2g + 3 \) and uses techniques like those used in the proof of Theorem 3 of [8] to obtain (I) and (II).

**Proposition 5.** Let \( \pi: V \rightarrow P^{2r-1} \) be as above, and let \( S_1 = \sum_x v_x \cdot x \), where the summation extends over all points of \( \text{Supp} \ (\Omega^1_{\pi_! P^{2r-1}}) \).

If \( \text{char} \ (k) \neq 2 \), then \( v_x = 1 \) for all \( x \).

If \( \text{char} \ (k) = 2 \), then \( v_x = 2 \) for all \( x \).

**Proof.** Let \( x \in S_1(\pi) \) and \( y = \pi(x) \). Since \( \pi \) is finite, \( \mathcal{O}_x \cong R_m \), where \( R \) is a semi-local ring which is a finite \( \mathcal{O}_y \)-module, and \( m \) is maximal. Hence, \( \hat{R} \cong R \otimes_{\mathcal{O}_y} \hat{\mathcal{O}}_y \), and \( \hat{\mathcal{O}}_x \cong R/a \), where \( a \) is generated by idempotent elements. Therefore, \( \Omega^1_{\mathcal{O}_x/\mathcal{O}_y} \cong \Omega^1_{\hat{\mathcal{O}}_x/\mathcal{O}_y} \otimes \hat{\mathcal{O}}_x \), so that the 0th Fitting ideal of \( \Omega^1_{\hat{\mathcal{O}}_x/\mathcal{O}_y} \) is \( I \cdot \hat{\mathcal{O}}_x \), where \( I \) is the 0th Fitting ideal of \( \Omega^1_{\hat{\mathcal{O}}_x/\mathcal{O}_y} \). Let \( A = k[[t_1, \ldots, t_{2r-1}]], B = k[[t_1, \ldots, t_r]] \), and let \( f: A \rightarrow B \) be given as above. Then \( \mathcal{O}_x(\mathcal{O}_x/I) = \mathcal{O}_B(J) \), where \( J \) is the 0th Fitting ideal of \( \Omega^1_{B/A} \). We have an exact sequence of \( B \)-modules

\[
\Omega^1_{A/k} \otimes_A B \xrightarrow{u} \Omega^1_{B/k} \xrightarrow{} \Omega^1_{B/A} \rightarrow 0.
\]

(Cf. EGA 0IV, 20.7.17.3 and 01, 7.3.5.) The first two terms are free, and \( u \) is given by

\[
\begin{align*}
u(dt_i \otimes 1) &= dt_i & \text{for } i = 1, \ldots, r-1, \\
u(dt_i \otimes 1) &= t_r dt_{i-r+1} + t_i \cdot t_{r+1} dt_r & \text{for } i = r, \ldots, 2r-2, \\
u(dt_{2r-1} \otimes 1) &= (2t_r + 3t_r^2) dt_r.
\end{align*}
\]

The 0th Fitting ideal is thus \( J = (t_1, \ldots, t_{r-1}, t_r) \) if \( \text{char} \ (k) \neq 2 \), and \( J = (t_1, \ldots, t_{r-1}, t_r^2) \) if \( \text{char} \ (k) = 2 \). The length of \( B/J \) is 1 (resp. = 2) if \( \text{char} \ (k) \neq 2 \) (resp. = 2). Q.E.D.

**Remark.** As a consequence of Proposition 5, the number of points in \( S_1(\pi) \) is independent of the choice of projection center, \( L \), provided (I) and (II) are satisfied.
We will now see how the number of points varies as $V$ is specialized. Thus, let $A$ be a noetherian ring, and $X$ a closed subscheme of $P^n_A = \text{Proj } A[T_0, \ldots, T_n]$; assume that $p: X \to \text{Spec } (A)$ is smooth and has absolutely irreducible fibres of dimension $r$. Assume that $n \geq 2r - 1$ and that $X$ does not meet the closed subscheme given by $T_0 = \cdots = T_{2r-1} = 0$. We define $N$ to be the normal bundle of $X$ in $P = P^n_A$; thus $N = (I/I^2)^*$, $I = \text{Ker } (O_p \to O_X)$.

We consider geometric points $\eta: \text{Spec } (k) \to \text{Spec } (A)$ ($k = \bar{k}$) such that the projection $\pi_\eta$ of the geometric fibre $X_\eta \subset P^n_k$ from the linear subspace $T_0 = \cdots = T_{2r-1} = 0$ satisfies (I) and (II) above. Now, the bundle $N_\eta$ induced on $X_\eta$ by $N$ is just the normal bundle of $X_\eta$ in $P^n_k$. Thus, by Theorem 2, the degree of the cycle $S_\eta(\pi_\eta)$ is $\text{deg } (c_1((N \otimes O_X(-1))))$. By Theorem 1, this is independent of $\eta$. Using Proposition 5, we obtain the following conclusion.

**Proposition 6.** Let $A$ and $X \subset P^n_A$ be as above. Assume that $\text{Spec } (A)$ is connected. For $i = 1, 2$, let $\eta_i: \text{Spec } (k_i) \to \text{Spec } (A)$ be geometric points such that the corresponding projections $\pi_{\eta_i}$ both satisfy (I) and (II). If $\text{char } (k_1)$ and $\text{char } (k_2)$ are both $\neq 2$ or both $= 2$, then $\#(\text{points in } S_1(\pi_{\eta_1})) = \#(\text{points in } S_1(\pi_{\eta_2}))$. If $\text{char } (k_1) = 2$ and $\text{char } (k_2) \neq 2$, then $\#(\text{points in } S_1(\pi_{\eta_1})) = \frac{1}{2} \#(\text{points in } S_1(\pi_{\eta_2})).$

If $r \geq 2$, we substitute $\#(\text{pinch-points of } \pi_{\eta_i}(X_{\eta_i}))$ for $\#(\text{points in } S_1(\pi_{\eta_i}))$ to obtain a statement about the behavior under specialization of the number of pinch-points. If $r = 1$, we obtain a similar statement about the behavior under specialization of the number of ramification points of the covering $V \to P^1$.

**Example.** Let $V^2 \subset P^5$ be the Veronese surface, i.e. the image of $P^2$ embedded by the complete linear system of conics. Explicitly, let points of $P^5$ have homogeneous coordinates $(y_{ij})$ with $0 \leq i \leq j \leq 2$. Then $(x_0, x_1, x_2) \in P^2$ is sent to the point of $P^5$ with $y_{ij} = x_i x_j$. Let $\pi: V \to P^3$ be induced by projection from the line $y_{01} = y_{12} = y_{00} + y_{11} + y_{22} = 0$. Thus, the composed map $P^2 \to P^3$ sends $(x_0, x_1, x_2)$ to $(x_1 x_2, x_0 x_2, x_0 x_1, x_0^2 + x_1^2 + x_2^2)$, and the image is the surface $V' \subset P^3$ whose equation is $t_0^2 t_1^2 + t_1^2 t_2^2 + t_2^2 t_3^2 - t_0 t_1 t_2 t_3 = 0$. The singular locus of $V'$ consists of the three lines $\Lambda_0$, $\Lambda_1$, and $\Lambda_2$ given respectively by $t_1 = t_2 = 0$, $t_0 = t_2 = 0$, and $t_0 = t_1 = 0$. If $\text{char } (k) \neq 2$, then $V'$ has 6 pinch-points, two of which lie on each of the lines $\Lambda_i$; if $\text{char } (k) = 2$, then $V'$ has 3 pinch-points, one on each $\Lambda_i$.

To see this, note that $C_i = \pi^{-1}(\Lambda_i)$ is a plane conic for each $i$. The projection center meets the plane of $C_i$ in a point which lies on two tangent lines of $C_i$ if $\text{char } (k) \neq 2$, but on just one tangent line of $C_i$ if $\text{char } (k) = 2$. (The thing to note is that if $\text{char } (k) = 2$, there is a point, not on the projection center, which lies on every tangent line of $C_i$.) It might also be noted that the plane conic provides the simplest example of the case $r = 1$ of Proposition 6.

**References**


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